• Qualitative plots of bound-state wave functions

• Finite square well:
  \(-E_n, \psi_n(x)\)
Recap

**II\(_{2,2}\)** Square Well of Finite Depth:

1) Solve time-independent SE for segments
2) Join pieces of \(\psi(x)\)

**Boundary conditions:** \(\psi(x)\) and \(\frac{d\psi}{dx}\) must be continuous everywhere!

\[\Rightarrow \text{Need to have decaying exponential waves in region I, III}\]

\[\Rightarrow \text{If } \psi(x) \xrightarrow{x \to \pm \infty} 0, \text{ boundary conditions can only be satisfied for certain energies of the particle!}\]
II.2,3 Qualitative Plots of Bound-State Wave Functions:

given potential \( V(x) \) \( \Rightarrow \) sketch solution \( \Psi(x) \) qualitatively

\[ \Rightarrow \text{Guidelines:} \]

1) Curvature \( \frac{d^2 \Psi}{dx^2} \)
2) # of nodes in stationary state
3) Amplitude of the wave function
4) Symmetry
1) **Curvature:**

\[
\text{time-independent S.E.:} \quad \frac{d^2 \psi}{dx^2} = -\frac{2m}{\hbar^2} \left( E - V(x) \right) \psi(x)
\]

\[
\implies \left| \frac{2m}{\hbar^2} (E - V(x)) \right| = \left| \frac{d^2 \psi}{dx^2} \right| / |\psi(x)|
\]

a) **\( E > V(x) \): inside well**

- **higher \( E-V \) means more curvature**
  \( \iff \) shorter "wavelength"
- if \( \psi(x) > 0 \) \( \iff \) negative curvature
- if \( \psi(x) < 0 \) \( \iff \) positive curvature
- always bends back toward \( x \)-axis
  \( \Rightarrow \) wave function oscillates in space

\[\psi \uparrow\]

\[x\]
6) if \( E < V(x) \): outside well

- Larger \( V - E \) means larger curvature
- \( \Rightarrow \) higher energy states extend further into “forbidden region”
- if \( \psi(x) > 0 \) \( \Rightarrow \) positive curvature
- if \( \psi(x) < 0 \) \( \Rightarrow \) negative curvature
- \( \Rightarrow \) always bend away from axis
- \( \Rightarrow \) exponential behavior
A particle in a finite square well is associated with the stationary state wave function shown below.

What state is it in?

A. Ground state
B. 1\textsuperscript{st} excited state
C. 2\textsuperscript{nd} excited state
D. 3\textsuperscript{rd} excited state
E. 4\textsuperscript{th} excited state
2) **# of nodes in stationary state (node = zero crossing)**

as energy increases =) 
short wavelength
=) more oscillations
=) one more zero crossing in \( \Psi(x) \) for each new level

=) **# of nodes = \( n - 1 \)**

with \( n = \text{energy level number} \)

- \( n = 1 \): ground state
- \( n = 2 \): 2nd energy state
  (1st excited state)
- \( n = 3 \): 3rd energy state ...
A particle of energy $E$ is bound in a stationary state in the potential shown below.

Where is it more likely to find the particle if its position would be measured?

A. Deep part of the well
B. Shallow part of the well
C. Both the same
D. No idea
3) Amplitude of wavefunction for \( E \geq V(x) \)

\[ = \text{ inside of well} \]

- **Example**

\[ \begin{array}{c}
\text{I} : \quad \text{II} \\
\text{E} \end{array} \]

\[ \rightarrow x \]

- **i) Classical** when \( E - V = kE \text{ small} \)

\[ \Rightarrow \text{small velocity} \]

\[ \Rightarrow \text{probability of finding classical particle \( P(x) \) or \( \frac{1}{\text{velocity}} \) large if velocity small} \]

- **ii) by correspondence principle**

\[ \text{expect: } P(x) = (\psi(x))^2 \text{ to be large when } E - V \text{ is small!} \]

- **iii) Proof:**

- **Region I:** \( \psi(x) = A_i \sin (k_i x + \delta_i) \) \( k_i = \frac{\sqrt{2m(E-V)}}{\hbar} \)

- **Region II:** \( \psi(x) = A_2 \sin (k_2 x + \delta_2) \)

- **At boundary \( x=0 \): \( \psi(x) \), \( d\psi/dx \) continuous!

\[ \Rightarrow \psi(0) = A_i \sin (\delta_i) \]

\[ \Rightarrow \psi_{\text{solution}} = d\psi/dx \bigg|_{x=0} = A_i k_i \cos (k_i x + \delta_i) \bigg|_{x=0} = A_i k_i \cos (\delta_i) \]
\[ \psi(0) = A_1 \sin (\delta_1) = A_2 \sin (\delta_2) \]

\[ \frac{\gamma}{k_i} = A_i \cos (\delta_i) \quad \text{for } i = 1, 2 \]

\[ = \] **Square and add:**

\[ (\psi(0))^2 + \left( \frac{\gamma}{k_i} \right)^2 = A_i^2 \]

\[ \Rightarrow A_i = \sqrt{(\psi(0))^2 + \left( \frac{\gamma}{k_i} \right)^2} = \sqrt{(\psi(0))^2 + \frac{\gamma^2}{2m}(E - V_i)} \]

\[ \Rightarrow \text{large } E - V \text{ gives small oscillation amplitude } A_i \text{ for } \gamma \neq 0. \]

\[ \Rightarrow \text{small } E - V \text{ gives large } \]  

**Note:** A is small in deep part of well!

- Also true for continuously varying potentials

\[ \Rightarrow \text{small } E - V : \text{large amplitudes, longer } \]

"local wavelengths" (no longer sinusoidal!)
Examples: Step-potential

- $n=4$: Large $E-V$ $\Rightarrow$ Small amplitude of oscillation, short $\lambda$.
- $n=3$: Small $E-V$ $\Rightarrow$ Large amplitude, long $\lambda$.
- $n=2$: Large $E-V$ $\Rightarrow$ Small amplitude of oscillation, short $\lambda$.
- $n=1$: Small $E-V$ $\Rightarrow$ Large amplitude, long $\lambda$.

Continuously varying potential

$V(x)$

$E_n$

$\psi(x)$
4) **Symmetry:**

Symmetric potential (about origin \( x = 0 \))

\[ V(x) = V(-x) \]

\[ \psi(x) = \psi(-x) \]

\( \implies \) Probability of finding particle in stationary state:

\[ P(x) = P(-x) \]

\[ |\psi(x)|^2 = |\psi(-x)|^2 \]

\( \implies \) 2 types:

- **Even functions:** \( \psi(x) = \psi(-x) \)
  \( \implies \) even number of nodes

- **Odd functions:** \( \psi(x) = -\psi(-x) \)
  \( \implies \) odd number of nodes
Summary: Guidelines for qualitative plots of $\psi(x)$

1) The amount of curvature (rate of oscillation or decay) increases with increasing $|V-E|$
   1a) $\psi(x)$ oscillates when $E > V$ (curvature towards $x$-axis)
   1b) $\psi(x)$ has curvature away from $x$-axis, when $E < V$

2) The $n$th energy level has $n-1$ zero crossings

3) When $E > V$, larger $E-V$ gives smaller wave amplitude

4) If the potential $V(x)$ is symmetric, then $\psi(x)$
   is either symmetric or antisymmetric (alternating)

5) $\psi(x) \xrightarrow{x \to \pm \infty} 0$ for allowed bound states
II.4 Square Well of Finite Depth, part II:

\[ \psi(x) \uparrow \]

\[ x < -\frac{L}{2} \]
\[ \psi(x) = C e^{\alpha x} \]
\[ \alpha = \sqrt{2m(V_0 - E)} / \hbar \]
\[ \psi(x) \xrightarrow{x \to -\infty} 0 \]
Symmetric well:

\[ \xrightarrow{\text{even solutions}} \]
\[ \psi(x) = D e^{\alpha x} \]

\[ x \geq -\frac{L}{2} \]
\[ \psi(x) = A \sin(kx) + B \cos(kx) \]
\[ k = \sqrt{2mE} / \hbar \]
\[ \psi(x) \xrightarrow{x \to \infty} 0 \]
Spatial oscillating wave

\[ \xrightarrow{\text{odd solutions}} \]
\[ \psi(x) = -D e^{\alpha x} \]

\[ x > \frac{L}{2} \]
\[ \psi(x) = 0 \]
Decaying exponential wave

\[ \alpha = \sqrt{2m(V_0 - E)} / \hbar \]
\[ \psi(x) \xrightarrow{x \to \infty} 0 \]

From boundary conditions and normalization:

\[ \text{find } A, B, D, \text{ and } E (\Rightarrow k, \alpha) \]
• Even functions/states:

- Inside well: $-\frac{L}{2} \leq x \leq \frac{L}{2}$ : $\Psi(x) = B \cos(kx)$
  \[
  \frac{d\Psi}{dx} = -Bk \sin(kx)
  \]

- Outside well: $x > \frac{L}{2}$ : $\Psi(x) = D e^{-\alpha x}$
  \[
  \frac{d\Psi}{dx} = -\alpha D e^{-\alpha x}
  \]

- Continuity of $\frac{d\Psi}{dx}$ at $x = \frac{L}{2}$:
  \[
  Bk \sin\left(k \frac{L}{2}\right) = \alpha D e^{-\alpha \frac{L}{2}}
  \]

- Continuity of $\Psi(x)$ at $x = \frac{L}{2}$:
  \[
  B \cos\left(k \frac{L}{2}\right) = D e^{-\alpha \frac{L}{2}}
  \]

= divide these eqns.

\[
\Rightarrow k \tan\left(\frac{kL}{2}\right) = \alpha
\]

\[
\Rightarrow \tan\left(\frac{kL}{2}\right) = \frac{\alpha}{k}
\]
\[ \tan \left( \frac{L}{2\hbar} \sqrt{2mE} \right) = \frac{\sqrt{2m(V_0 - E)}}{\sqrt{2mE}} = \sqrt{\frac{V_0 - E}{E}} = \sqrt{\frac{V_0}{E} - 1} \]