- Physical meaning of the Wave Function
  - Normalization
  - Expectation Values
  - Operators
- Stationary States
  - Properties
  - time-independent Schrödinger equation
Recap:

**II$_{1,3}$ Plausibility Argument leading to Schrödinger's Equation:**

Postulate: \[ \frac{\text{wave equation}}{\partial t} \quad i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x,t) \Psi(x,t) \]

wave function for free particle with const. $E$, $p$

\[ \psi(x,t) = e^{i(kx - \omega t)} \]

plane complex wave

Time-dependent Schrödinger equation: \[ i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi(x,t) \]

\[ \hat{H}: \text{Hamiltionian operator} \quad \hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x,t) \]

**II$_{1,4}$ Physical Significance of the Wave Function $\Psi(x,t)$:**

\[ \int_{a}^{b} |\Psi|^2 \, dx = \int_{a}^{b} \Psi^\ast \Psi \, dx = \left\{ \begin{array}{l} \text{probability of finding the particle between } a \text{ and } b \text{ at time } t \\ = P(x,t) \, dx: \text{probability density} \end{array} \right. \]
Measurement:

- $|\psi|^2$ just before measurement
- $|\psi|^2$ immediately after measurement has found particle at point "c"

Wave function "collapses" upon measurement!
For a proper wave function, \[
\int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = ?
\]

A. 0  
B. 1  
C. infinite  
D. Something else

Probability of finding the particle somewhere must be = 1
- **Normalization**

\[ \int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 1 \]

(normalization condition)

(Probability of finding particle somewhere \(= 1\))

**Note:** If \(\Psi(x,t)\) is solution of S. E. \(\Rightarrow C \cdot \Psi(x,t)\) is too!

\[ \int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = a \text{ normalized wave function} = \frac{1}{\sqrt{a}} \Psi(x,t) \]

**Note:** If a wave function is normalized at time \(t=0\) it will stay normalized as time goes on!
Examples: (1) Plane complex wave: free particle with constant $E, p$.
\[ \Psi = A e^{i(kx - \omega t)} \]
\[ \implies P(x,t) = |A|^2 = \text{const!} \]

\[ \text{Problem: can not be normalised} \implies \text{not physical!} \]
Examples: ① Plane complex wave: free particle with constant $E,p$
\[ \Psi = A e^{i(kx-\omega t)} \]
\[ \Rightarrow P(x,t) = |A|^2 = \text{const} \]
Wave packet:
Particle in a 1D-box:

$n = 3$ state
Particle in a 1D-box:

\[\psi(x)\]

\[n=2\]

\[n=1\]

\[n=3\] state

\[|\psi|^2 = \psi^* \psi \rightarrow \text{time independent!} \Rightarrow \text{stationary state}\]

Note: \(\psi(x,t)\) itself is still time dependent!
**II.5 Expectation Values**

- **Start with position:**
  - Wave function spreads over certain space → cannot give definite position value for particle
  - Specify average measured position of a particle = expectation value of the x-coordinate

\[
\langle x \rangle = \int_{-\infty}^{+\infty} x \, P(x,t) \, dx
\]

- Value of probability of observing that value

\[
\Rightarrow \quad \text{Born's postulate } P(x,t) = \psi^* \psi
\]

\[
\Rightarrow \quad \langle x \rangle = \int_{-\infty}^{+\infty} \psi^* (x,t) \times \psi (x,t) \, dx
\]

**Note:** The expectation value is the average of repeated measurements on an ensemble of identically prepared systems.

It is **not** what you would get if you measure the position of one particle over and over again!
A particle is associated with the following wave function:

\[ \Psi(x,t) = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right)e^{-i\alpha t} \quad \text{for } -L/2 < x < L/2 \]

\[ \Psi(x,t) = 0 \quad \text{elsewhere} \]

If the position \( x \) of the particle would be measured, the result would be:

A. \( x=0 \)
B. Something between \(-L/2\) and \(+L/2\)
C. Something between \(-\infty\) and \(+\infty\)
D. Something else
A particle is associated with the following wave function:
\[ \Psi(x, t) = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right)e^{-i\omega t} \]
for \(-L/2 < x < L/2\)

\[ \Psi(x, t) = 0 \]
elsewhere

What is the expectation value of the position \( \langle x \rangle \)?

A. \( \langle x \rangle = 0 \)
B. \( \langle x \rangle = L/2 \)
C. \( \langle x \rangle = -L/2 \)
D. \( \langle x \rangle \approx L/4 \)
E. Something else

Symmetry of probability density about \( x = 0 \) -> \( \langle x \rangle = 0 \)
* Expectation value of the particle momentum:

want: \( <p> = m <v> = m \frac{d<x>}{dt} \)

start with: \( \frac{d<x>}{dt} = \int \psi^* \frac{\partial}{\partial x} \psi^* dx \)

\( \frac{\partial}{\partial t} |\psi(x, t)|^2 = \frac{\partial}{\partial t} (\psi^* \psi) = \psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \psi \)

use Schrödinger's equation:

\( \frac{\partial \psi}{\partial t} = \frac{i \hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V \psi \)

complex conjugate:

\( \frac{\partial \psi^*}{\partial t} = \frac{-i \hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar} V \psi^* \)

\( \Rightarrow \frac{\partial}{\partial t} |\psi|^2 = \frac{i \hbar}{2m} \left( \psi^* \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi^*}{\partial x^2} \psi \right) = \frac{\partial}{\partial x} \left( \frac{i \hbar}{2m} \psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \psi^* \right) \)
\[ \frac{d \langle x \rangle}{dt} = \int_{-\infty}^{+\infty} x \frac{\partial}{\partial t} |\psi|^2 \, dx = \frac{i \hbar}{2m} \int_{-\infty}^{+\infty} x \frac{\partial}{\partial x} \left\{ \psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right\} \, dx \]

Use integration by parts:
\[ \frac{d}{dx} (fg) = f \frac{df}{dx} + df \cdot g = \int_a^b f \frac{df}{dx} \, dx = -\int_a^b \frac{df}{dx} \, g \, dx + f g \bigg|_a^b \]

\[ \frac{d \langle x \rangle}{dt} = -\frac{i \hbar}{2m} \int_{-\infty}^{+\infty} \left\{ \psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right\} \, dx + f g \bigg|_{-\infty}^{+\infty} = 0, \quad |\psi| \xrightarrow{x \to \pm \infty} 0 \]

2nd integration by parts on 2nd term:
\[ \frac{d \langle x \rangle}{dt} = -\frac{i \hbar}{2m} \int_{-\infty}^{+\infty} 2 \psi^* \frac{\partial \psi}{\partial x} \, dx = -\frac{i \hbar}{m} \int_{-\infty}^{+\infty} \psi^* \frac{\partial \psi}{\partial x} \, dx = \langle v \rangle \]

Finally:
\[ \langle p \rangle = m \langle v \rangle = -\frac{i \hbar}{m} \int_{-\infty}^{+\infty} \left( \psi^* \frac{\partial \psi}{\partial x} \right) \, dx \]

Expectation value for velocity
\[ \langle P \rangle = \int_{-\infty}^{\infty} \Psi^* \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi \, dx \] Expectation value of particle momentum

- operator \( \frac{\hbar}{i} \frac{\partial}{\partial x} \) \( \Leftrightarrow \) "represents" momentum in quantum mechanics

\[ x \Leftrightarrow \) "represents" position

\( \Rightarrow \) For other quantities:

- Recipe:
  1) express quantity as function of position and momentum
  2) replace "p" by \( \frac{\hbar}{i} \frac{\partial}{\partial x} \), "multiply" \( x \) by \( x \)

\[ \Rightarrow \langle Q(x, p) \rangle = \int_{-\infty}^{\infty} \Psi^* (x, t) \hat{Q} (x, \frac{\hbar}{i} \frac{\partial}{\partial x}) \Psi (x, t) \, dx \]

Example:

\[ \langle x^2 \rangle = \int_{-\infty}^{\infty} \Psi^* x^2 \Psi \, dx = \) variance:

\[ \sigma_x^2 \equiv \langle x^2 \rangle - \langle x \rangle^2 \]
- Potential Energy: \( \langle V \rangle = \int_{-\infty}^{+\infty} \Psi^* V(x,t) \Psi \, dx \)

- Kinetic Energy: \( KE = \frac{p^2}{2m} \Rightarrow \langle KE \rangle = \int_{-\infty}^{+\infty} \Psi^* \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \Psi \, dx \)

- Total Energy: \( \langle E \rangle = \langle KE \rangle + \langle V \rangle = \int_{-\infty}^{+\infty} \Psi^* \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x,t) \right) \Psi \, dx \)

\[ \hat{H} = \text{Hamiltonian Operator} \]