USPAS course on

Recirculated and Energy Recovered Linacs

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Electron Sources: Single Particle Dynamics, Space Charge Limited Emission

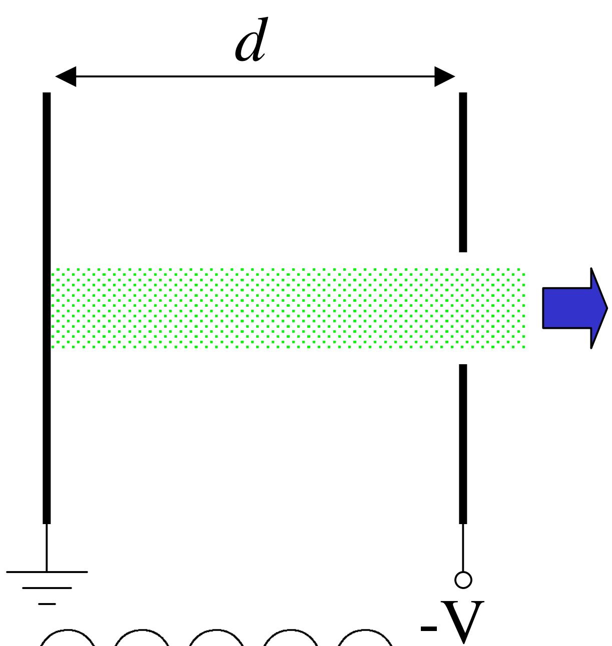
Contents

- Child-Langmuir limit
- Space charge limit with short pulses
- Busch's theorem
- Paraxial ray equation
- Electrostatic and magnetostatic focusing
- RF effects on emittance
- Drift bunching

Child-Langmuir limit

So far we have discussed current density available from a cathode.

Child-Langmuir law specifies maximum current density for a space-charge limited, nonrelativistic, 1-D beam *regardless* of available current density from the cathode. The law has a limited applicability to R&ERLs guns (applies to continuous flow, few 100s kV DC guns), but provides an interesting insight.



1D problem.

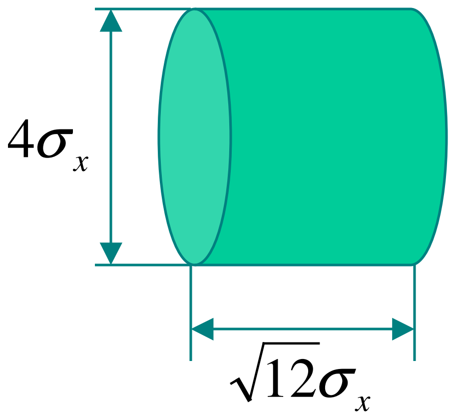
$$J = \frac{4\varepsilon_0}{9}\sqrt{\frac{2e}{m}}\frac{V^{3/2}}{d^2}$$

$$J[\mathrm{A/cm^2}] = 2.33 E^{3/2}[\mathrm{MV/m}]/\sqrt{d[\mathrm{cm}]}$$

Bunch charge limit in guns

Let's estimate bunch charge limit of a short pulse in a gun.

Assume 'beer-can' with rms $\sigma_{x,y}$ σ_t



also that E_{cath} does not change much over the bunch duration (usually true for photoguns)

If $\frac{eE_{cath}\times(c\sigma_t)}{mc^2} << 1$ or $\frac{E_{cath}[\text{MV/m}]\times(c\sigma_t)[\text{mm}]}{511} << 1$

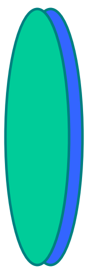
motion during emission stays nonrelativistic.

Aspect ratio of emitted electrons near the cathode after the laser pulse has expired:

$$A = \frac{\perp}{\parallel} = \frac{2\sigma_x}{3(c\sigma_t)} \frac{mc^2}{eE_{cath}(c\sigma_t)} = \frac{341}{E_{cath}[\text{MV/m}]} \frac{\sigma_x[\text{mm}]}{(c\sigma_t[\text{mm}])^2}$$

Bunch charge limit in guns (contd.)

More often than not A >> 1 in photoinjectors, i.e. the bunch looks like a pancake near the cathode (!).



From PHYS101 (note a factor of 2 due to image charge)

$$E_{s.c.} = \frac{\sigma}{\varepsilon_0} \rightarrow \begin{aligned} q &= 4\pi\varepsilon_0 E_{cath}\sigma_x^2 \\ &= 0.11 \times E_{cath}[\mathrm{MV/m}]\sigma_x[\mathrm{mm}]^2 \ \mathrm{nC} \end{aligned}$$

if emittance is dominated by thermal energy of emitted electrons, the following scaling applies (min possible emit.)

$$\varepsilon_n[\mathrm{mm\text{-}mrad}] \geq 4\sqrt{q[\mathrm{nC}]E_{th}[\mathrm{eV}]/E_{cath}[\mathrm{MV/m}]}$$

Beam dynamics

Beam dynamics without collective forces is simple.

$$\vec{\nabla}\cdot\vec{B}=0 \quad \vec{\nabla}\times\vec{H}=\frac{\partial\vec{D}}{\partial t}+\vec{J} \quad \vec{\nabla}\cdot\vec{D}=\rho \quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

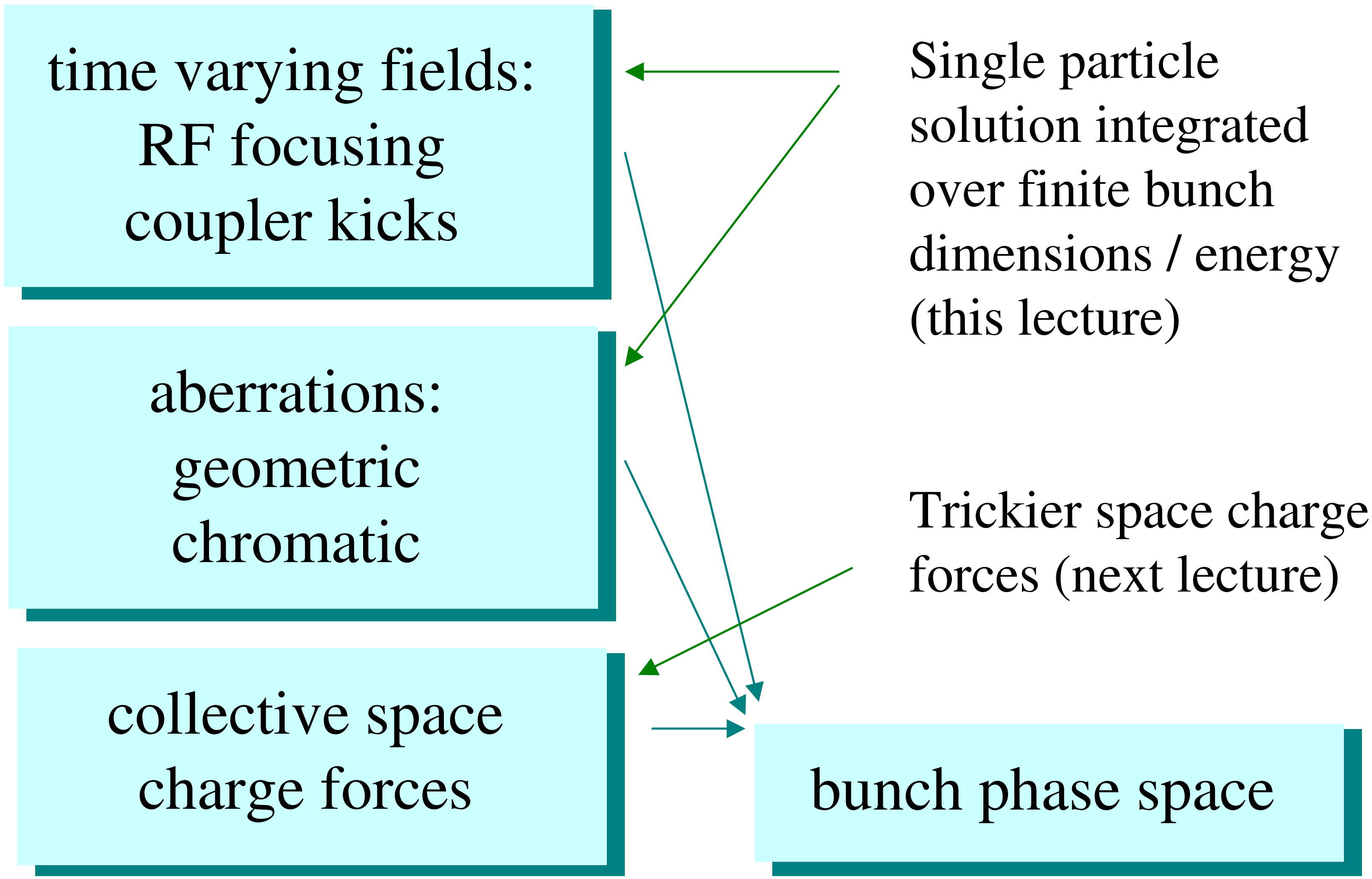
$$\vec{B}=\mu\vec{H}$$

$$\vec{D}=\varepsilon\vec{E} \qquad \frac{d}{dt}(\gamma m\vec{\beta}c)=e(\vec{E}+\vec{v}\times\vec{B})$$

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$$

Calculating orbits in known fields is a single particle problem.

Effects on the phase space



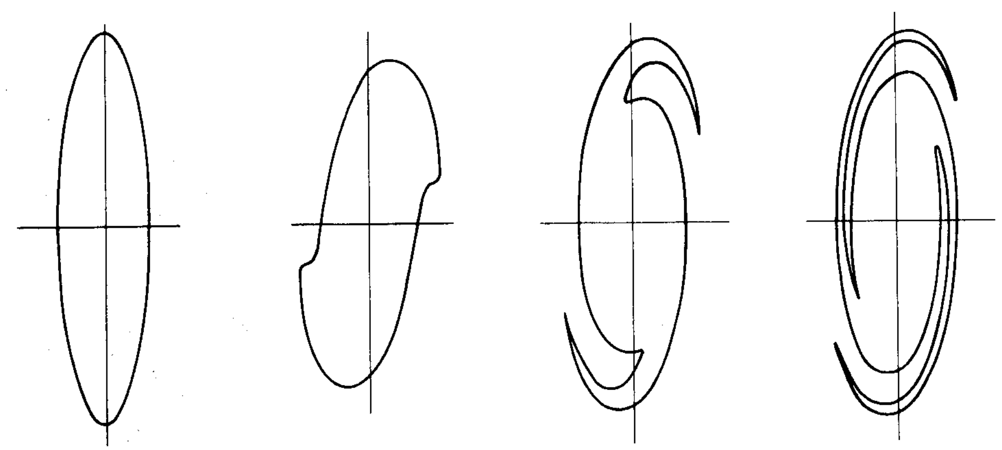
Emittance

Since emittance is such a central concept / parameter in the accelerator physics, it warrants few comments.

For Hamiltonian systems, the phase space density is conserved (a.k.a. Liouville's theorem). Rms (normalized) emittance most often quoted in accelerators' field is based on the same concept and defined as following [and similarly for (y, py) or (E, t)]

$$\varepsilon_{n,x} = \frac{1}{mc}\sqrt{\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2} = \beta\gamma\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2} = \beta\gamma\varepsilon_x$$

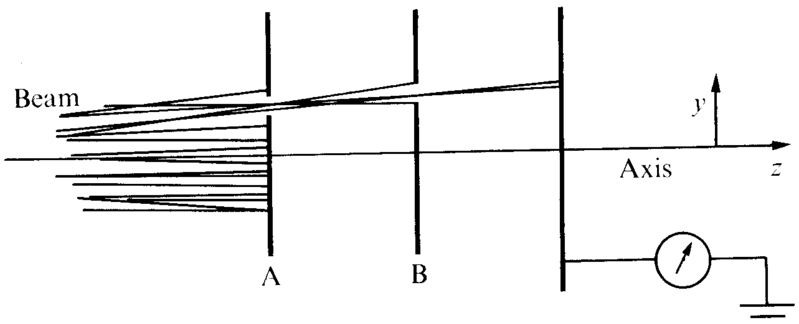
Strictly speaking, this quantity is not what Lioville's theorem refers to, i.e. it does not have to be conserved in Hamiltonian systems (e.g. geometric aberrations 'twist' phase space, increasing *effective* area, while *actual* phase space area remains constant). *Rms emittance is conserved for linear optics (and no coupling) only.*



Emittance measurement

Usefulness of rms emittance: it enters envelope equations & can be readily measured, but it provides *limited* info about the beam.

slits / pinholes



The combination of two slits give position and divergence → direct emittance measurement. Applicable for space charge dominated beams (if slits are small enough).

lens / quadrupole scan

We'll see later that envelope equation in drift is

$$\sigma'' \approx \cancel{\frac{I}{2I_0}\frac{1}{\gamma^3\sigma}} + \frac{\varepsilon^2}{\sigma^3}$$

Vary lens strength and measure size to fit in eqn with 3 unknowns to find ε.

OK to use if $\sigma << \varepsilon_n \sqrt{2\gamma I_0 / I}$

Busch's theorem

Consider axially symmetric magnetic field

$$F_\theta = -e(\dot{r}B_z - \dot{z}B_r) = \frac{1}{r}\frac{d}{dt}(\gamma m r^2 \dot{\theta})$$

Flux through a circle centered on the axis and passing through *e*

$$\Psi = \int_0^r 2\pi r B_z dr$$

When particle moves from (r,z) to (r+dr, z+dz) from $\vec{\nabla}\cdot\vec{B} = 0$

$$\frac{d\Psi}{dt} = 2\pi r(\dot{r}B_z - \dot{z}B_r) \Rightarrow \quad \dot{\theta} = \frac{-e}{2\pi\gamma m r^2}(\Psi - \Psi_0)$$

$\dot{\theta} = 0$ (at Ψ_0)

Busch's theorem simply states that canonical angular momentum is conserved

$$P_\theta = erA_\theta + \gamma m r^2\dot{\theta} \quad (\Psi \to 2\pi r A_\theta,\ \Psi_0 \to 2\pi P_\theta / e \to \text{get Busch's formula})$$

Magnetized beam (immersed cathodes)

If magnetic field $B_z \neq 0$ at the cathode, the bunch acquires angular velocity

$$\dot{\theta} = -\frac{eB_z}{2\gamma m} \;\rightarrow\; \sigma_{p_\perp} = \gamma m \sigma_{x,y} \dot{\theta}$$

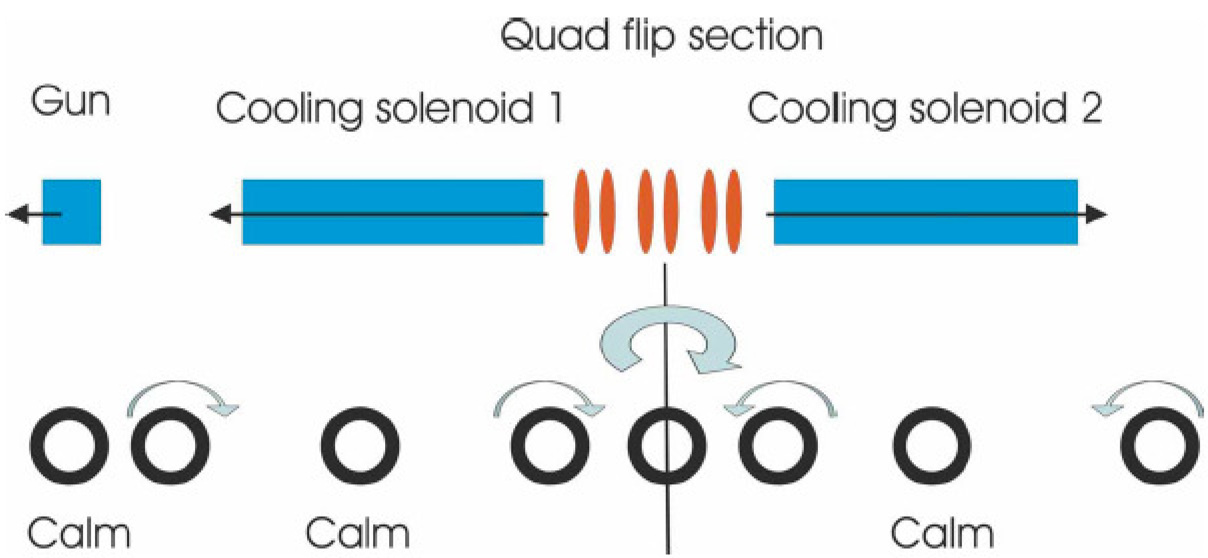
$$\varepsilon_{n,mag} \sim \frac{\sigma_{p_\perp}}{mc} \sigma_x \sim \frac{eB_0}{2mc} \sigma_x^2$$

$$\varepsilon_{n,mag}[\text{mm-mrad}] \sim 0.3B[\text{mT}]\sigma_x[\text{mm}]^2$$

Normally, magnetic field at the cathode is a nuisance. However, it is useful for a) magnetized beams; b) round to flat beam transformation.

Magnetized beam (example: ion cooler)

Similarly, rms emittance inside a solenoid is increased due to Busch's theorem. This usually does not pose a problem (it goes down again) except when the beam is used in the sections with non-zero longitudinal magnetic field. In the latter case, producing magnetized beam from the gun becomes important.



Paraxial ray equation

Paraxial ray equation is equation of 'about'-axis motion (angle with the axis small & only first terms in off-axis field expansion are included).

$$\dot{\gamma} = \gamma' \dot{z} = \gamma' \beta c$$
$$\dot{r} = r' \dot{z} = r' \beta c$$
$$\ddot{r} = (r' \beta c)' \beta c = r'' \beta^2 c^2 + r' \beta' \beta c^2$$

$$\frac{d}{dt}(\gamma m \dot{r}) - \gamma m r \dot{\theta}^2 = e(E_r + r\dot{\theta}B_z)$$

with $-\dot{\theta} = \frac{q}{2\gamma m}\left(B_z - \frac{\Psi_0}{\pi r^2}\right)$ and $\dot{\gamma} = \beta e E_z / mc$

$$\ddot{r} + \frac{\beta e E_z}{\gamma m c}\dot{r} + \frac{e^2 B_z^2}{4\gamma^2 m^2} r - \frac{e^2 \Psi_0^2}{4\pi^2 \gamma^2 m^2}\frac{1}{r^3} - \frac{e E_r}{\gamma m} = 0$$

eliminating time and using $E_r \approx -\frac{1}{2} r E_z' = -\frac{1}{2} r \gamma'' mc^2 / e$

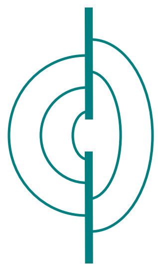
$$r'' + \frac{\gamma' r'}{\beta^2 \gamma} + \left(\frac{\gamma''}{2\beta^2\gamma} + \frac{\Omega_L^2}{\beta^2 c^2}\right) r - \left(\frac{P_\theta}{\beta\gamma m c}\right)^2 \frac{1}{r^3} = 0$$

$$P_\theta \equiv e r A_\theta + \gamma m r^2 \dot{\theta}$$
$$\Omega_L \equiv -eB_z / 2\gamma m$$
$$\theta_L = \int_0^z \Omega_L \frac{dz}{\beta c}$$

Focusing: electrostatic aperture and solenoid

With paraxial ray equation, the focal length can be determined

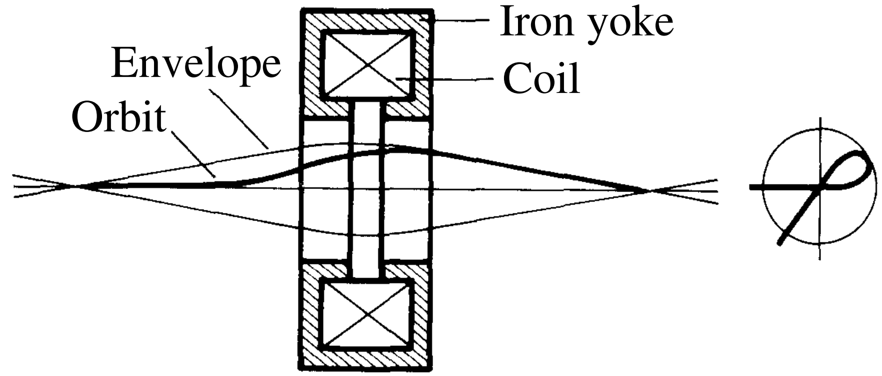
electrostatic aperture



$$f = 4V \frac{1+\frac{1}{2}eV/mc^2}{1+eV/mc^2}\frac{1}{E_2 - E_1}$$

eV is equal to beam K.E., E_1 and E_2 are electric fields before and after the aperture

solenoid



$$\frac{1}{f} = \int \left(\frac{\Omega_L}{\beta c}\right)^2 dz$$

$$= \int \left(\frac{eB_z}{2\beta\gamma mc}\right)^2 dz \approx \frac{1}{4}\left(\frac{e}{cp}\right)^2 B_z^2 L$$

$$cp/e[1\,\mathrm{MeV/c}] \rightarrow (B\rho)[33.4\,\mathrm{G\text{-}m}]$$

RF effect

SW longitudinal field in RF cavities requires transverse components from Maxwell's equations → cavity can impart tranverse momentum to the beam

Chambers (1965) and Rosenzweig & Serafini (1994) provide a fairly accurate (≥ 5 MeV) matrix for RF cavities (Phys. Rev. E **49** (1994) 1599 – beware, formula (13) has a mistypo)

Edges of the cavities do most of the focusing. For $\gamma >> 1$

$$\frac{1}{f} = -\frac{\gamma'}{\gamma_2}\left[\frac{\cos^2\varphi}{\sqrt{2}} + \frac{1}{\sqrt{8}}\right]\sin\alpha \qquad \text{with } \alpha = \frac{\ln(\gamma_2/\gamma_1)}{\sqrt{8}\cos\varphi}$$

γ_1, γ_2, γ', φ Lorentz factor before, after the cavity, cavity gradient and off-crest phase

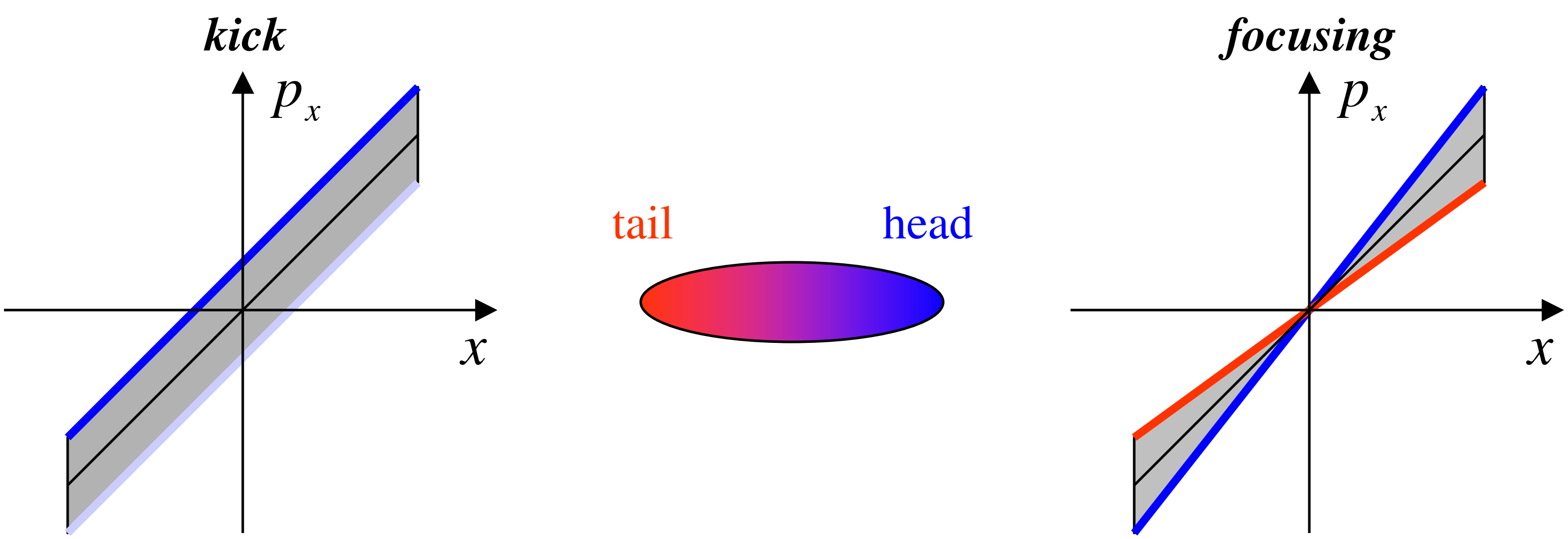
On crest, and when $\Delta\gamma = \gamma' L << \gamma$:

$$\frac{1}{f} \approx -\frac{3}{8}\frac{\gamma'^2 L}{\gamma^2}$$

Emittance growth from RF focusing and kick

$$\varepsilon_n = \frac{1}{mc}\sqrt{\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2}$$

$$p_x(x,z) = p_x(0,0) + \frac{\partial p_x}{\partial x}x + \underbrace{\frac{\partial p_x}{\partial z}z}_{kick} + \underbrace{\frac{\partial^2 p_x}{\partial x \partial z}xz}_{focusing} + \ldots$$



RF focusing and kick

$$\varepsilon_n^2 = \varepsilon_0^2 + \varepsilon_{kick}^2 + \varepsilon_{focus}^2$$

- Kick effect on emittance is energy independent (modulo beam size) and can be cancelled downstream
- RF focusing effect scales $\propto \frac{1}{\gamma}$ (in terms of p_x) and generally is not cancelled

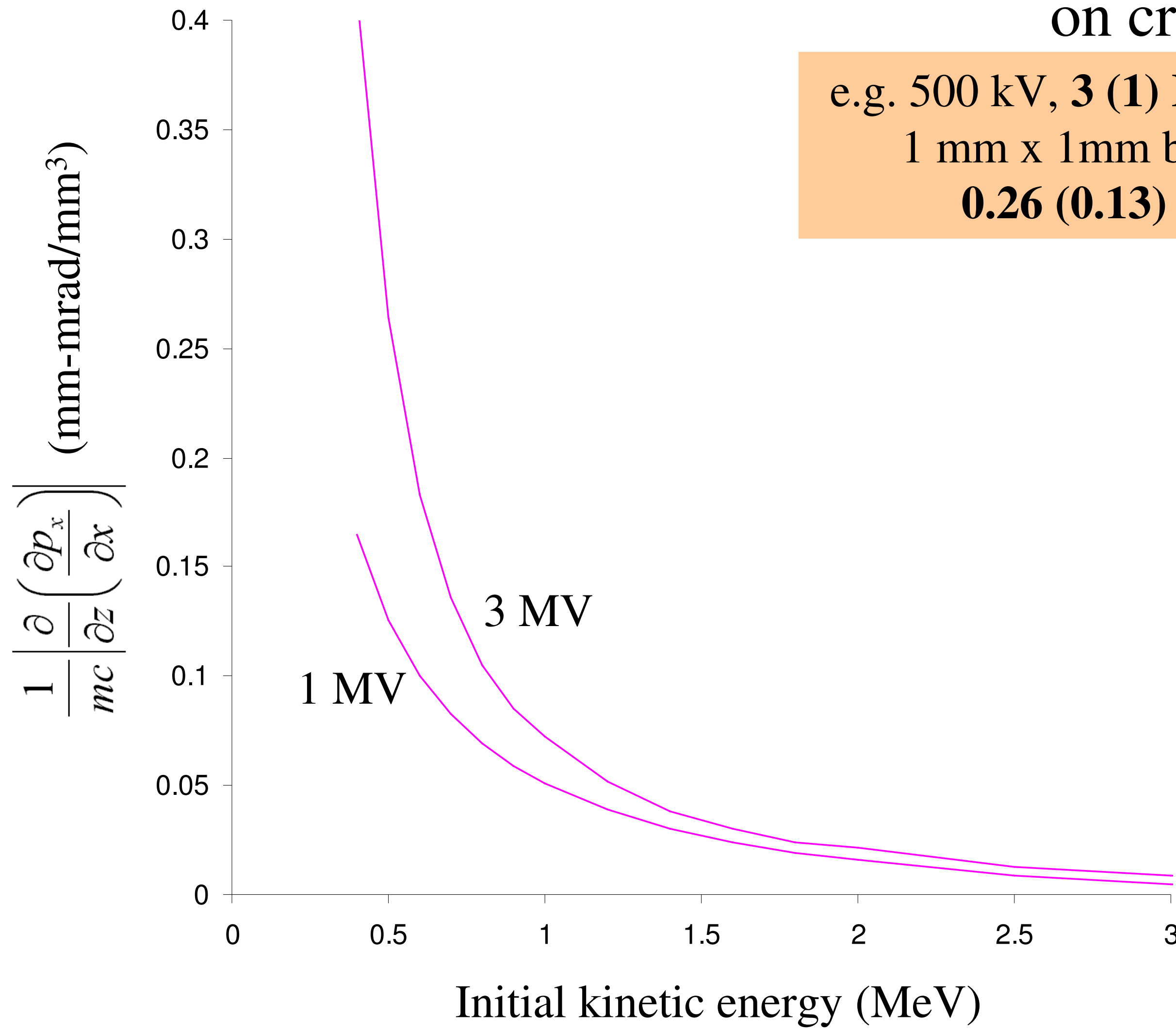
$$\varepsilon_{kick} = \frac{1}{mc}\left|\frac{\partial p_x}{\partial z}\right|\sigma_x \sigma_z$$

$$\varepsilon_{focus} = \frac{1}{mc}\left|\frac{\partial^2 p_x}{\partial z \partial x}\right|\sigma_x{}^2 \sigma_z$$

Example: RF focusing in 2-cell SRF injector cavity

on crest

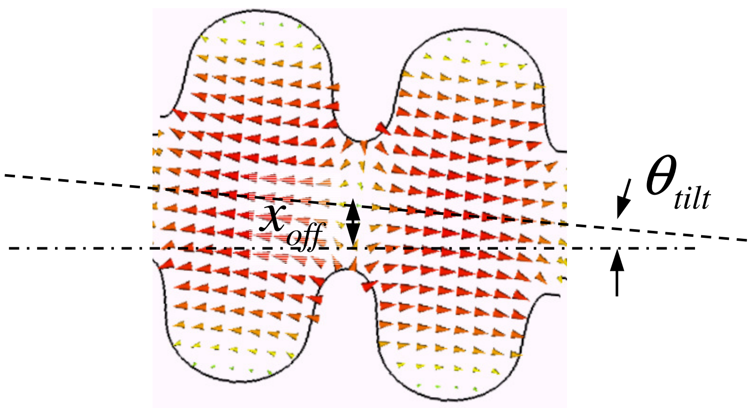
e.g. 500 kV, **3 (1) MV** in the 1st cell,
1 mm x 1mm bunch receives
0.26 (0.13) mm-mrad



RF tilt and offset

$$\varepsilon_{kick} \approx \sigma_x \sigma_z \left[\underbrace{\theta_{tilt} \frac{\Delta E}{mc^2} k_{RF} \sin\varphi}_{\text{tilt}} + \underbrace{x_{off} \frac{1}{mc} \frac{\partial}{\partial z} \left(\frac{\partial p_x}{\partial x} \right)}_{\text{offset}} \right]$$

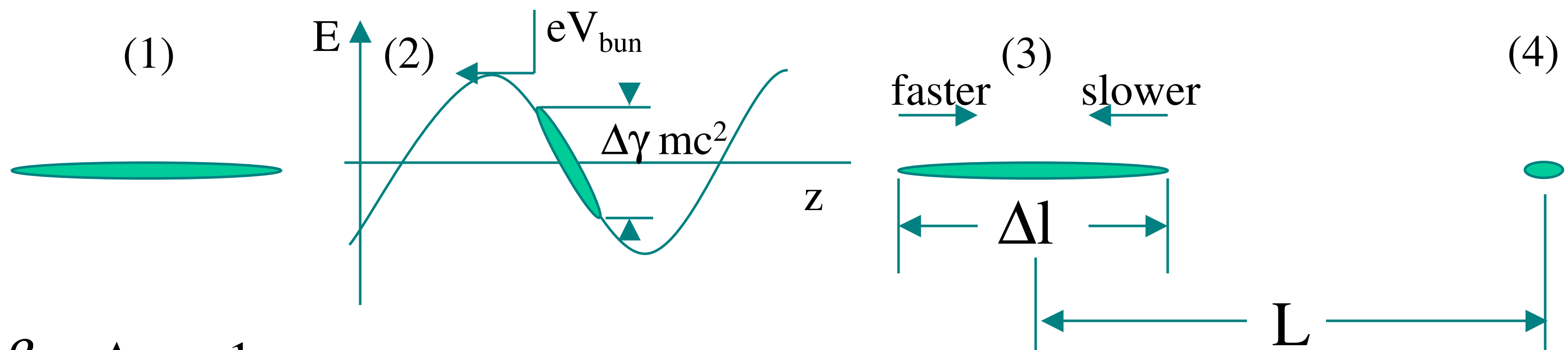
e.g. **3 MeV** energy gain
for 1 mm x 1mm yields
0.16 sin φ mm-mrad
per mrad of **tilt**



- One would prefer on-crest running in the injector (and elsewhere!) from tolerances' point of view

Drift Bunching: Simple Picture

For bunch compression, two approaches are used: magnetic compression (with lattice) and drift bunching. Magnetic compression relies on path vs. beam energy dependence, while drift bunching relies on velocity vs. energy dependence (i.e. it works only near the gun when $\gamma \geq 1$).



$$\frac{\Delta\beta}{\beta} \approx \frac{\Delta\gamma}{\gamma}\frac{1}{\gamma^2 - 1}$$

$$L = c\beta\frac{\Delta l}{c\Delta\beta} = \frac{\Delta l}{(\Delta\gamma/\gamma)}(\gamma^2 - 1) = \frac{\lambda_{RF}}{2\pi}\frac{E}{eV_{bun}}\left(\frac{E^2}{(mc^2)^2} - 1\right)$$