Holographic classification of topological Insulators

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Beauty of Integrability

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Outline

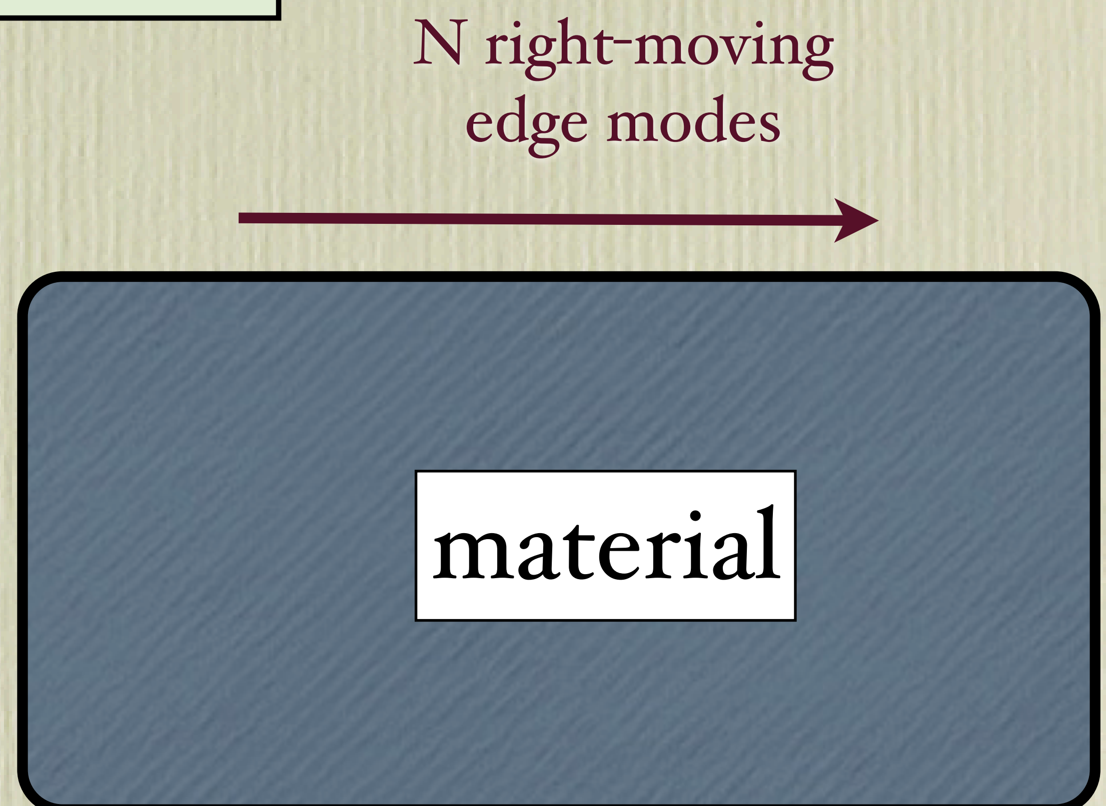
- What are Topological Insulators?
- Two important examples in 2 spatial dimensions: IQHE and Quantum Spin Hall Effect.
- Classification in any spatial dimension. The Periodic Table.
- New Topological Insulators in 2 dimensions.
- Role of interactions in 2d: New Luttinger L's.

What are Topological Insulators?

- Band insulators with a gap with special topological properties.
- Bulk wave functions have a topological invariant.
- This leads to gapless states on the boundary that are robust, i.e. protected against scattering with impurities, localization, etc.
- Illustrate with 2 important examples in 2 dimensions: IQHE, QSHE.

Integer Quantum Hall Effect

2-dimensional electron gas in a perpendicular magnetic field:

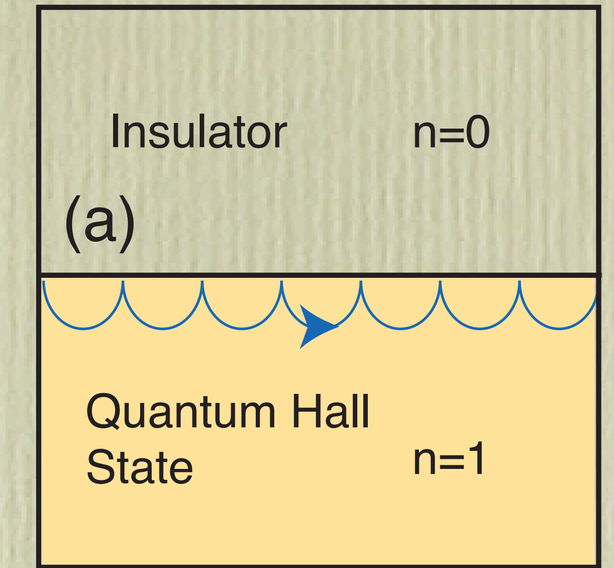


B into page breaks time reversal symmetry

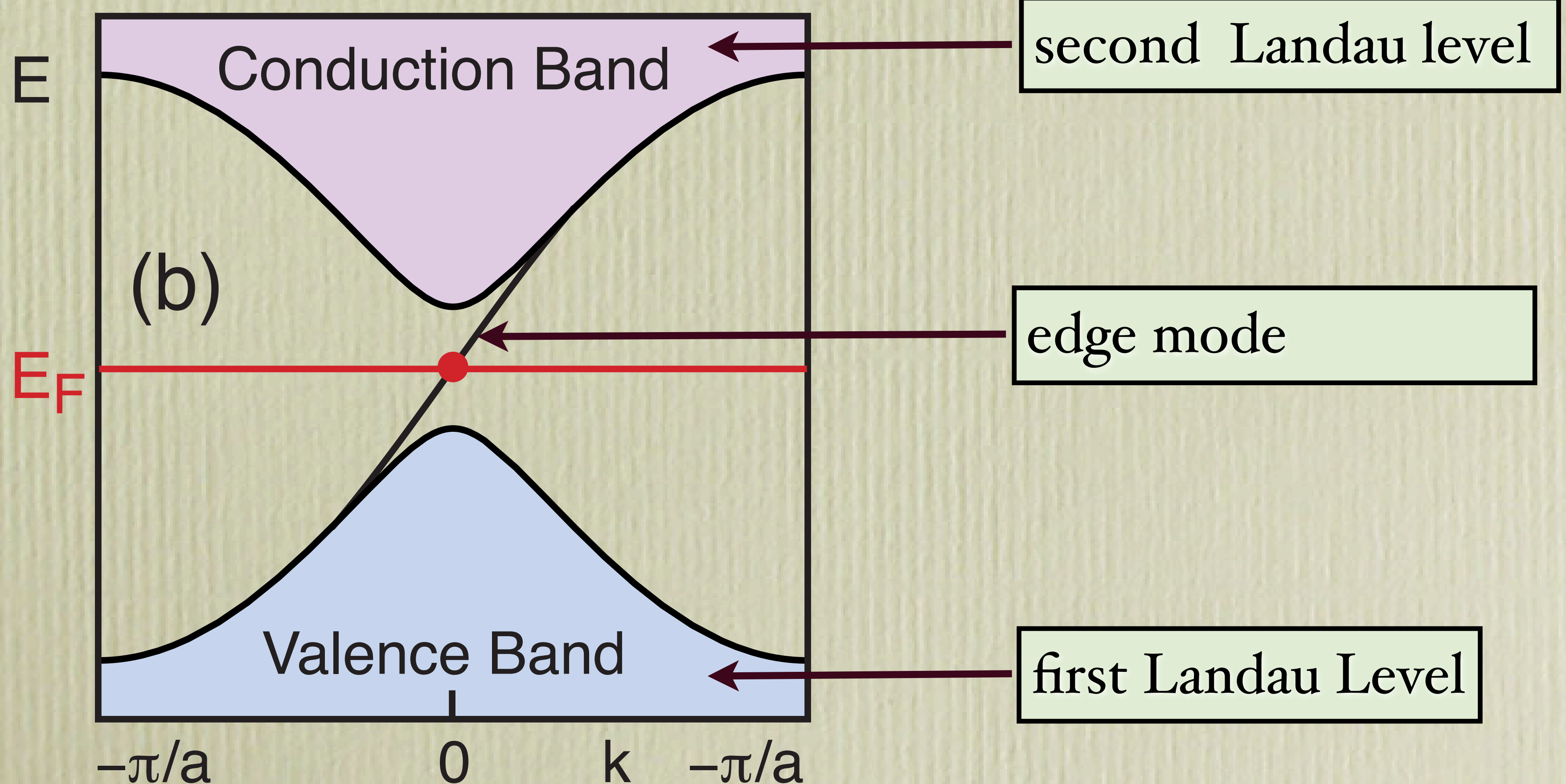
Hall conductivity is quantized: $\sigma_{xy} = N\, e^2/h$

N is an integer to 1 part in 10^9 !

Why? N right-moving edge modes.



Generalized viewpoint:



The BULK topological invariant

example of topology: you cannot smoothly deform a caju into a donut:

smooth deformations:

=

=

=

=

number of holes = Euler invariant
=integral over surface of some function.

The bulk topological invariant....

Bulk wavefunctions $|u(\mathbf{k})\rangle$ have analogous topological properties (TKNN invariant):

$$\mathbf{A} = \mathbf{i}\langle \mathbf{u}(\mathbf{k})|\nabla_{\mathbf{k}}|\mathbf{u}(\mathbf{k})\rangle$$

$$N = \frac{1}{2\pi}\int d^2\mathbf{k}\ \nabla \times \mathbf{A}$$ = integer = Chern #

= number of chiral edge modes

We have holography: bulk/boundary correspondence.

Quantum Spin Hall

- the first new realization of a topological insulator. (Kane-Mele). Top. inv. is Z_2
- Preserves time-reversal symmetry, spin orbit coupling plays the role of magnetic field.
- Physical realization in HgCdTe quantum wells.

Due to T-reversal, there are now both left and right moving edge states, but momentum is locked with spin.

(no B field)

Classification of TI's

- IQHE and QSHE differ in their time-reversal symmetries, and this is the main distinction.
- One can also consider particle-hole symmetry (for superconductors).
- Two approaches, one based on K-theory (Kitaev), the other on the existence of topological invariants (Ryu et. al.), both predict 5 classes of TI in any dimension.
- Our work: holographic approach, i.e. classification of symmetry protected zero modes on the boundary (Bernard,Kim,AL). Not necessarily equivalent.

The 10 symmetry classes

Under time reversal (T), particle-hole (C) and chirality (P), the hamiltonian transforms as:

$$\begin{aligned}
\mathbf{T}: \quad & T\mathcal{H}^*T^\dagger = \mathcal{H} \\
\mathbf{C}: \quad & C\mathcal{H}^TC^\dagger = -\mathcal{H} \\
\mathbf{P}: \quad & P\mathcal{H}P^\dagger = -\mathcal{H}
\end{aligned}$$

For hermitian H, $\mathcal{H}^T = \mathcal{H}^*$, and we work with the transpose.

AZ-classes	T	C	P
A	$\emptyset$	$\emptyset$	$\emptyset$
AIII	$\emptyset$	$\emptyset$	1
AII	-1	$\emptyset$	$\emptyset$
AI	$+1$	$\emptyset$	$\emptyset$
C	$\emptyset$	-1	$\emptyset$
D	$\emptyset$	$+1$	$\emptyset$
BDI	$+1$	$+1$	1
DIII	-1	$+1$	1
CII	-1	-1	1
CI	$+1$	-1	1

TABLE I: *The 10 Altland-Zirnbauer (AZ) hamiltonian classes. The* $\pm$ *signs refer to* $T^T = \pm T$ *and* $C^T = \pm C$*, whereas* $\emptyset$ *denotes non-existence of the symmetry.*

Notation BDI etc. goes back to Cartan's classification of symmetric spaces.

Principles of Classification

- Assume the boundary theory is first order in derivatives (Dirac). This can give a spectrum $E^2 = k^2 + M^2$ which is gapless if M=0.
- Classify zero modes of M according T,C,P and spatial dimension d.
- A well-posed mathematical problem, solved using generic properties of Clifford algebras.

Dirac hamiltonian:

$$\mathcal{H} = -i\sum_{a=1}^{\bar{d}} \gamma_a \frac{\partial}{\partial x_a} + M$$

. $\bar{d} = d - 1$

To obtain Dirac spectrum:

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}; \qquad \{\gamma_a, M\} = 0, \quad \forall a$$

The conditions for $\mathbf{P}, \mathbf{T}, \mathbf{C}$ symmetry are the following $\forall a$: :

$$\begin{aligned}
\mathbf{P} &: \quad \{P, \gamma_a\} = 0, \qquad \{P, M\} = 0 \\
\mathbf{T} &: \quad T\gamma_a^T = -\gamma_a T, \qquad TM^T = MT \\
\mathbf{C} &: \quad C\gamma_a^T = \gamma_a C, \qquad CM^T = -MC
\end{aligned}$$

Clifford algebra representation on a 2^n dimensional vector space

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$$

$$\begin{aligned}
\Gamma_1 &= \sigma_y \otimes \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_z \\
\Gamma_2 &= \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_z \\
\Gamma_3 &= \mathbf{1} \otimes \sigma_y \otimes \sigma_z \otimes \cdots \otimes \sigma_z \\
\Gamma_4 &= \mathbf{1} \otimes \sigma_x \otimes \sigma_z \otimes \cdots \otimes \sigma_z \\
&\vdots \\
\Gamma_{2n-1} &= \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \sigma_y \\
\Gamma_{2n} &= \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \sigma_x \\
\Gamma_{2n+1} &= \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_z \otimes \sigma_z
\end{aligned}$$

σ = Pauli matrices

The hamiltonian:

Dirac hamiltonian: $$\mathcal{H} = -i \sum_{a=1}^{\bar{d}} \gamma_a \frac{\partial}{\partial x_a} + M$$ $\bar{d} = d - 1$

For d odd: Let $d = 2n{+}1.$

choose $\gamma_a = \Gamma_a$ for $a = 1, 2, ...2n$

and $M = \Gamma_{2n+1}$

For d even: Let $d = 2n.$

$\gamma_a = \Gamma_a$ for $a = 1$ to $2n - 1,$

$M = M^T = \Gamma_{2n}$

Note, for d even there is one unused Γ_{2n+1} = P which leads to "left/right" chirality with the projectors: $p_\pm = (1 \pm \Gamma_{2n+1})/2,$

Implementing T, C,P

- In any dimension P = Γ_{2n+1}
- T and C are elements of the Clifford algebra.
- In any dimension T, C are either:

$$G = \Gamma_1 \Gamma_3 \Gamma_5 \cdots \Gamma_{2n-1}, \qquad \widetilde{G} = G\,\Gamma_{2n+1}$$

They satisfy:

$$G^T = (-1)^{n(n+1)/2} G, \qquad \widetilde{G}^T = (-1)^{n(n-1)/2} \widetilde{G}$$

$$\mathbf{T}: \quad T\gamma_a^T = -\gamma_a T,$$

$$\mathbf{C}: \quad C\gamma_a^T = \gamma_a C,$$

d mod 8	T	T^T/T	C	C^T/C	s_t	s_c
0	$\widetilde{G}$	$+1$	G	$+1$	-1	$+1$
1	$\widetilde{G}$	$+1$	G	$+1$	$+1$	$+1$
2	G	-1	$\widetilde{G}$	$+1$	-1	$+1$
3	G	-1	$\widetilde{G}$	$+1$	-1	-1
4	$\widetilde{G}$	-1	G	-1	-1	$+1$
5	$\widetilde{G}$	-1	G	-1	$+1$	$+1$
6	G	$+1$	$\widetilde{G}$	-1	-1	$+1$
7	G	$+1$	$\widetilde{G}$	-1	-1	-1

To obtain all AZ classes, one needs to tensor in an additional space, for example:

$$T' = i\tau_y \otimes G, \qquad C' = i\tau_y \otimes \widetilde{G}.$$

τ = another set of Pauli matrices

Classifying zero modes.

- i.e. we can classify zero eigenvalues of the mass M based on the constraints that come from:

$$\{P, M\} = 0$$

$$TM^T = MT$$

$$CM^T = -MC$$

The algebra:

The general form of $\mathbf{T}, \mathbf{C}$ are $T = \tau_t \otimes X_t$ and $C = \tau_c \otimes X_c$, where $X_{t,c}$ are either $G, \widetilde{G}$ $\tau_{t,c} = 1$ or $i\tau_y$.

$M = V \otimes \Gamma$, where $\Gamma = \Gamma_{2n+1}, \Gamma_{2n}$ for $d = 2n+1, 2n$ respectively.

constraints on V coming from $\mathbf{T}, \mathbf{C}$,

$$\tau_t V^T = s_t V \tau_t, \qquad \tau_c V^T = -s_c V \tau_c$$

$$X_{t,c}\Gamma = s_{t,c}\Gamma X_{t,c}.$$

There are 9 ways for a zero mode to arise:

case	τ_t	τ_c	s_t	s_c	constraints on V	type
1	1	$\emptyset$	-1	$\emptyset$	eq. 14	$\mathbb{Z}_2$
2	$\emptyset$	1	$\emptyset$	$+1$	eq. 14	$\mathbb{Z}_2$
3	1	1	-1	$+1$	eq. 14	$\mathbb{Z}_2$
4	1	1	-1	-1	$V=0$	$\mathbb{Z}$
5	1	1	$+1$	$+1$	$V=0$	$\mathbb{Z}$
6	$i\tau_y$	$i\tau_y$	-1	-1	$V=0$	$2\mathbb{Z}$
7	$i\tau_y$	$i\tau_y$	$+1$	$+1$	$V=0$	$2\mathbb{Z}$
8	$i\tau_y$	1	$+1$	$+1$	eq. 15	$\mathbb{Z}_2$
9	1	$i\tau_y$	-1	-1	eq. 15	$\mathbb{Z}_2$

$$V^T = -V \implies \det V = 0 \text{ if } \dim(V) \text{ is odd} \tag{14}$$

$$V = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \text{ with } a^T = -a \implies \det V = 0 \text{ if } \dim(a) \text{ is } 1 \tag{15}$$

Periodic Table:

	d mod 8							
AZ class	0	1	2	3	4	5	6	7
A	$\mathbb{Z}$	$\emptyset$	$\mathbb{Z}$	$\emptyset$	$\mathbb{Z}$	$\emptyset$	$\mathbb{Z}$	$\emptyset$
AIII	$\emptyset$	$\mathbb{Z}$	$\emptyset$	$\mathbb{Z}$	$\emptyset$	$\mathbb{Z}$	$\emptyset$	$\mathbb{Z}$
AI	$\mathbb{Z}, \mathbb{Z}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$2\mathbb{Z}$	$\emptyset$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	$\emptyset$	$\emptyset$	$\emptyset$	$2\mathbb{Z}$	$\emptyset$	$\mathbb{Z}_2$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}, \mathbb{Z}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$2\mathbb{Z}$	$\emptyset$
DIII	$\emptyset$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\emptyset$	$\emptyset$	$\emptyset$	$2\mathbb{Z}$
AII	$2\mathbb{Z}$	$\emptyset$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}, \mathbb{Z}_2$	$\emptyset$	$\emptyset$	$\emptyset$
CII	$\emptyset$	$2\mathbb{Z}$	$\emptyset$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\emptyset$	$\emptyset$
C	$\emptyset$	$\emptyset$	$2\mathbb{Z}$	$\emptyset$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}, \mathbb{Z}_2$	$\emptyset$
CI	$\emptyset$	$\emptyset$	$\emptyset$	$2\mathbb{Z}$	$\emptyset$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Red entries are new. Blue indicates chiral classes.
AL and D. Bernard

Exceptionality of two dimensions

- Here the structure is even richer since there are 2 inequivalent ways of implementing time reversal symmetry.
- There are 17 inequivalent classes of Dirac fermions rather than just the 10 AZ classes.
- There are 11 topological insulators!

$$\mathcal{H} = \begin{pmatrix} V_+ + V_- & -i\partial_x + A \\ -i\partial_x + A^\dagger & V_+ - V_- \end{pmatrix}$$

1d-classes	**T**	**C**	**P**	$V_\pm$	A	**zero-mode**
A	$\emptyset$	$\emptyset$	$\emptyset$	$V_\pm^\dagger = V_\pm$		$\mathbb{Z}$
$\mathrm{AIII}_{(1)}$	$\emptyset$	$\emptyset$	$\mathbf{1} \otimes \sigma_z$	$V_\pm = 0$		$\mathbb{Z}$
$\mathrm{AIII}'_{(1)}$	$\emptyset$	$\emptyset$	$\mathbf{1} \otimes i\sigma_y$	$V_+ = 0$		
$\mathrm{AIII}_{(2)}$	$\emptyset$	$\emptyset$	$\tau_z \otimes \sigma_z$	$\tau_z V_\pm = -V_\pm \tau_z$	$\tau_z A = A \tau_z$	
$\mathrm{AIII}'_{(2)}$	$\emptyset$	$\emptyset$	$\tau_z \otimes i\sigma_y$	$\tau_z V_\pm = \mp V_\pm \tau_z$		
$\mathrm{AII}_{(1)}$	$\mathbf{1} \otimes i\sigma_y$	$\emptyset$	$\emptyset$	$V_\pm = \pm V_\pm^T$	$A^T = -A$	$\mathbb{Z}_2$
$\mathrm{AII}_{(2)}$	$i\tau_y \otimes \sigma_z$	$\emptyset$	$\emptyset$	$\tau_y V_\pm^T = V_\pm \tau_y$	$\tau_y A^* = -A\tau_y$	
$\mathrm{AI}_{(1)}$	$i\tau_y \otimes i\sigma_y$	$\emptyset$	$\emptyset$	$\tau_y V_\pm^T = \pm V_\pm \tau_y$	$\tau_y A^T = -A\tau_y$	
$\mathrm{AI}_{(2)}$	$\mathbf{1} \otimes \sigma_z$	$\emptyset$	$\emptyset$	$V_\pm^T = V_\pm$	$A^* = -A$	
C	$\emptyset$	$i\tau_y \otimes \mathbf{1}$	$\emptyset$	$\tau_y V_\pm^T = -V_\pm \tau_y$	$\tau_y A^* = -A\tau_y$	$\mathbb{Z}$
C′	$\emptyset$	$i\tau_y \otimes \sigma_x$	$\emptyset$	$\tau_y V_\pm^T = \mp V_\pm \tau_y$	$\tau_y A^T = -A\tau_y$	
D	$\emptyset$	$\mathbf{1} \otimes \mathbf{1}$	$\emptyset$	$V_\pm = -V_\pm^T$	$A^* = -A$	$\mathbb{Z},\ \mathbb{Z}_2$
D′	$\emptyset$	$\mathbf{1} \otimes \sigma_x$	$\emptyset$	$V_\pm = \mp V_\pm^T$	$A^T = -A$	
$\mathrm{BDI}_{(1)}$	$i\tau_y \otimes i\sigma_y$	$\mathbf{1} \otimes \mathbf{1}$	$i\tau_y \otimes i\sigma_y$	$V_\pm = -V_\pm^T = \mp \tau_y V_\pm \tau_y$	$A = -A^* = -\tau_y A^T \tau_y$	
$\mathrm{BDI}'_{(1)}$	$i\tau_y \otimes i\sigma_y$	$\tau_x \otimes \sigma_x$	$\tau_z \otimes \sigma_z$	$V_\pm = \pm \tau_y V_\pm^T \tau_y = \mp \tau_x V_\pm^T \tau_x$	$\tau_{x,y} A^T = -A\tau_{x,y}$	
$\mathrm{BDI}_{(2)}$	$\mathbf{1} \otimes \sigma_z$	$\mathbf{1} \otimes \mathbf{1}$	$1 \otimes \sigma_z$	$V_\pm = 0$	$A^* = -A$	$\mathbb{Z}$
$\mathrm{DIII}_{(1)}$	$\mathbf{1} \otimes i\sigma_y$	$\mathbf{1} \otimes \mathbf{1}$	$1 \otimes i\sigma_y$	$V_+ = 0,\ V_-^T = -V_-$	$A = -A^* = -A^T$	$\mathbb{Z}_2$
$\mathrm{DIII}_{(2)}$	$i\tau_y \otimes \sigma_z$	$\mathbf{1} \otimes \mathbf{1}$	$i\tau_y \otimes \sigma_z$	$V_\pm = -V_\pm^T = -\tau_y V_\pm \tau_y$	$A = -A^* = -\tau_y A^T \tau_y$	$\mathbb{Z}_2$
$\mathrm{DIII}'_{(2)}$	$i\tau_y \otimes \sigma_z$	$\tau_x \otimes \sigma_x$	$\tau_z \otimes i\sigma_y$	$V_\pm = \tau_y V_\pm^T \tau_y = \mp \tau_x V_\pm^T \tau_x$	$A = -\tau_y A^* \tau^y = -\tau_x A^T \tau_x$	
$\mathrm{CII}_{(1)}$	$\mathbf{1} \otimes i\sigma_y$	$i\tau_y \otimes \mathbf{1}$	$i\tau_y \otimes i\sigma_y$	$V_\pm = \pm V_\pm^T = \mp \tau_y V_\pm \tau_y$	$A = -A^T = -\tau_y A^* \tau_y$	$\mathbb{Z}_2$
$\mathrm{CII}'_{(1)}$	$\tau_x \otimes i\sigma_y$	$i\tau_y \otimes \sigma_x$	$\tau_z \otimes \sigma_z$	$V_\pm = \pm \tau_x V_\pm^T \tau_x = \mp \tau_y V_\pm^T \tau_y$	$\tau_{x,y} A^T = -A\tau_{x,y}$	
$\mathrm{CII}_{(2)}$	$i\tau_y \otimes \sigma_z$	$i\tau_y \otimes \mathbf{1}$	$1 \otimes \sigma_z$	$V_\pm = 0$	$A = -\tau_y A^* \tau_y$	$\mathbb{Z}$
$\mathrm{CI}_{(1)}$	$i\tau_y \otimes i\sigma_y$	$i\tau_y \otimes \mathbf{1}$	$1 \otimes i\sigma_y$	$V_+ = 0,\ \tau_y V_-^T = -V_- \tau_y$	$A = -\tau_y A^T \tau_y = -\tau_y A^* \tau_y$	
$\mathrm{CI}_{(2)}$	$\mathbf{1} \otimes \sigma_z$	$i\tau_y \otimes \mathbf{1}$	$i\tau_y \otimes \sigma_z$	$V_\pm = V_\pm^T = -\tau_y V_\pm \tau_y$	$A = -A^* = -\tau_y A^* \tau_y$	
$\mathrm{CI}'_{(2)}$	$\tau_x \otimes \sigma_z$	$i\tau_y \otimes \sigma_x$	$\tau_z \otimes i\sigma_y$	$V_\pm = \tau_x V_\pm^T \tau_x = \mp \tau_y V_\pm^T \tau_y$	$A = -\tau_x A^* \tau_x = -\tau_y A^T \tau_y$	

Table of TI's in d=2. (red are new)

D. Bernard, E-A Kim and AL.

$\bar{d}=1$ classes	zero modes	topological invariant	examples
A	$\mathbb{Z}$	$\mathbb{Z}$	QH edge states
C	$\mathbb{Z}$	$\mathbb{Z}$	spin QH edge states in $d+id$-wave SC[17,18]
D	$\mathbb{Z}$	$\mathbb{Z}$	thermal QH edge states in spinless chiral p-wave SC[17]

TABLE III. $\bar{d}=1$ chiral Dirac hamiltonian classes.

$\bar{d}=1$ classes	**T**	**C**	**P**	zero modes	top. inv.	locking	examples
$\mathbf{AIII}_{(1)}$	$\emptyset$	$\emptyset$	σ_z	$\mathbb{Z}$			
$\mathrm{AII}_{(1)}$	$i\sigma_y$	$\emptyset$	$\emptyset$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	Y	HgTe/(Hg,Cd)Te
D	$\emptyset$	$\mathbf{1}$	$\emptyset$	$\mathbb{Z}_2$			
$\mathbf{BDI}_{(2)}$	σ_z	$\mathbf{1}$	σ_z	$\mathbb{Z}$			"strained graphene"
$\mathrm{DIII}_{(1)}$	$i\sigma_y$	$\mathbf{1}$	$i\sigma_y$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	Y	$(p+ip)\times(p-ip)$-wave SC
$\mathbf{DIII}_{(2)}$	$i\tau_y\otimes\sigma_z$	$\mathbf{1}$	$i\tau_y\otimes\sigma_z$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	N	particle-hole symmetric KM model
$\mathbf{CII}_{(1)}$	$\mathbf{1}\otimes i\sigma_y$	$i\tau_y\otimes\mathbf{1}$	$i\tau_y\otimes i\sigma_y$	$\mathbb{Z}_2$		Y	doubled KM
$\mathbf{CII}_{(2)}$	$i\tau_y\otimes\sigma_z$	$i\tau_y\otimes\mathbf{1}$	$\mathbf{1}\otimes\sigma_z$	$\mathbb{Z}$		N	

TABLE IV. $\bar{d}=1$ non-chiral Dirac hamiltonian classes with symmetry protected zero modes. The spin-momentum locking column is left blank when spins cannot be assigned because the time-reversal operator do not involve either $i\sigma_y$ or $i\tau_y$. New classes are shown in boldface (red online). The example in quotation marks is a *suggested* possible realization.

Interactions in 2d

- For the IQHE, bulk interactions lead to FQHE. Boundary Dirac theory deformed into a Luttinger liquid.
- Quartic interactions on the boundary are marginal.
- Can show: ALL quartic interactions on the boundary consistent with the T,C,P symmetries are EXACTLY marginal, like the Luttinger L.
- This strongly suggests fractional Topological Insulators. Likely to be integrable on boundary.

Summary

- this holographic approach reproduces other approaches based on topology or K-theory but suggests new topological insulators.
- On additional TI in every even dimension.
- 6 additional TI in 2 dimensions!
- physical realizations?