1) Thermal emission is a process in which the thermal energy provides non-zero density of electrons at energies larger than the potential barrier allowing them to escape. The current density associated with this process can be written as

$$J_z = \int_{E \geq E_{\min}} en(E) v_z(E) dE$$

$n(E)dE$ is density of electrons per unit volume, $v_z(E)$ is velocity distribution along *z*-component (perpendicular to the surface). The integral is evaluated for energies sufficient to escape the barrier, i.e. $E \geq E_{\min} = e(\varphi_W + \varphi_F)$. Derive Richardson-Dushman equation recalling that

$$n(E)dE = \frac{8\pi m^3}{h^3} \frac{v^2 dv}{1 + \exp\left(\frac{E - e\varphi_F}{kT}\right)} \cong \frac{2m^3}{h^3} \exp\left(-\frac{E - e\varphi_F}{kT}\right) 4\pi v^2 dv .$$

Solution:

Using the fact that $E = \frac{1}{2} m v^2 = \frac{1}{2} m (v_x^2 + v_y^2 + v_z^2)$ and $4\pi v^2 dv \to dv_x dv_y dv_z$, rewrite current density

$$J_z = \frac{2em^3}{h^3} e^{\frac{e\varphi_F}{kT}} \int_{-\infty}^{\infty} e^{-\frac{mv_x^2}{2kT}} dv_x \int_{-\infty}^{\infty} e^{-\frac{mv_y^2}{2kT}} dv_y \int_{v_{z,\min}}^{\infty} e^{-\frac{mv_z^2}{2kT}} v_z dv_z$$

here $\frac{mv_z^2}{2} \geq e(\varphi_F + \varphi_W)$. Using $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi / a}$ and $\int x e^{-ax^2} dx = -\frac{e^{-ax^2}}{2a}$ we get

$$J_z = \frac{2em^3}{h^3} e^{\frac{e\varphi_F}{kT}} \frac{\pi 2kT}{m} \frac{e^{\frac{-mv_{z,\min}^2}{2kT}}}{m / kT} = \frac{4em\pi k^2}{h^3} T^2 e^{\frac{-e\varphi_W}{kT}}$$

2) a) Derive Child-Langmuir formula. b) For initial tests with Cornell ERL gun it is planned to use a CW laser to investigate the photocathode lifetime issues. Gun power supply will be limited to 300 kV for these tests. Estimate illuminated laser spot size required to produce 100 mA average current. Assume a planar diode geometry and 5 cm cathode-anode gap.

Solution:
a) We assume one-dimensional problem. Potential satisfies Poisson equation

$$\frac{d^2V}{dz^2} = -\frac{\rho}{\varepsilon_0}.$$

Current density and charge density are related by $J_z = \rho v_z$, while the velocity is found through energy conservation

$$\frac{1}{2}mv_z^2 = eV.$$

Eliminating ρ and v_z, Poisson distribution is rewritten as

$$\frac{d^2V}{dz^2} = -\frac{J_z}{\varepsilon_0}\sqrt{\frac{m}{2eV}}.$$

First, we need to determine if J_z depends on the coordinate. From charge conservation we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial J_z}{\partial z} = 0,$$

thus, $J_z = const$ in steady state ($\partial\rho/\partial t = 0$). Therefore, one solves the differential equation for V. Use sample solution $V(z) = Az^B$ (note that $V(0) = 0$, and $dV(0)/dz = -E_z = 0$, or field at the cathode vanishes, just like Child law argues), substituting and solving for constants yields

$$V = \left(-\frac{9J_z}{4\varepsilon_0}\sqrt{\frac{m}{2e}}\right)^{2/3} z^{4/3}, \text{ or } J_z = -\frac{4\varepsilon_0}{9}\sqrt{\frac{2e}{m}}\frac{V^{3/2}}{z^2}.$$

b) Answer: 0.65 mm^2.

3) In computer simulations of the space charge inside the bunch, one uses 'macroparticles' with the same charge to mass ratio to reduce the required computational resources. Discuss what happens to simulated beam's Debye length and plasma frequency as opposed to real case scenario. In this respect, what artificial effects may be introduced in simulations?

Solution:
Plasma frequency is given by

$$\omega_p = \sqrt{\frac{q^2 n}{\varepsilon_0 m \gamma^3}}$$

with q and m being (macro)particle's charge and mass. 'Macroparticles' are chosen to have the total charge equal to the actual value Q, therefore the particle density $n \propto Q/q$, and $\omega_p \propto \sqrt{q/m}$. Plasma frequency of simulated distribution is the same as in the actual beam.

Debye length is defined as

$$\lambda_D \equiv \frac{\sigma_{v_x}}{\omega_p}.$$

Because the velocities of 'macroparticles' are the same as for the actual particles, it follows that Debye length in simulated bunch is identical to the actual beam. What is different in simulated vs. actual case? The number of particles in the Debye sphere is smaller in the simulated bunch (or, equivalently, inter-particle distance is larger than in the actual beam). It follows from slide 6 in the lecture, that 'graininess' of Coulomb forces become more significant in this case, and, therefore, fields of individual particles tend to be overemphasized.