Supergroups for disordered dirac fermions

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Outline

• Introduction

• Classification of universality

• Supersymmetric disorder averaging

• $\mathfrak{gl}(1|1)$ supercurrent algebra as a critical point from super spin charge separation

• solution of the the $\mathfrak{gl}(1|1)$ level k model.

• Critical points and logarithmic perturbations

• multi-fractal and localization length exponents

• Conclusions
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• Critical points and logarithmic perturbations

• Multi-fractal and localization length exponents

• Conclusions

based on 0710.2906[hep-th] and 0710.3778[cond-math] (October)
Motivations from Mathematics and Physics
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- **Anderson transitions in 2+1 dimensions**
  - physics of metal–insulator transitions
  - the challenge: computing quenched disorder averages.
  - important physical examples: Quantum Hall Transition, Graphene
  - new universality classes beyond percolation
Supergroups in Mathematical Physics


- Sigma models on Lie supergroups arise in string theory on AdS spaces, e.g. \( \text{psl}(2|2) \) sigma models.

- Spin chains built on supergroups arise in the integrability approach to \( N=4 \) susy Yang-Mills.

- Various problems in statistical mechanics: percolation, self-avoiding walks, polymers, ....
• Supergroups in Mathematical Physics

• Anderson transitions: supergroups arise in Efetov’s supersymmetric method of computing quenched disorder averages.

• Sigma models on Lie supergroups arise in string theory on AdS spaces, e.g. $\text{psl}(2|2)$ sigma models.
  
  Berkovits, Vafa, Witten 1999

• Spin chains built on supergroups arise in the integrability approach to $\mathbb{N}=4$ susy Yang-Mills.
  
  Beisert and Staudacher 2005

• Various problems in statistical mechanics: percolation, self-avoiding walks, polymers, ....
Anderson localization and the Quantum Hall Transition
Consider electrons moving in a random potential:
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Quantum Hall Effect
Quantum Hall Effect

* free electrons in a magnetic field and random impurity potential
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Quantum Hall Effect

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\[ \xi_c \sim (E - E_c)\nu, \quad \Delta B \propto T^{1/\nu}, \quad \nu \approx 7/3 \]
Why Dirac fermions? Nearly all interesting cases have 1-st order actions.

Most general Dirac Hamiltonian in 2d:

\[ H = \begin{pmatrix}
V_+ + V_- & -i\partial_{\bar{z}} + A_{\bar{z}} \\
-i\partial_z + A_z & V_+ - V_-
\end{pmatrix} \]

V, A = V(x), A(x) = random potentials
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Classification according to discrete symmetries:

- Chirality: \(H = -PHP^{-1}, \quad P^2 = 1\)
- Particle-hole: \(H = -CH^TC^{-1}, \quad C^T = \pm C\)
- Time-reversal: \(H = KHK^*K^{-1}, \quad K^T = \pm K\)
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* Guruswamy, AL, Ludwig (1999)
Supersymmetric Disorder Averaging

Consider a free hamiltonian in a random potential $V(x)$:

\[ H = -\frac{\nabla^2}{2m} + V(x) \]

e.g. Schrodinger for simplicity:

We are interested in disorder averaged Green functions:

\[ \langle \psi(x)\psi^\dagger(x') \rangle = \int DVP[V] \langle \psi(x)\psi^\dagger(x') \rangle_V \]

The problem: properly normalize the Green function at fixed $V$ by $Z(V)$:

The trick: represent $Z$ with bosonic ghosts:

\[ \frac{1}{Z(V)} = \int D\beta \ e^{-S(\psi\rightarrow\beta,V)} \]

We can now perform the functional integral over the random potential $V$:

\[ \langle \psi(x)\psi^\dagger(y) \rangle = \int D\psi D\beta e^{-S_{\text{eff}}} \psi(x)\psi^\dagger(y) \]

$S_{\text{eff}}$ is an interacting quantum field theory of fermions and ghosts.
Where are the critical points?
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• Other approaches:

  • Replica sigma models (Pruisken 1984)

  • Supergroup sigma models (Zirnbauer 1999)
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  - Supergroup sigma models (Zirnbauer 1999)

**OUR NEW APPROACH:** Resolve the RG flow in 2 stages; use super spin charge separation; new results for gl(1|1) current algebra; explicit form of logarithmic operators in terms of symplectic fermions.
Supergroup symmetries in the N-copy theory

\[ = \int dx \quad \Psi^* \ H \ \Psi \]

For any realization of the disorder the action has a \( gl(N\mid N) \) symmetry.

The important super subgroup symmetry which commutes with permutations of the copies is:

\[ gl(1\mid 1)_N \]
Supergroup symmetries in the N-copy theory

Introduce N-copies of the theory in order to compute multiple moments:

fields: \( \Psi_\pm^\alpha = (\psi_\pm^\alpha, \beta_\pm^\alpha) \), \( \alpha = 1, \ldots, N \)

\[ = \int \text{d}x \quad \Psi^\dagger \quad \mathcal{H} \quad \Psi \]

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The action at fixed realization of disorder:

\[
S_{\text{susy}} = \int \frac{d^2x}{2\pi} \left[ \bar{\Psi}_-(\partial_z - iA_z(x))\Psi_+ + \bar{\Psi}_-(\partial_z - iA_z(x))\Psi_+ - iV(x) (\bar{\Psi}_-\Psi_+ + \Psi_-\bar{\Psi}_+) \\
- iM(x) (\bar{\Psi}_-\Psi_+ - \Psi_-\bar{\Psi}_+) \right]
\]

\[ = \int dx \; \Psi^* \mathcal{H} \Psi \]

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\[ gl(1|1)_N \]
The $gl(1|1)_N$ affine Lie algebra symmetry is generated by the chiral currents:

$$H = \sum_\alpha \psi_+^\alpha \psi_-^\alpha, \quad J = \sum_\alpha \beta_+^\alpha \beta_-^\alpha, \quad S_\pm = \pm \sum_\alpha \psi_+^\alpha \beta_-^\alpha$$

which satisfy the operator product expansion: (k=N = level)

$$H(z)H(0) \sim \frac{k}{z^2}, \quad J(z)J(0) \sim -\frac{k}{z^2}$$

$$H(z)S_\pm(0) \sim J(z)S_\pm(0) \sim \pm \frac{1}{z} S_\pm$$

$$S_+(z)S_-(0) \sim \frac{k}{z^2} + \frac{1}{z} (H - J)$$

Additional symmetries that commute with the above: $su(N)$ at level k=0

$$L_\psi^a = \psi_+^{\alpha_+} t_\alpha^a \psi_-^{\alpha_+}, \quad L_\beta^a = \beta_+^{\alpha_+} t_\alpha^a \beta_-^{\alpha_+}, \quad L^a = L_\psi^a + L_\beta^a$$

**Important symmetry:** \[ gl(1|1)_N \oplus su(N)_0 \]
* First separate the theory into two commuting sets of degrees of freedom. This involves a remarkable identity for the Sugawara stress-tensors:

\[
T_{\text{free}}^{N-\text{copy}} = -\frac{1}{2} \sum_{\alpha=1}^{N} (\psi_{-}^{\alpha} \partial_{z} \psi_{+}^{\alpha} + \beta_{-}^{\alpha} \partial_{z} \beta_{+}^{\alpha}) = T_{\text{gl}(1|1)_{k=N}} + T_{\text{su}(N)_{0}}
\]
Critical points from Super Spin-Charge Separation

Strategy for resolving the renormalization group (RG) flow: Based on the idea that the RG flow to low energies decouples massive degrees of freedom.

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* In the first stage of the RG flow, carry out the flow for the couplings in \( S_{\text{eff}} \) corresponding to these two sets of degrees of freedom:

\[
S = S_{\text{eff}} + \int \frac{d^2x}{2\pi} \left( g_A \ J_A \cdot \overline{J}_A + g_B \ J_B \cdot \overline{J}_B \right)
\]

where \( J_A = gl(1|1) \) currents, \( J_B = su(N) \) currents. The r-loop beta functions are:

\[
\frac{dg_A}{dl} = -g_A^2, \quad \frac{dg_B}{dl} = +g_B^2
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$$\frac{dg_{A}}{d\ell} = -g_{A}^{2}, \quad \frac{dg_{B}}{d\ell} = +g_{B}^{2}$$

Since $g_{A}$ ($g_{B}$) is marginally irrelevant (relevant) only the $su(N)$ degrees of freedom are gapped out in the flow. First stage: flow to $g\ell(1|1)_{N}$

Wednesday, August 10, 2011
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Since \( g_A \) (\( g_B \)) is marginally irrelevant (relevant) only the \( su(N) \) degrees of freedom are gapped out in the flow. First stage: flow to \( gl(1|1)_N \)

* Introduce additional forms of disorder as relevant perturbations of \( gl(1|1)_N \)
Solution of the $\text{gl}(1|1)_k$ theory

AL 0710.2906 [hep-th], builds on Schomerus and Saleur 2006
Solution of the \( \text{gl}(1|1)_k \) theory

Free field representation: two scalar field and a symplectic fermion:

Action:

\[
S = \frac{1}{8\pi} \int d^2x \sum_{a,b=1}^{2} \left( \eta_{ab} \partial_{\mu} \phi^a \partial_{\mu} \phi^b + \epsilon_{ab} \partial_{\mu} \chi^a \partial_{\mu} \chi^b \right)
\]

\[
\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

Representation of the current algebra:

\[
H = i\sqrt{k} \partial_z \phi^1, \quad J = i\sqrt{k} \partial_z \phi^2
\]

\[
S_+ = \sqrt{k} \partial_z \chi^1 e^{i(\phi^1 - \phi^2)/\sqrt{k}}, \quad S_- = -\sqrt{k} \partial_z \chi^2 e^{-i(\phi^1 - \phi^2)/\sqrt{k}}
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**Solution of the $\text{gl}(1|1)_k$ theory**

**Free field representation:**
two scalar field and a symplectic fermion:

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\]

**Twist fields:**
\[
\chi^1(e^{2\pi i z}) \mu_\lambda(0) = e^{-2\pi i \lambda} \chi^1(z) \mu_\lambda(0)
\]
\[
\chi^2(e^{2\pi i z}) \mu_\lambda(0) = e^{2\pi i \lambda} \chi^2(z) \mu_\lambda(0)
\]

\[
\Delta(\mu_\lambda) = \frac{\lambda(\lambda - 1)}{2} \equiv \Delta(\chi)
\]
**VErTEX OPERATORS:** fields transforming in finite dimensional reps of $\mathfrak{gl}(\mathfrak{t}|\mathfrak{t})_k$

The corresponding vertex operator:

$$V_{(h,j)} = (h - j)^{1/4} \begin{pmatrix} -\mu_{\lambda} e^{i(h\phi^1 - j\phi^2)/\sqrt{k}} \\ \sigma_{\lambda}^2 e^{i((h-1)\phi^1 - (j-1)\phi^2)/\sqrt{k}} \end{pmatrix}, \quad \lambda = \frac{h - j}{k}$$

Conformal scaling dimension:

$$\Delta_{(h,j)} = \frac{(h - j)^2}{2k^2} + \frac{(h - j)(h + j - 1)}{2k}$$

Closed operator algebra:

$$-k < h-j < k \quad \text{h,j,k = integers}$$
**VERTEX OPERATORS:**

Fields transforming in finite dimensional reps of \( gl(1|1)_k \)

2-dimensional reps \(<h,j>:\)

\[
H = \begin{pmatrix} h & 0 \\ 0 & h-1 \end{pmatrix}, \quad J = \begin{pmatrix} j & 0 \\ 0 & j-1 \end{pmatrix}
\]

\( S_+ = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \)

\((bc = h-j)\)

The corresponding vertex operator:

\[
V_{<h,j>} = (h-j)^{1/4} \left( -\mu \lambda \ e^{i(h\phi^1-j\phi^2)/\sqrt{k}} \right)
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Conformal scaling dimension:

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\Delta_{<h,j>} = \frac{(h-j)^2}{2k^2} + \frac{(h-j)(h+j-1)}{2k}
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Closed operator algebra:

\(-k < h-j < k\)

\(h,j,k = \) integers
Logarithmic vertex operators for indecomposable representations.

4-dimensional indecomposable reps \(<\mathbf{0}>_4:\) \(\langle 1, 0 \rangle \otimes \langle 0, 1 \rangle = \langle 0 \rangle_{(4)}\)

Corresponding vertex operator \((\Delta=0):\)

\[
V_{\langle 0 \rangle_{(4)}} = \begin{pmatrix}
\chi^1 e^{i(\phi^1 - \phi^2)/\sqrt{k}} \\
\sqrt{k} \\
\chi^2 \chi^2 / \sqrt{k} \\
\chi^2 e^{-i(\phi^1 - \phi^2)/\sqrt{k}}
\end{pmatrix}
\]

Logarithmic property: Virasoro zero mode is not diagonal \((\text{Jordan block form})\)

\[
L_0 = -\frac{1}{k} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(similar properties found for \(\text{osp}(2|2)\) by Maassarani and Serban 1997)
Logarithmic vertex operators for indecomposable representations.

4-dimensional indecomposable reps $<0>_{4}$: $\langle 1, 0 \rangle \otimes \langle 0, 1 \rangle = \langle 0 \rangle_{(4)}$

Corresponding vertex operator $(\Delta=0)$:

$$V_{\langle 0 \rangle_{(4)}} = \begin{pmatrix} \chi^1 e^{i(\phi^1-\phi^2)/\sqrt{k}} \\ \sqrt{k} \\ \chi^1 \chi^2 / \sqrt{k} \\ \chi^2 e^{-i(\phi^1-\phi^2)/\sqrt{k}} \end{pmatrix}$$

Logarithmic property: Virasoro zero mode is not diagonal (Jordan block form)

( due to the log pair (1, $\chi^1 \chi^2$) )

$$L_0 = -\frac{1}{k} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(similar properties found for osp(2|2) by Maassarani and Serban 1997)
Logarithmic perturbations

Quantum numbers:

* under the $\text{gl}(1|1) \times \text{su}(N)$ symmetries:

$$\psi_{\pm}, \beta_{\pm} \implies (\langle 1, 0 \rangle \oplus \langle 0, 1 \rangle) \otimes [\text{vec}]$$

* currents = bilinears in these fields. Examining the quantum numbers:

For $N<2$ the most relevant operator corresponds to $\langle 0 \rangle_{(4)}$. Leads to:

$$S = S_{\text{gl}(1|1)_N} + g \int \frac{d^2x}{8\pi} \Phi_{(0)}^{(4)}$$

$$= \int \frac{d^2x}{8\pi} \left( \sum_{a,b=1}^2 \eta_{ab} \partial_\mu \phi^a \partial_\mu \phi^b + \epsilon_{ab} \partial_\mu \chi^a \partial_\mu \chi^b + g \chi^1 \chi^2 \cos \left( (\phi^1 - \phi^2)/\sqrt{N} \right) \right)$$
Logarithmic perturbations

Additional disorder as perturbations of the $\text{gl}(1|1)$ cft:

* in the original theory they correspond to left/right current interactions.

* after gapping out the $\text{su}(N)_0$ degrees of freedom, additional disorder corresponds to relevant perturbations consistent with quantum numbers.

Quantum numbers:

* under the $\text{gl}(1|1) \times \text{su}(N)$ symmetries:

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For $N<2$ the most relevant operator corresponds to $\langle 0 \rangle_{(4)}$. Leads to:

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S = S_{\text{gl}(1|1) \times \text{su}(N)} + g \int \frac{d^2x}{8\pi} \Phi_{(0)}(4) \\
= \int \frac{d^2x}{8\pi} \left( \sum_{a,b=1}^2 \eta_{ab} \partial_\mu \phi^a \partial_\mu \phi^b + \epsilon_{ab} \partial_\mu \chi^a \partial_\mu \chi^b + g \chi^1 \chi^2 \cos \left( (\phi^1 - \phi^2)/\sqrt{N} \right) \right)
\]
* The above action defines a $\mathfrak{gl}(1|1)$ version of sine-Gordon theory.

* The logarithmic perturbations do not drive the theory to a new fixed point:

$$e^{ia(\phi^1 - \phi^2)(z)} e^{ib(\phi^1 - \phi^2)(0)} \sim \text{regular}$$
* The above action defines a $gl(1|1)$ version of sine-Gordon theory.

* The logarithmic perturbations do not drive the theory to a new fixed point:

\[ e^{i a(\phi^1 - \phi^2)}(z) \ e^{i b(\phi^1 - \phi^2)}(0) \sim \text{regular} \]

Thus: The critical exponents should be in the $gl(1|1)_N$ conformal field theory.
Multi-fractal exponents

* a probe of disorder averaged higher moments; must be computed in the N-copy theory

**density of states operator:**

\[ \rho(x) = \bar{\Psi} - \Psi + \Psi - \bar{\Psi} \]

**q-th moment:**

\[ P^{(q)} = \frac{\int d^2 x \rho(x)^q}{(\int d^2 x \rho(x))^q} \]

**scaling at the critical point:**

\[ P^{(q)} \sim L^{-\tau_q} \quad (L = \text{size}) \]

**Relation to scaling dimension of operators:**

\[ \tau_q = \hat{\Gamma}_q + 2(q - 1) \]
Multi-fractal exponents

* a probe of disorder averaged higher moments; must be computed in the N-copy theory

density of states operator: \( \rho(x) = \overline{\Psi_- \Psi_+} + \Psi_- \overline{\Psi_+} \)

q-th moment: \( P^{(q)} = \frac{\int d^2 x \rho(x)^q}{(\int d^2 x \rho(x))^q} \)

scaling at the critical point: \( P^{(q)} \sim L^{-\tau_q} \) \( (L = \text{size}) \)

Relation to scaling dimension of operators: \( \tau_q = \hat{\Gamma}_q + 2(q - 1) \)

\( \hat{\Gamma}_q \leftrightarrow \text{scaling dimension of } \rho^q \)
We compute $\hat{\Gamma}_q$ in the $\mathbb{N}=2$ copy theory since for $q>q_c$ the multi-fractal spectrum is known to cross over to a non-parabolic spectrum and $2 < q_c < 3$.

The most relevant operator in $\rho^q$ corresponds to the $<o,q>$ $\text{gl}(\mathfrak{h})$ rep.
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The most revelant operator in $\rho^q$ corresponds to the $<0,q>$ $\text{gl}(1|1)$ rep.

Result:

$$\hat{\Gamma}_q = \frac{q(1-q)}{4}$$
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The most relevant operator in $\rho^q$ corresponds to the $<o,q>$ $\text{gl}(1|1)$ rep.

**Result:**

$$\widehat{\Gamma}_q = \frac{q(1-q)}{4}$$

agrees to 1-2\% with numerical results of Klesse\&Metzer (1995); Evers, Mildenberger and Mirlin (2001)
Localization exponent
Localization exponent

This exponent corresponds to tuning a parameter in the action to critical point, i.e. it’s a quantum critical point.

\[ \delta S_\nu = \int \frac{d^2 x}{2\pi} \lambda \mathcal{O}_\nu(x) \]

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What is the operator \( \mathcal{O}_\nu \)? No simple quantum number arguments to identify it.

Hint from spin quantum Hall: here \( \text{gl}(1|1)_N \) becomes \( \text{osp}(2|2)_{-2N} \)

Use the exact embedding: \( \text{gl}(1|1)_2 \subset \text{osp}(2|2)_{-2} \)
By comparing conformal dimensions: $\text{gl}(1|1)_2 = \text{percolation}$

In the $N=2$ theory, the localization length exponent for percolation is $\langle 2,1 \rangle$ field.

Natural generalization in the $\text{gl}(1|1)_N$ theory is the field $\langle N,N-1 \rangle$.
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Real experiments: \( 2.3 \pm 0.1 \), S. Koch et. al. (1991)

Numerical simulations: \( 2.33-2.35 \pm 0.03 \), Huckestein (1995); D.-H. Lee and Wang (1996)
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- relies on new results for $\text{gl}(\mathbb{1}\mathbb{1})_k$ current alg.
The End