THe Riemann Hypothesis : a New perspective

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Paris, May 11, 2015 (LPTHE)

Outline

- Riemann Zeta in Quantum Statistical Physics.
- Riemann Hypothesis
- Zeta and the distribution of Prime Numbers.
- Zeta and Random Matrix Theory.
- Generalized RH for Dirichlet and modular L-functions
- 3 Conjectures and 2 concrete strategies toward a proof.

Introductory material reviewed in my Riemann Center lectures: arXiv:1407.4358

Preface

- Main results involve an interplay between the Euler Product Formula and the functional equation.
- Conjectures are "derived" but we do not prove rigorous mathematical theorems.
- The approach is universal, i.e. applies to at least two infinite classes of "zeta" functions.
- It's a "constructive" approach, i.e. leads to new formulas, etc.

Riemann Zeta Function was present at the birth of Quantum Mechanics:

On the Law of Distribution of Energy in the Normal Spectrum

Max Planck

Annalen der Physik, vol. 4, p. 553 ff (1901)

On the other hand, according to equation (12) the energy density of the total radiant energy for $\theta = 1$ is:

$$\begin{aligned} u^* &= \int_0^\infty u d\nu = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/k} - 1} \\ &= \frac{8\pi h}{c^3} \int_0^\infty \nu^3 (e^{-h\nu/k} + e^{-2h\nu/k} + e^{-3h\nu/k} + \cdots) d\nu \end{aligned}$$

Bose-Einstein distribution

and by termwise integration:

$$\begin{aligned} u^* &= \frac{8\pi h}{c^3} \cdot 6 \left(\frac{k}{h}\right)^4 \left(1 + \frac{1}{24} + \frac{1}{34} + \frac{1}{44} + \cdots\right) \\ &= \frac{48\pi k^4}{c^3 h^3} \cdot 1.0823 \end{aligned}$$

A very bad typo of the English translation. Should read:

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots.. = \zeta(4) = \frac{\pi^4}{90} = 1.0823$$

The Riemann Zeta Function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \ldots, \qquad \Re(z) > 1$$

It can be analytically continued to the whole complex z=plane. For example, by considering "fermions":

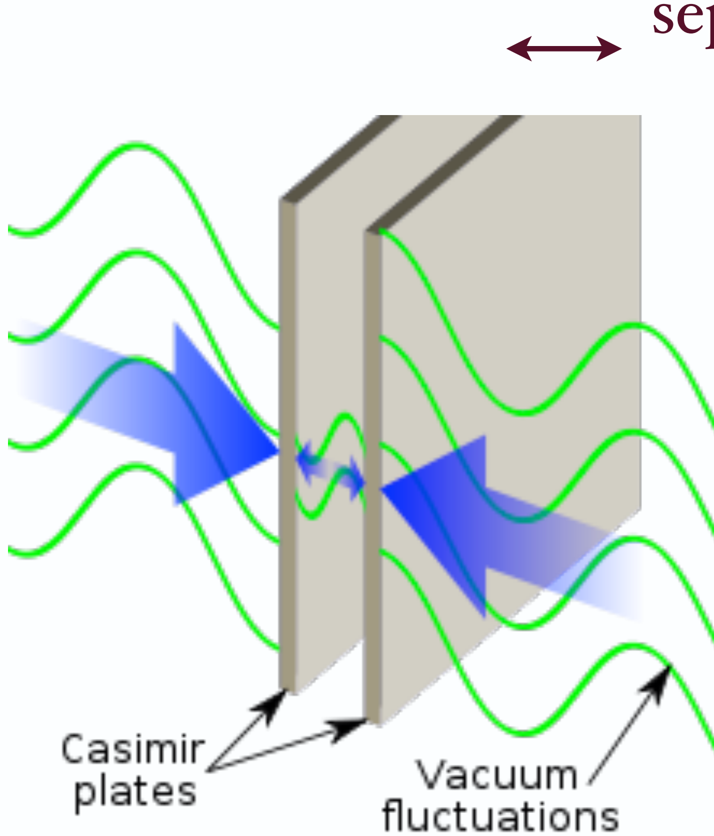
$$\zeta(z) = \frac{1}{\Gamma(z)(1-2^{1-z})} \int_0^\infty dt\, \frac{t^{z-1}}{e^t + 1}, \qquad \Re(z) > 0$$

$(\Gamma(n+1) = n!)$

Trivial zeros: $\zeta(-2) = \zeta(-4) = \zeta(-6) \ldots = 0$

The Casimir effect and Zeta

separation = ℓ



Force= $F(\ell) = -dE_{\text{vac}}(\ell)/d\ell.$

energy density: $$\rho_{\text{vac}}^{\text{cas}} = -\pi^2/720\ell^4.$$

This effect has been measured.

For now note: 720 = 6 x 120

Cylindrical version of Casimir effect

Just change boundary conditions: join plates at edges to have periodic b.c.

One compactified spatial dimension of circumference β

2 dim'l space + time

Relation to Casimir: $\rho^{\rm cas}_{\rm vac}(\ell) = 2\rho^{\rm cyl}_{\rm vac}(\beta = 2\ell)$

$$\rho^{\rm cyl}_{\rm vac} = \frac{1}{2\beta}\sum_{n\in\mathbb{Z}}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\sqrt{\mathbf{k}^2+(2\pi n/\beta)^2} = -\beta^{-4}\pi^{3/2}\Gamma(-3/2)\zeta(-3) + \text{const.}$$

quantized modes on circle

divergent as UV cutoff $k_c \to \infty$.
This is the Cosmological constant problem.

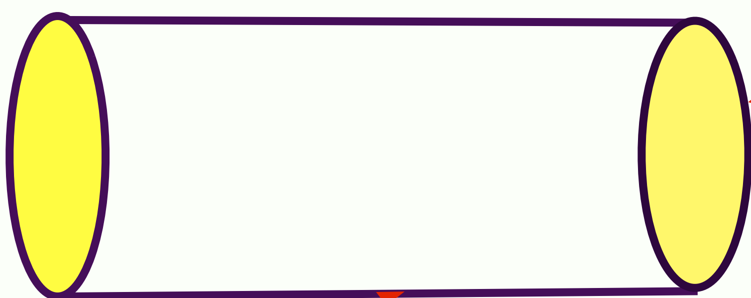
$$\zeta(-3) = 1 + 2^3 + 3^3 + 4^3 + \ldots\ldots\ldots = \ ?$$

$$= \frac{1}{120}$$

By analytic continuation!

Quantum Statistical Mechanics viewpoint.

Passing to euclidean time $t = -i\,\tau$, ϱ_{vac} is just the finite temperature free energy on the cylinder with circumference $\beta = 1/T$.



Quantum Statistical. Mech. gives a very different convergent expression.

$$\rho_{\rm vac}^{\rm cyl} = \frac{1}{\beta} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \log\left(1 - e^{-\beta k}\right) = -\beta^{-4} \frac{\zeta(4)}{2\pi^{3/2}\Gamma(3/2)} = -\frac{\pi^2}{90} T^4.$$

black body

$$= -\beta^{-4}\pi^{3/2}\Gamma(-3/2)\zeta(-3)$$?

YES!
Due to the amazing functional equation:

$$\chi(z) \equiv \pi^{-z/2}\Gamma(z/2)\zeta(z) = \chi(1-z)$$

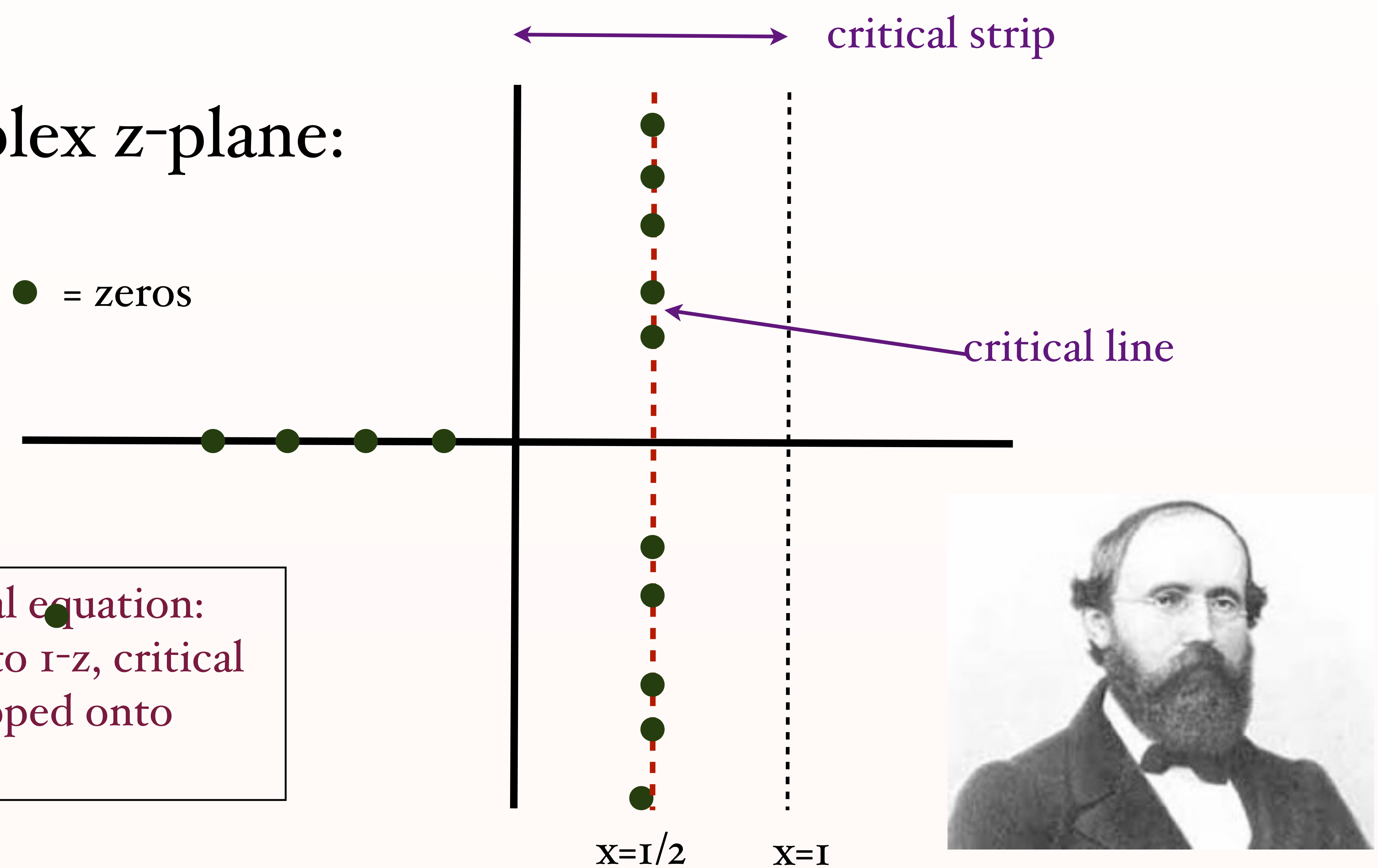
AL Int. J. Mod. Phys. A23 (2008)

Riemann Hypothesis: All non-trivial zeros of Zeta have real part 1/2. That is they are of the form:

$$\zeta(\rho) = 0, \qquad \rho = \frac{1}{2} + iy$$

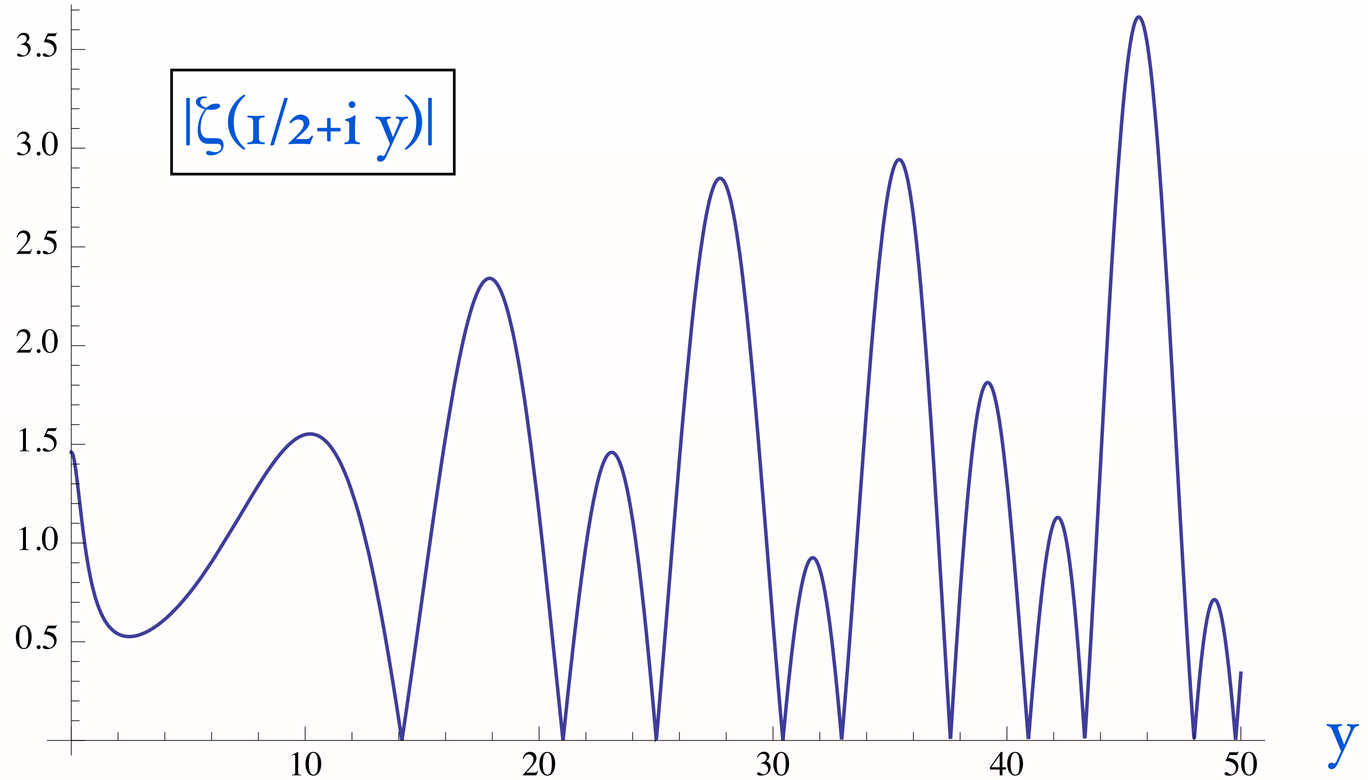
1859

complex z-plane:



functional equation: under z to 1-z, critical strip mapped onto itself

Some Riemann Zeros:



Can enumerate zero along y-axis:

$$\text{n} - \text{th zero on critical line}: \quad \rho_n = \frac{1}{2} + i y_n$$

n	y_n
1	14.134725141734693790457251983562470270784257115699243175685 5
2	21.022039638771554992628479593896902777334340524902781754629 5
3	25.010857580145688763213790992562821818659549672557996672496 5
4	30.424876125859513210311897530584091320181560023715440180962 1
5	32.935061587739189690662368964074903488812715603517039009280 0

Known: the first 10^{13} zeros are on the critical line. (numerically).

The distribution of Prime Numbers and Zeta

Prime number theorem

How many primes less than x?

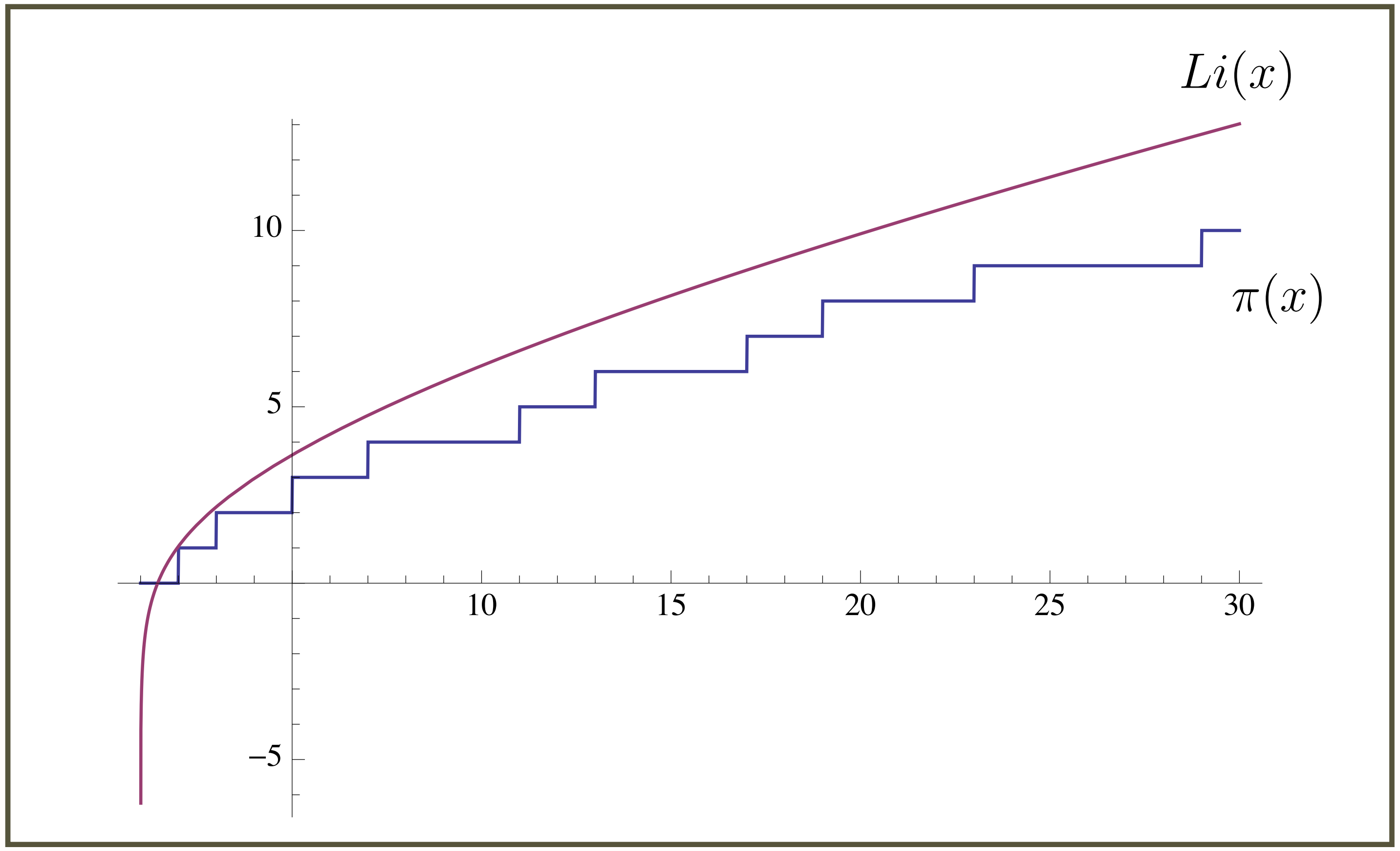
Gauss, a 15 years old boy, guessed in 1792

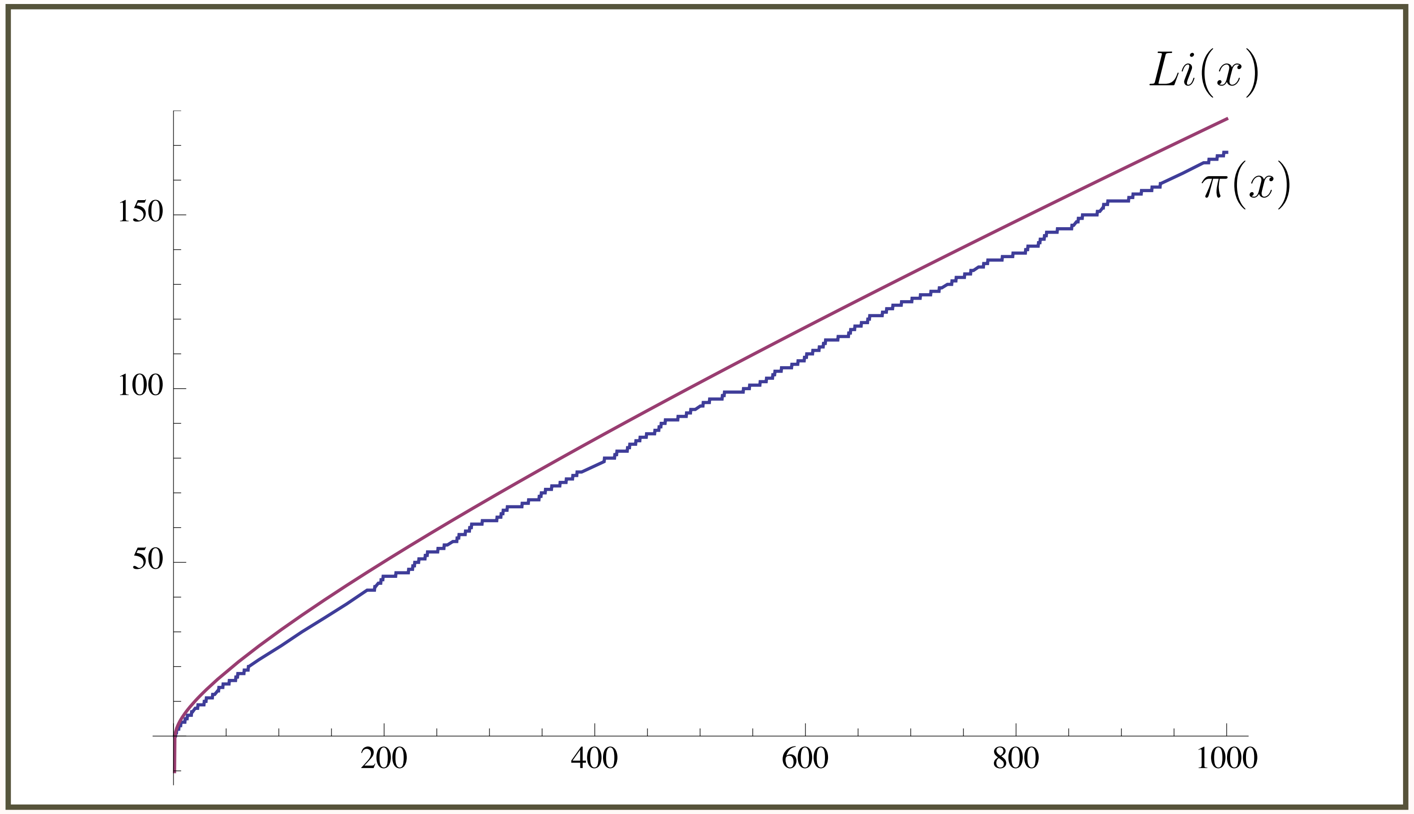
$$\pi(x) = \sum_{p \leq x} 1 \approx \frac{x}{\log x} \approx \mathrm{Li}(x)$$

$$\mathrm{Li}(x) = \int_0^x \frac{dt}{\log t}$$

- Chebyshev (1850) tried to prove using $\zeta(z)$
- Only proven 100 years later (1896) by Hadamard/de la Vallé Poussin $\zeta(1 + iy) \neq 0$

Works quite well:





Zeta and the Primes

The Golden Key: Euler product formula: (1737)

$$\zeta(z) = \prod_p \frac{1}{1 - p^{-z}},$$

$p = \text{prime}$

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \dots$$

$$\frac{1}{2^z}\zeta(z) = \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \dots$$

$$(1 - \frac{1}{2^z})\zeta(z) = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \dots$$

$$(1 - \frac{1}{3^z})(1 - \frac{1}{2^z})\zeta(z) = 1 + \frac{1}{5^z} + \frac{1}{7^z} + \dots$$

Remark: pole at z=1 implies there are an infinite number of primes.

Riemann's Main Result

$$\pi(x) = \sum_{n \geq 1} \frac{\mu(n)}{n} J(x^{1/n}).$$

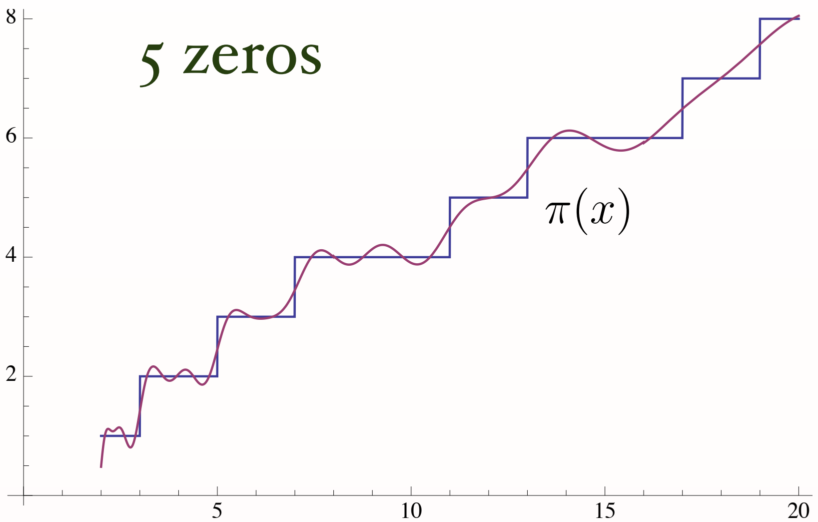
$$J(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}\left(x^{\rho}\right) + \int_{x}^{\infty} \frac{dt}{\log t} \, \frac{1}{t\left(t^{2}-1\right)} - \log 2,$$

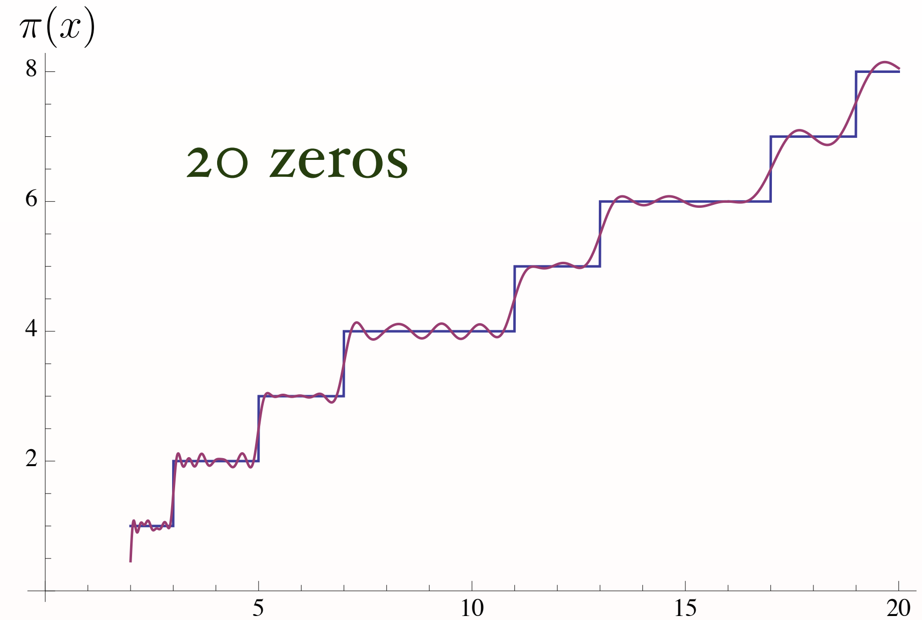
ϱ = a zero on the critical strip

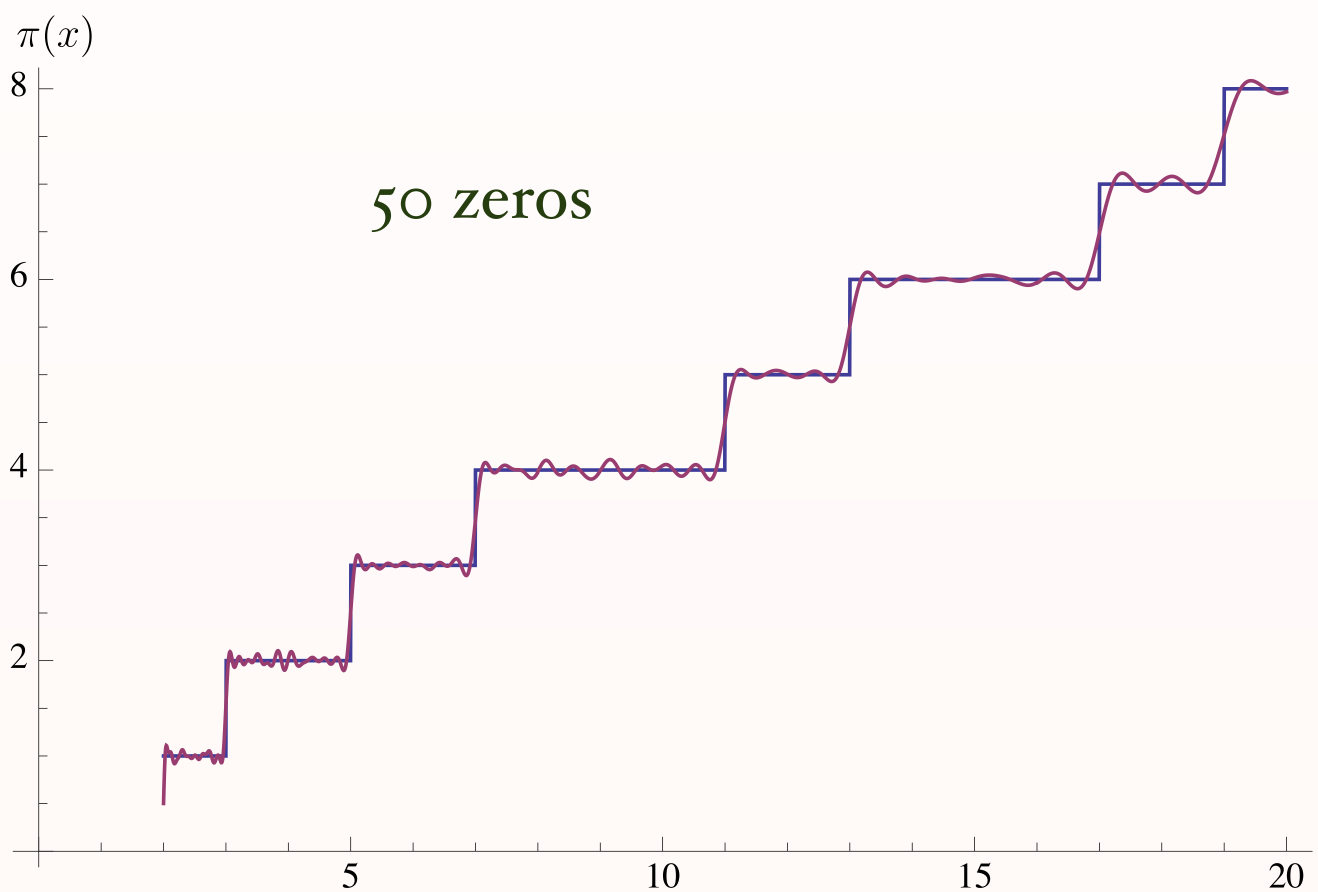
Derived using clever real and complex analysis.

Here, $\mu(n)$ is the Möbius function, equal to 1 (-1) if n is a product of an even (odd) number of distinct primes, and equal to zero if it has a multiple prime factor. The above expression is actually a finite sum, since for large enough n, $x^{1/n} < 2$ and $J = 0$.

Remark: if there are no zeros with real part equal to 1, Li(x) is the leading term.


5 zeros
π(x)
8
6
4
2
5
10
15
20


π(x)
20 zeros
8
6
4
2
5
10
15
20


π(x)
50 zeros
8
6
4
2
5
10
15
20

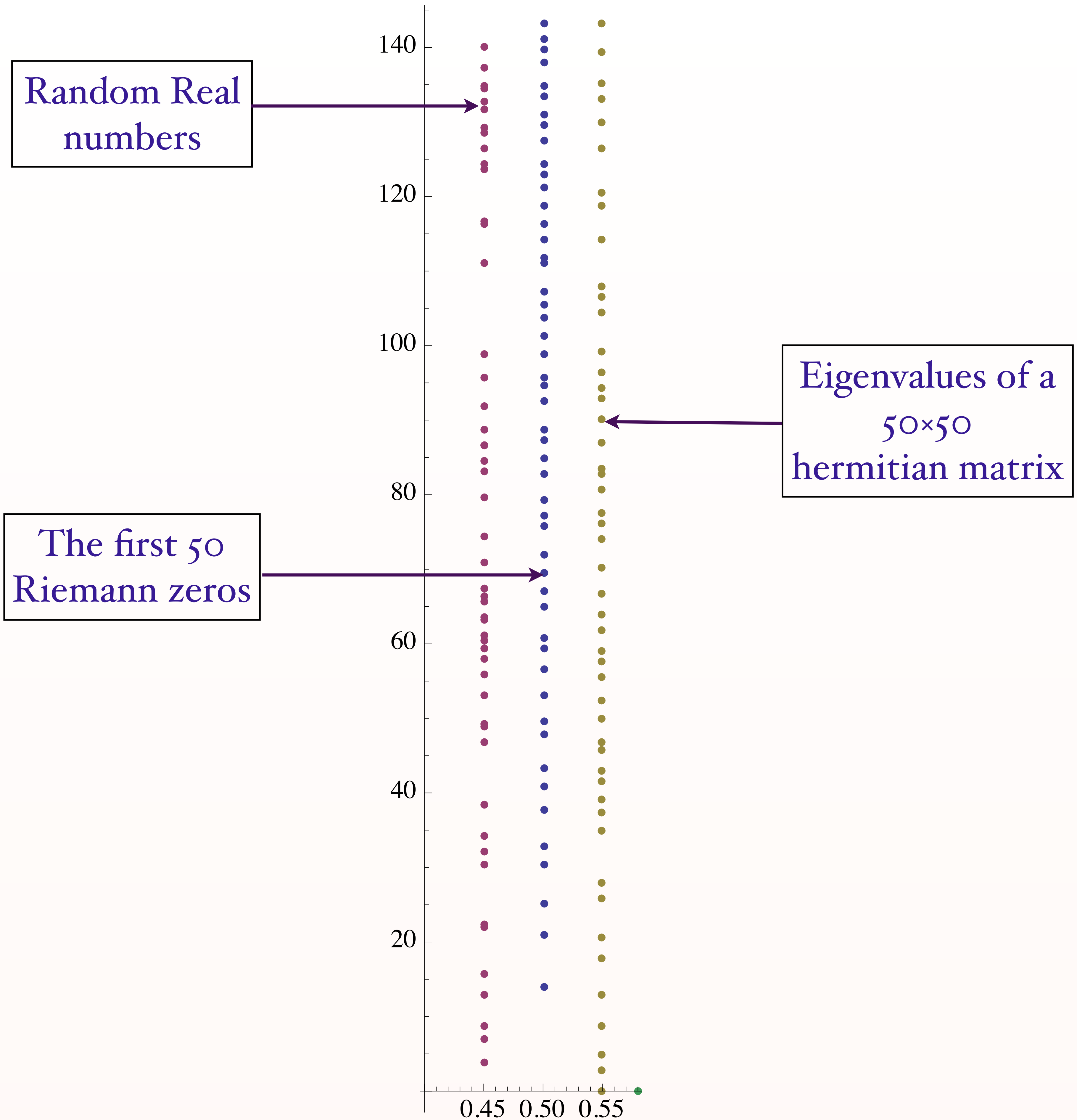
Zeta and Random Matrix Theory

The distribution of zeros on the critical line appears random, but is not completely random.

Dyson studied the properties of eigenvalues of random hamiltonians H. Though H is random, the spacing of its eigenvalues has predictable properties. ("level repulsion")

Montgomery studied the "pair correlation function" of the zeros of zeta. Dyson pointed out that was for the same as the GUE! (1973). Verified numerically for high zeros by Odlyzko (1987)

Gaussian Unitary Ensemble = exponential of random hamiltonian


Random Real numbers
Eigenvalues of a 50×50 hermitian matrix
The first 50 Riemann zeros
140
120
100
80
60
40
20
0.45
0.50
0.55

Zeta is the trivial case of Dirichlet L-functions

Axiomatic definition:

Dirichlet characters of modulus k:

1. $\chi(n+k) = \chi(n)$.

2. $\chi(1) = 1$ and $\chi(0) = 0$.

3. $\chi(nm) = \chi(n)\chi(m)$.

4. $\chi(n) = 0$ if $(n,k) > 1$ and $\chi(n) \neq 0$ if $(n,k) = 1$.

5. If $(n,k) = 1$ then $\chi(n)^{\varphi(k)} = 1$, where $\varphi(k)$ is the Euler totient arithmetic function. This implies that $\chi(n)$ are roots of unity.

Example:

n	1	2	3	4	5	6	7
$\chi_{7,2}(n)$	1	$e^{2\pi i/3}$	$e^{\pi i/3}$	$e^{-2\pi i/3}$	$e^{-\pi i/3}$	-1	0

L-function:

$$L(z,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^z} = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^z}\right)^{-1}$$

Also satisfies functional eqn.
relating L(z) to L(1-z)

Strategy 1: Validity of the Euler Product Formula to the right of critical line.

L-function: $$L(z,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^z} = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^z}\right)^{-1}$$

* Converges absolutely for Re(z) > 1

* One proves there are no zeros with Re(z>1) using the EPF.

* If the EPF is valid for Re(z)>1/2, then this combined with the functional equation implies Riemann Hypothesis is true.

Conjecture 1: The Euler Product Formula is valid in the Cesaro averaged sense for Re(z)>1/2. (G.Franca and AL 1410.3520)

The argument involved:

1. A reorganization of the series for the EP (Abel transform).
2. Prime number theorem, i.e. $p_n > n \log n$ and average gap = log n.
3. A central limit theorem for the Random Walk of Primes series
4. The Cesaro average of the Euler product converges for Re(z)>1/2

The Central Limit Theorem:

RWPrimes series: $$B_N = \sum_{n=1}^{N} \cos(\lambda_n)$$

$$\lambda_n = t \log p_n - \arg \chi(p_n), \quad \text{where} \quad t = \Im(z)$$

Multiplicative independence of the primes, reflected in their pseudo-randomness, makes the cosines behave as independently distributed random variables, so like a random walk where each step a random number between -1 and 1. $B_N = O(N)$ would imply convergence for Re(z)>1, but this CLT implies:

$$B_N = O(\sqrt{N})$$ (not fully proven yet)

The significance of Re(z) > 1/2, i.e. right half of critical strip, arises from this square root!

Numerical Evidence is compelling

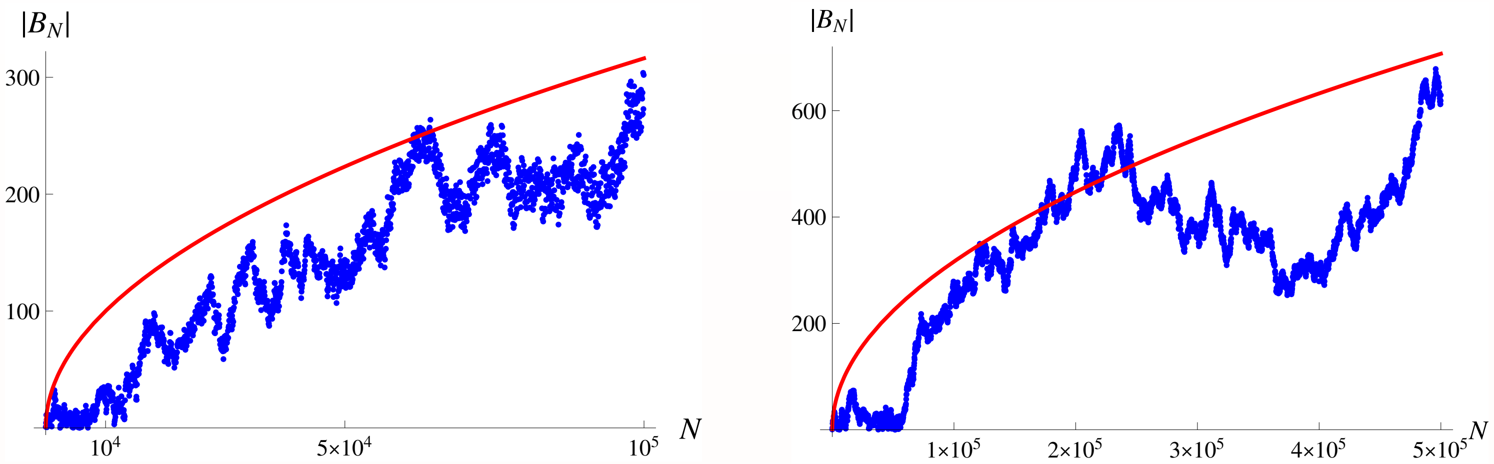
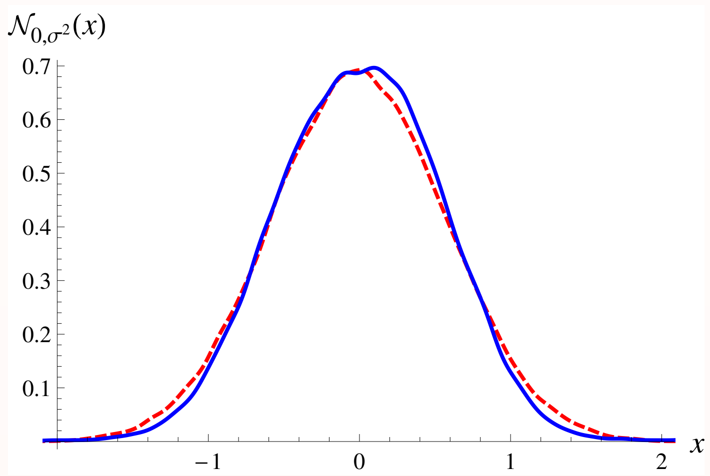


FIG. 1. The absolute value of the partial sum B_N versus N, for a fixed t. **Left:** We use (23) with $t = 5 \cdot 10^3$. Note that N is below the cut-off (30). **Right:** Here we use (21) ($u = 1$) with the character $\chi = \chi_{7,2}$ shown in (A3), and $t = 5 \cdot 10^2$. In this case we can freely take the limit $N \to \infty$.



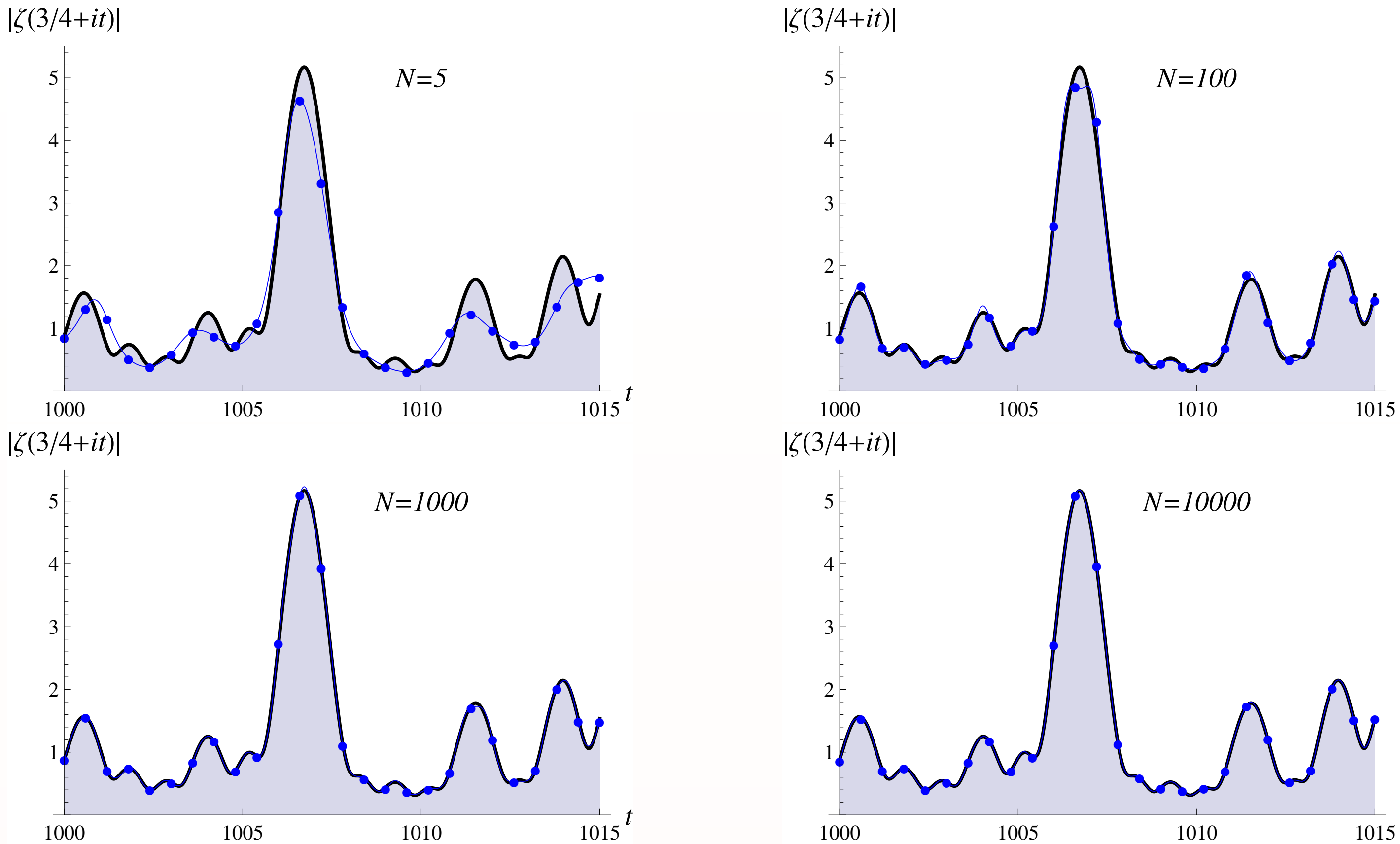


FIG. 4. The black line is the actual $|\zeta(3/4+it)|$, analytically continued into the strip, and the blue line is the partial product $|\mathcal{P}_N(3/4+it)|$. Dots are added to the line to aid visualization.

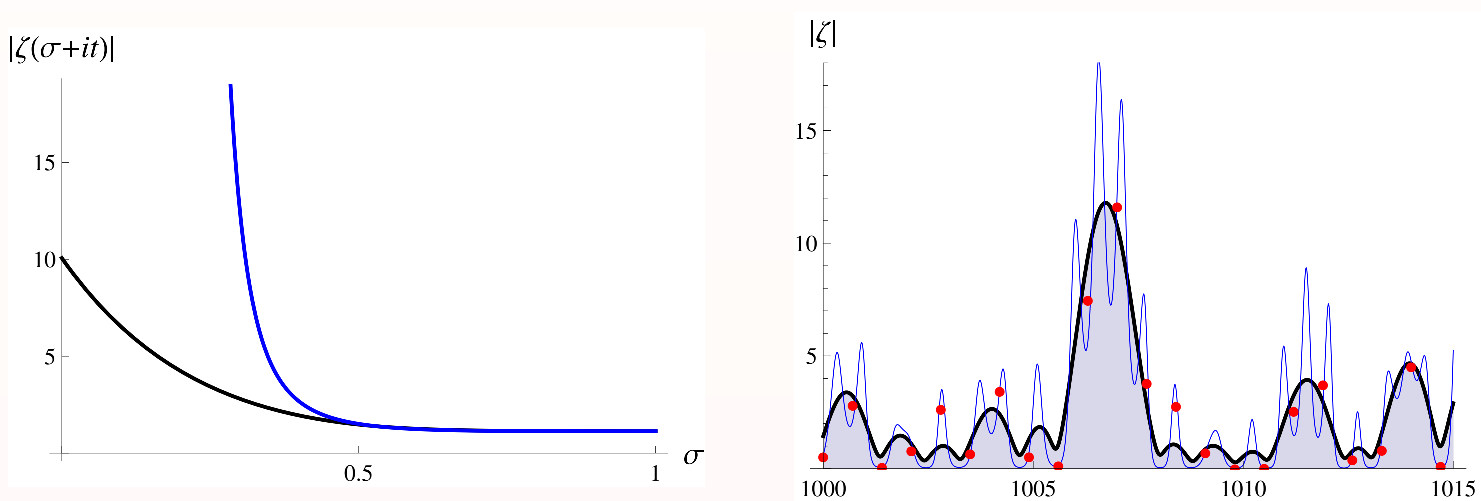


FIG. 6. **Left:** the black line corresponds to $|\zeta(\sigma+it)|$ against $0<\sigma<1$, for $t=500$. The blue line is the partial product $|\mathcal{P}_N(\sigma+it)|$ with $N=10^4$. **Right:** the black line is the exact $|\zeta|$, and the blue line is the partial product $|\mathcal{P}_N|$ (with $N=8\cdot10^3$), against t. We took $\sigma=0.4$. The red dots are the Cesàro average $|\langle\mathcal{P}_N\rangle|$. If we increase N the results are even worse.

N	$\lvert\langle\mathcal{P}_N\rangle\rvert$	$\lvert\mathcal{P}_N\rvert$
$1\cdot 10^3$	0.976752	0.972210
$2\cdot 10^3$	0.976690	0.981506
$3\cdot 10^3$	0.977653	0.976654
$4\cdot 10^3$	0.977865	0.975735
$5\cdot 10^3$	0.977926	0.984674
$6\cdot 10^3$	0.977463	0.977893
$7\cdot 10^3$	0.978208	0.976510
$8\cdot 10^3$	0.977593	0.978773
$9\cdot 10^3$	0.978290	0.981781
$1\cdot 10^4$	0.977900	0.971017
$1\cdot 10^5$	0.977703	0.971203
$1\cdot 10^6$	0.977925	0.971491
$1\cdot 10^7$	0.978168	0.978027
$1\cdot 10^8$	0.977823	0.984481
$2\cdot 10^8$	0.956304	0.885545
$3\cdot 10^8$	0.924928	0.794254
$\lvert\zeta(0.95+i\,20)\rvert = 0.977848$		

N	$\lvert\langle\mathcal{P}_N\rangle\rvert$	$\lvert\mathcal{P}_N\rvert$
$1\cdot 10^3$	1.690988	1.694894
$2\cdot 10^3$	1.692350	1.694156
$3\cdot 10^3$	1.692590	1.690354
$4\cdot 10^3$	1.692399	1.688480
$5\cdot 10^3$	1.691996	1.687150
$6\cdot 10^3$	1.691666	1.689158
$7\cdot 10^3$	1.691508	1.688145
$8\cdot 10^3$	1.691400	1.691700
$9\cdot 10^3$	1.691381	1.692973
$1\cdot 10^4$	1.691345	1.690480
$1\cdot 10^5$	1.691373	1.692136
$1\cdot 10^6$	1.691429	1.691577
$1\cdot 10^7$	1.691414	1.691703
$1\cdot 10^8$	1.691385	1.693287
$2\cdot 10^8$	1.745257	1.923738
$3\cdot 10^8$	1.852499	2.203470
$\lvert\zeta(0.95+i\,100)\rvert = 1.691397$		

TABLE I. Convergence $\langle\mathcal{P}_N\rangle$, and $\mathcal{P}_N$, for the ζ-function. Note that even for $N \gg N_c \sim t^2$ the results are good, but eventually it starts to deviate from the correct value as shown in the two last entries.

Zeta and other L's based on principal characters are the exception and actually trickier since all characters are 1 or 0. Partly due to the pole at z=1.

Transcendental equations for individual zeros.

AL Int. J. Mod. Phys. A28 (2013)
G. França, AL, Comm. Numb. Theory and Phys. 2015

Everyone here knows one function with an infinite number of zeros along a line in the complex z-plane........

$\cos(z) = 0$
for $z = (n+1/2)\pi$

Our result: There are an infinite number of zeros of zeta along critical line in one-to-one correspondence with the zeros of cosine.

The n-th zero satisfies a Transcendental Equation that depends only on n.

How to derive this equation:

Write:

$$\chi(z) = \chi(x+iy) = A(x,y)e^{i\theta(x,y)}$$

$$\chi(1-z) = \chi(1-x-iy) = A'(x,y)e^{i\theta'(x,y)}$$

If ρ is a zero : $\quad \chi(\rho) + \chi(1-\rho) = 0$

$$\Longrightarrow e^{i\theta} + e^{-i\theta'} = 0$$

The particular solution $\theta = \theta', \quad \cos(\theta) = 0$

gives an infinite number of zeros on the critical line

There exists an infinite number of zeros of zeta satisfying

$$\theta(x=1/2, y) = (n+1/2)\,\pi$$

Conjecture 2: $\theta(x=1/2, y) = (n+1/2)\,\pi$ is a transcendental equation for the ordinate y_n of the n-th Riemann zero:

There is an exact equation, but let me present its large y limit:

The n − th zero is of the form $\rho = \frac{1}{2} + i y_n$

$$\frac{y_n}{2\pi} \log\left(\frac{y_n}{2\pi e}\right) + \lim_{\delta \to 0^+} \frac{1}{\pi} \arg \zeta\left(\tfrac{1}{2} + \delta + i y_n\right) = n - \frac{11}{8} \qquad (n = 1, 2, \ldots)$$

Smooth part

small fluctuating part

To a very good approximation, the n-th zero satisfies: $y_n \approx \widetilde{y}_n$

$$\frac{\widetilde{y}_n}{2\pi} \log\left(\frac{\widetilde{y}_n}{2\pi e}\right) = n - \frac{11}{8}.$$

The solution is explicitly given in terms of an elementary function: the Lambert W-function:

$$\widetilde{y}_n = \frac{2\pi\left(n - \frac{11}{8}\right)}{W\left[e^{-1}\left(n - \frac{11}{8}\right)\right]}$$

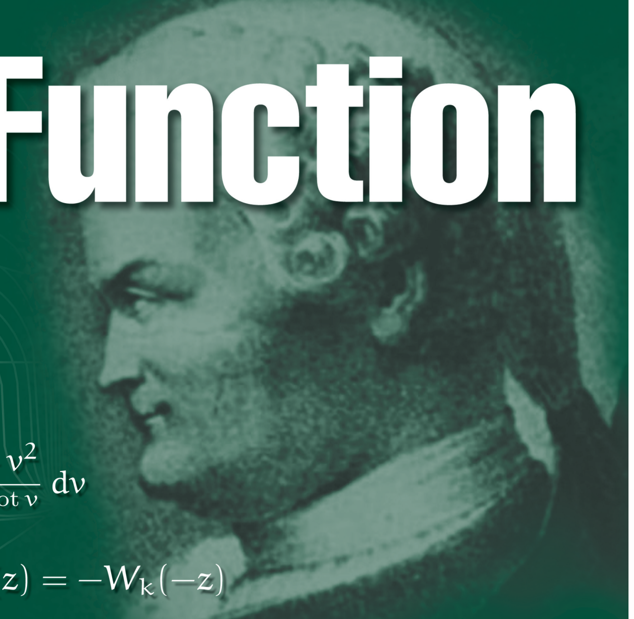
W is defined to satisfy:

$$W(z)e^{W(z)} = z$$

Lambert W was first studied by Lambert in the 1758. Euler recognized its importance in 1779 in a paper on transcendental equations, and credited Lambert.

It's importance was only realized in the 1990's, when it finally obtained the name the Lambert W-function.

The Lambert W Function

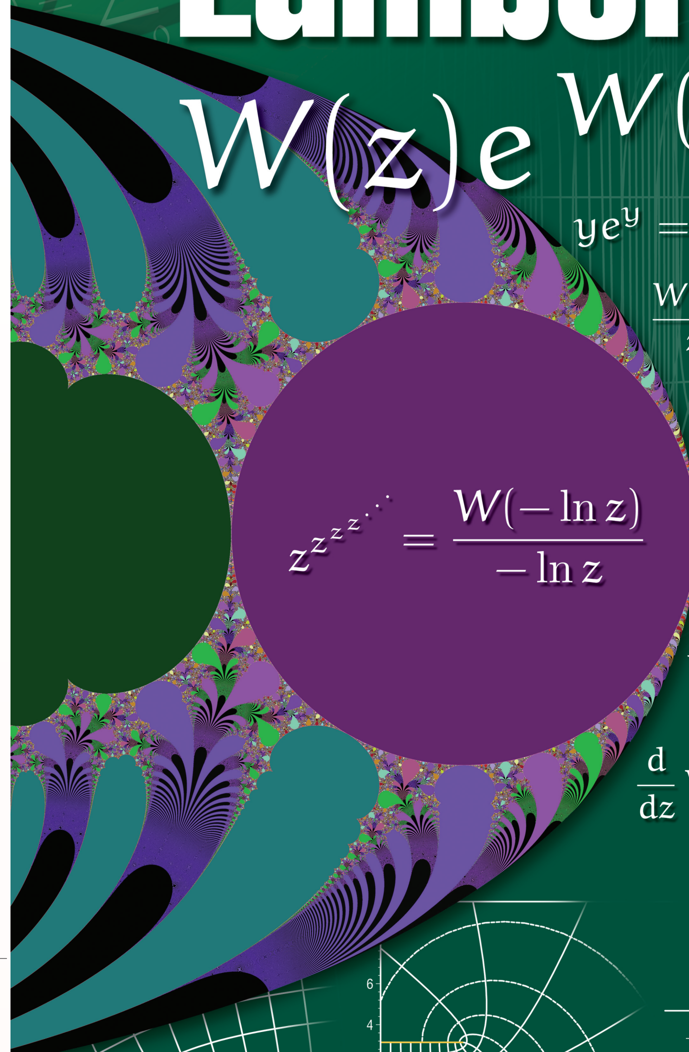


$$W(z)e^{W(z)} = z$$

$$ye^{y} = z \iff y = W_k(z)$$

$$\frac{W(z)}{z} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-v\cot v)^2 + v^2}{z + v\csc v\, e^{-v\cot v}}\, dv$$

$$ye^{-y} = z \iff y = T_k(z) = -W_k(-z)$$



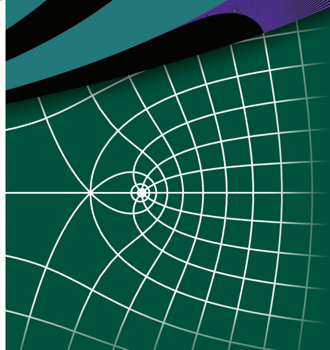
$$z^{z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}} = \frac{W(-\ln z)}{-\ln z}$$

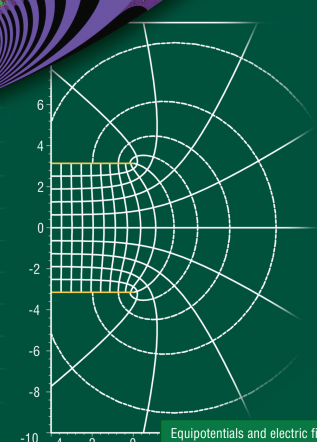
A Fractal Related to W

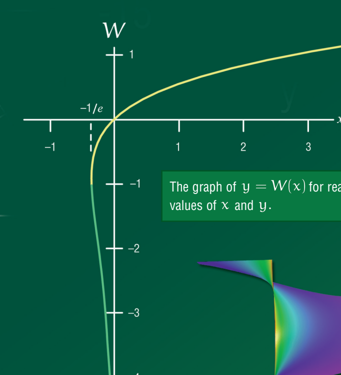
Each colour represents a cycle length in the iteration $a_{n+1} = z^{a_n}$, with $a_0 = 1$. A pixel at coordinate $\zeta = x + iy$ where $\zeta = T(\ln z)$ is given the colour corresponding to the length of the attracting cycle.

$$W(z) = \sum_{n\geq 1} \frac{(-n)^{n-1}}{n!} z^n$$

$$\frac{d}{dz}W(z) = \frac{W(z)}{z(1+W(z))} \quad \text{if } z \neq 0, -1/e$$



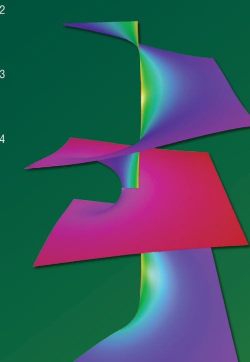




The graph of $y = W(x)$ for real values of x and y.

Images of circles and rays under the maps $z \mapsto W_k(z)$. Equivalently, images of horizontal and vertical lines under the map $z \mapsto \omega(z) = W_{K(z)}(e^z)$.

Equipotentials and electric field lines at the edge of a capacitor consisting of two charged thin plates a distance 2π apart.
$\zeta = z - 1 + \omega(z-1)$



A portion of the Riemann surface for $W(z)$, drawn by plotting a surface with height $\mathrm{Im}\,(W(x+iy))]$ at coordinates (x, y) and colouring the surface with Re $(W(x+iy))$; the apparent intersection on the line $-1/e \leq x \leq 0$, $y = 0$ is of surfaces with different colours and therefore not a true intersection.

$$\int W(z)\, dz = \frac{z\left(W^2(z) - W(z) + 1\right)}{W(z)} + C$$

$$\int_0^\infty x^{s-1} W(x)\, dx = \frac{(-s)^{-s}\,\Gamma(s)}{s} \quad \text{if } -1 < \mathrm{Re}(s) < 0$$

$$\int 2 \sin W(x)\, dx = \left(x + \frac{x}{W(x)}\right) \sin W(x) - x \cos W(x) + C$$

$$\int_0^\infty e^{-st} W(e^t)\, dt = s^{s-2}\Gamma(1-s, sW(1)) + \frac{W(1)}{s} \quad \text{if } \mathrm{Re}(s) > 0$$

Johann Heinrich Lambert

Johann Heinrich Lambert was born in Mulhouse on the 26th of August, 1728, and died in Berlin on the 25th of September, 1777. His scientific interests were remarkably broad. The self-educated son of a tailor, he produced fundamentally important work in number theory, geometry, statistics, astronomy, meteorology, hygrometry, pyrometry, optics, cosmology and philosophy. Lambert was the first to prove the irrationality of π. He worked on the parallel postulate, and also introduced the modern notation for the hyperbolic functions.

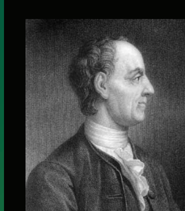
In a paper entitled "Observationes Variae in Mathesin Puram", published in 1758 in *Acta Helvetica*, he gave a series solution of the trinomial equation, $x^m + px = q$, for x. His method was a precursor of the more general Lagrange inversion theorem. This solution intrigued his contemporary, Euler, and led to the discovery of the Lambert W function.

Lambert wrote Euler a cordial letter on the 18th of October, 1771, expressing his hope that Euler would regain his sight after an operation; he explains in this letter how his trinomial method extends to series reversion.

The Lambert W function is *implicitly elementary*. That is, it is implicitly defined by an equation containing only elementary functions. The Lambert W function is not, itself, an elementary function. It is also not a *Liouvillian* function, which means that it is not expressible as a finite sequence of exponentiations, root extractions, or antidifferentiations (quadratures) of any elementary function.

The Lambert W function has been applied to solve problems in the analysis of algorithms, the spread of disease, quantum physics, ideal diodes and transistors, black holes, the kinetics of pigment regeneration in the human eye, dynamical systems containing delays, and in many other areas.

Leonhard Euler



Leonhard Euler was born on the 15th of April, 1707, in Basel, Switzerland, and died on the 18th of September, 1783, in St. Petersburg, Russia. Half his papers were written in the last fourteen years of his life, even though he had gone blind.

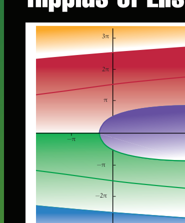
Euler was the greatest mathematician of the 18th century, and one of the greatest of all time. His work on the calculus of variations has been called "the most beautiful book ever written", and Pierre Simon de Laplace exhorted his students: "Lisez Euler, c'est notre maître â tous", advice that is still profitable today.

Many functions and concepts are named after him, including the Euler totient function, Eulerian numbers, the Euler-Lagrange equations, and the "eulerian" formulation of fluid mechanics. The mathematical formulae on this poster are typeset in the Euler font, designed by Hermann Zapf to evoke the flavour of excellent human handwriting.

Lambert's series solution of his trinomial equation, which Euler rewrote as $x^\alpha - x^\beta = (\alpha - \beta)vx^{\alpha+\beta}$, led to the series solution of the transcendental equation $x \ln x = v$. This was the earliest known occurrence of the series for the function now called the Lambert W function.

$$x^y = y^x \iff y = -\frac{x}{\ln x} W_k\left(-\frac{\ln x}{x}\right)$$

Hippias of Elis



Hippias of Elis lived, travelled and worked around 460 BC, and is mentioned by Plato. The Quadratrix (or trisectrix) of Hippias is the first curve ever named after its inventor. As drawn in the picture here, its equation is $x = -y \cot y$. This curve can be used to square the circle and to trisect the angle. Since these classical problems are unsolvable by straightedge and compass, we therefore conclude that the construction of the Quadratrix is impossible under that restriction. The Quadratrix is also the image of the real axis under the map $z \mapsto W_k(z)$, and the parts of the curve corresponding to the negative real axis delimit the ranges of the branches of W. We have here coloured the ranges of the different branches of W with different colours.

Sir Edward Maitland Wright $\omega(z) = W_{K(z)}(e^z)$

With Lambert W one can accurately estimate arbitrarily high zeros, even the $10^{1000000}$-th to million digit accuracy.

n	$\tilde{y}_n$	y_n
1	14.52	14.134725142
10	50.23	49.773832478
10^2	235.99	236.524229666
10^3	1419.52	1419.422480946
10^4	9877.63	9877.782654006
10^5	74920.89	74920.827498994
10^6	600269.64	600269.677012445
10^7	4992381.11	4992381.014003180
10^8	42653549.77	42653549.760951554
10^9	371870204.05	371870203.837028053
10^{10}	3293531632.26	3293531632.397136704

The 10^{999}-th zero to 1000 digits based on Lambert W:

2.7418985289770733523380199967281384304396404342236129703462008148794017483102
288989728527567413645122744311921172826961083680270092169498827568635959416113
429885386834142256620793027203450326850405406192401605278151278292126757823589
021159380557496232240667437943583994705834760582066723674368091278444158666608
455977853018177282026565267255273883601499075355217444189231104752684424593438
624806198537729334547336147304637269663107947384735659921127394121662743671648
211294886601858945279496294727955094639029288094054687941252225478426786182046
523221704263095085135100819383398596169703987228336044024659350088753385324537
829732202404696954235778305250096210562727012320495894109605623304319565563992
484717380637709436240220452151111044939346281951249654746987540134824713871321
328533373296657458895502274291514524646315414320664466625774466094199153901000
163674331154397634011868264241305320165870441692798635788965590575893640872077
63792090920744162661827244311481936682248189296258020149248439142

$\times 10^{996}$

Differ only in last digit shown

The $10^{999}+1$-th zero to 1000 digits based on Lambert W:

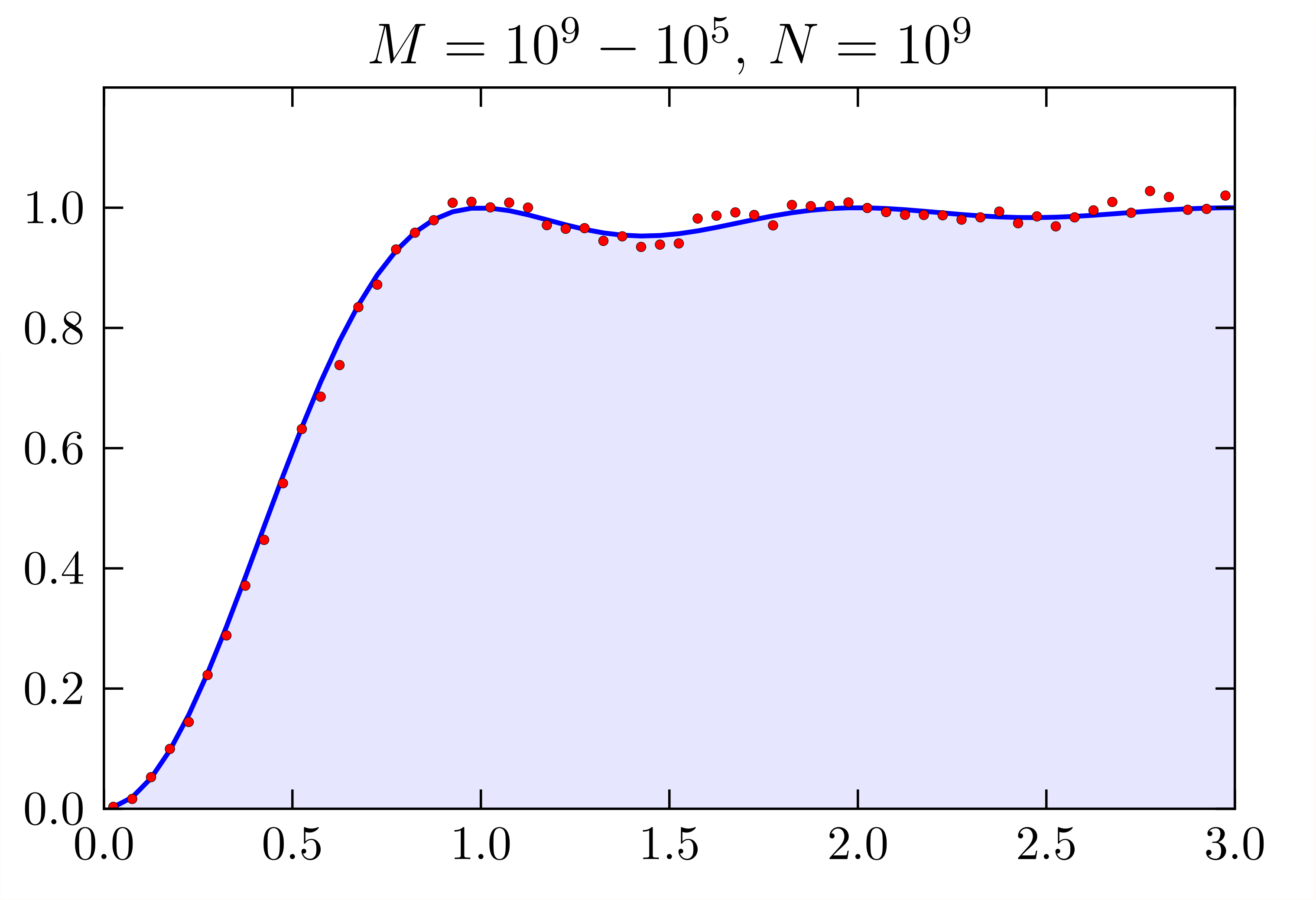
2.7418985289770733523380199967281384304396404342236129703462008148794017483102
288989728527567413645122744311921172826961083680270092169498827568635959416113
429885386834142256620793027203450326850405406192401605278151278292126757823589
021159380557496232240667437943583994705834760582066723674368091278444158666608
455977853018177282026565267255273883601499075355217444189231104752684424593438
624806198537729334547336147304637269663107947384735659921127394121662743671648
211294886601858945279496294727955094639029288094054687941252225478426786182046
523221704263095085135100819383398596169703987228336044024659350088753385324537
829732202404696954235778305250096210562727012320495894109605623304319565563992
484717380637709436240220452151111044939346281951249654746987540134824713871321
328533373296657458895502274291514524646315414320664466625774466094199153901000
163674331154397634011868264241305320165870441692798635788965590575893640872077
63792090920744162661827244311481936682248189296258020149248439145

$\times 10^{996}$

Solutions of the asymptotic transcendental equation
are accurate enough to reveal the GUE statistics:

10^5 zeros around the billion-th zero:

curve is the GUE prediction



(b)

Lambert approximation not good enough to see the statistics.

Solving the exact version of the transcendental equation gives zeros to any desired accuracy.

The 1000-th zero to 500 digits:

1419.42248094599568646598903807991681923210060106416601630469081468460
8676417593010417911343291179209987480984232260560118741397447952650637
0672508342889831518454476882525931159442394251954846877081639462563323
8145779152841855934315118793290577642799801273605240944611733704181896
2494747459675690479839876840142804973590017354741319116293486589463954
5423132081056990198071939175430299848814901931936718231264204272763589
1148784832999646735616085843651542517182417956641495352443292193649483
857772253460088

..........with very simple Mathematica commands.

Strategy 2 to prove the Riemann Hypothesis

Recall our main result:

The n − th zero is of the form $\rho = \frac{1}{2} + iy_n$

$$\frac{y_n}{2\pi}\log\left(\frac{y_n}{2\pi e}\right) + \lim_{\delta\to 0^+}\frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+\delta+iy_n\right) = n - \frac{11}{8} \qquad (n = 1, 2, \dots)$$

If there is a unique solution to this equation for every n, since they are enumerated by n, we can count how many zeros are on the critical line up to a height y=T.

$N_0(T)$ = number of zeros on the line with ordinate y<T. The above formula implies:

$$N_0(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + \frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+iT\right) + O\left(T^{-1}\right)$$

Now: N(T) = number of zeros on *the entire critical strip* has been known for over 100 years by performing a certain contour integral (*argument principle)* around the strip (Riemann, Backlund).

our $N_0(T)$ = the known N(T)

Thus: all zeros are on the line.

The converse of Riemann's result

Recall Riemann's main result: to calculate primes, one needs to know the zeros of zeta.

Our results give the converse: to calculate zeros, you need to know all the primes.

Recall trans. eqn:

The n − th zero is of the form $\rho = \frac{1}{2} + i y_n$

$$\frac{y_n}{2\pi} \log\left(\frac{y_n}{2\pi e}\right) + \lim_{\delta \to 0^+} \frac{1}{\pi} \arg \zeta\left(\tfrac{1}{2} + \delta + i y_n\right) = n - \frac{11}{8} \qquad (n = 1, 2, \ldots)$$

Conjecture 3: The validity of EPF for Re(z) > 1/2 smooths out S(t) and this leads to a unique solution to the transcendental equations for each n.

By EPF:
$$S_\delta(t) \equiv \frac{1}{\pi} \arg \zeta\left(\tfrac{1}{2} + \delta + it\right) = -\frac{1}{\pi} \lim_{N \to \infty} \Im\left[\sum_{n=1}^{N} \log\left(1 - p_n^{-1/2 - \delta - it}\right)\right].$$

Every individual zero knows about all the primes!

Conclusions

- According to our conjectures, the validity of the RH needs both the EPF and the functional equation.
- These two work together: The validity of the EPF and existence of solutions to the transcendental equations are closely related.
- Known counter-examples to RH have no EPF.
- We extended to another infinite class of L-functions based on modular forms. Brings in reasonably recent (1975) results of Deligne.
- A unified perspective on different of L-functions

$$\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{z}}\right)}}\right)}}\right)}}\right)}}\right)}}\right)}}\right)}}\right)}}\right)}}$$

The End