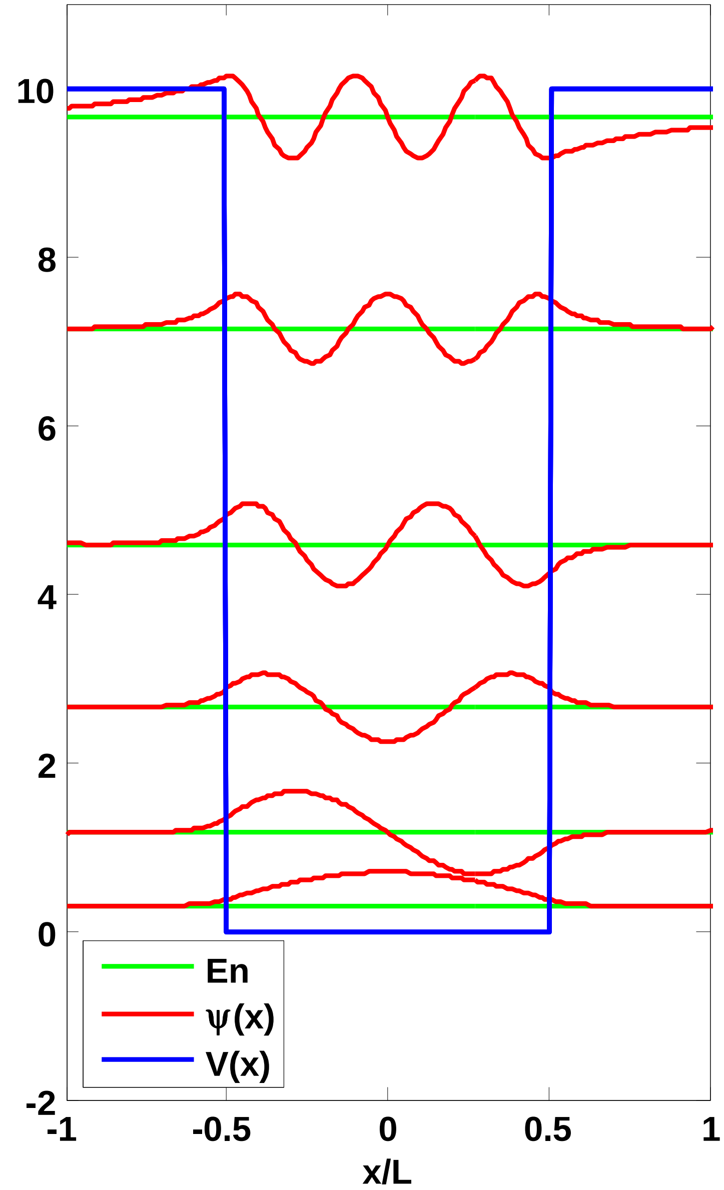
Lecture 17: 02/25/09

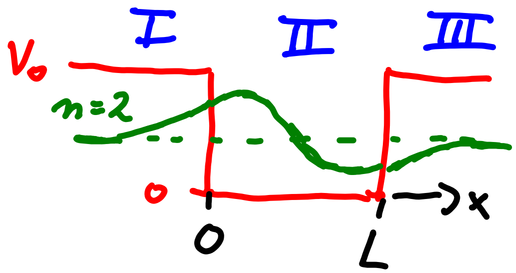
- Qualitative plots of bound-state wave functions

- Finite square well:
  - E_n, $\psi_n(x)$



Recap

$II_{2,2}$ Square Well of Finite Depth:



1) Solve time-indep. SE for segments
2) Join pieces of $\psi(x)$

Boundary conditions: $\psi(x)$ and $\frac{d\psi}{dx}$ must be continuous everywhere!

$\Rightarrow$ Need to have decaying exponential waves in region I, III

$\Rightarrow$ If $\psi(x) \xrightarrow[x \to \pm\infty]{} 0$, boundary conditions can only be satisfied for certain energies of the particle!

$II_{2,3}$ Qualitative Plots of Bound-State Wave Functions:

given potential $V(x)$ $\Rightarrow$ sketch solution $\psi(x)$ qualitatively

$\Rightarrow$ Guidelines:

1) Curvature $\frac{d^2\psi}{dx^2}$
2) # of nodes in stationary state
3) Amplitude of the wave function
4) Symmetry

1) Curvature:

time-indep. S.E. : $\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - V(x))\,\psi(x)$

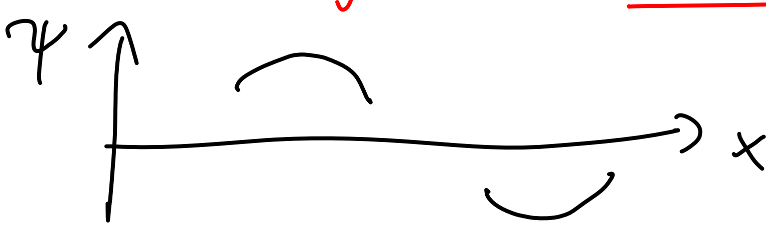
$\Rightarrow \left|\frac{2m}{\hbar^2}(E - V(x))\right| = \frac{\left|\frac{d^2\psi}{dx^2}\right|}{|\psi(x)|}$

a) $E > V(x)$: inside well

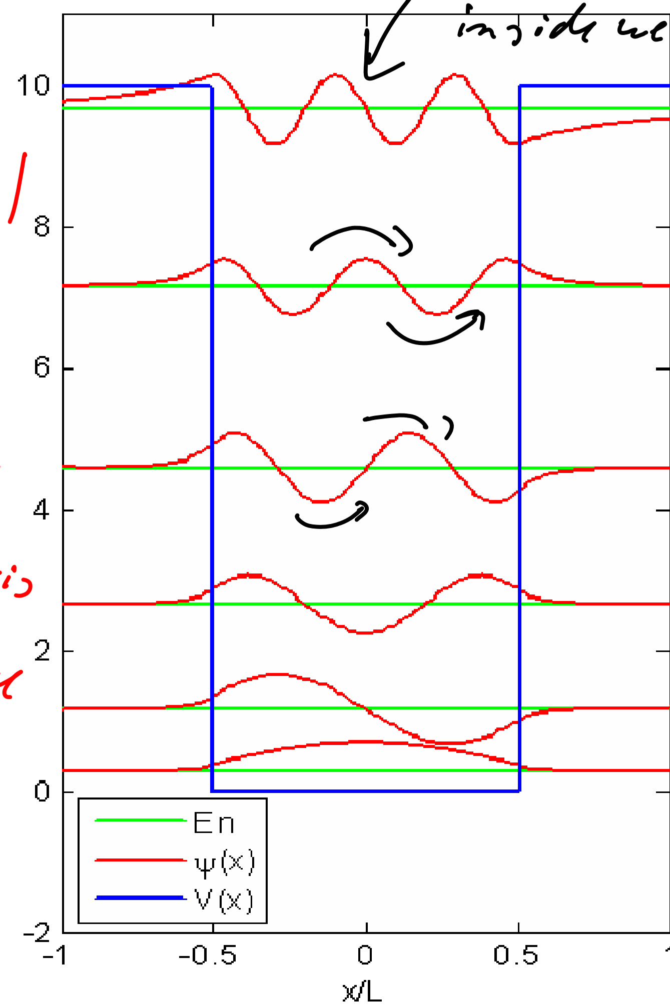
- higher $E-V$ means more |curvature| $\Rightarrow$ shorter "wavelength"
- if $\psi(x) > 0 \Rightarrow$ negative curvature
- if $\psi(x) < 0 \Rightarrow$ positive curvature

$\Rightarrow$ always bends back toward x-axis

$\Rightarrow$ wave function oscillates in space



higher $E_n \Rightarrow$ more curvature inside well



b) if $E < V(x)$: outside well

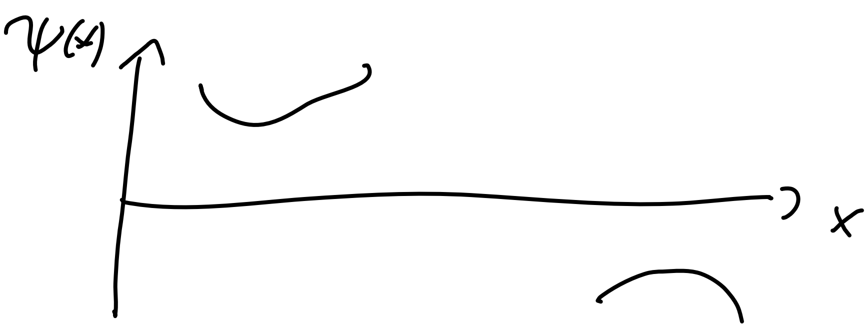
- larger $V-E$ means larger |curvature|

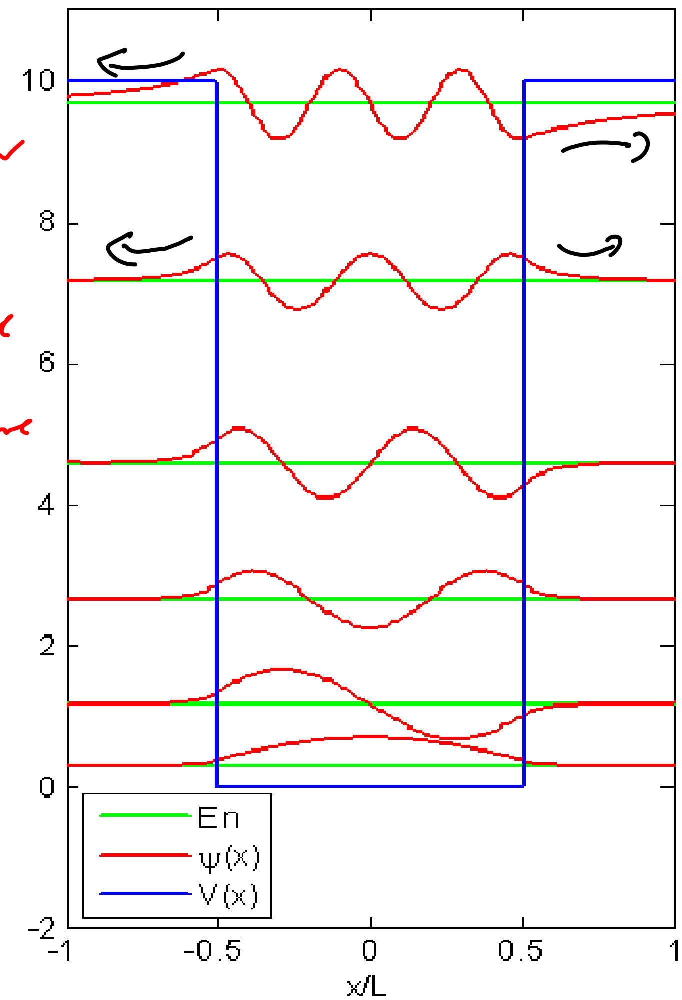
  $\Rightarrow$ higher E states extend further into "forbidden region"

  - if $\psi(x) > 0 \Rightarrow$ positive curvature
  - if $\psi(x) < 0 \Rightarrow$ negative curvature

  $\Rightarrow$ always bends away from axis

  $\Rightarrow$ exponential behavior



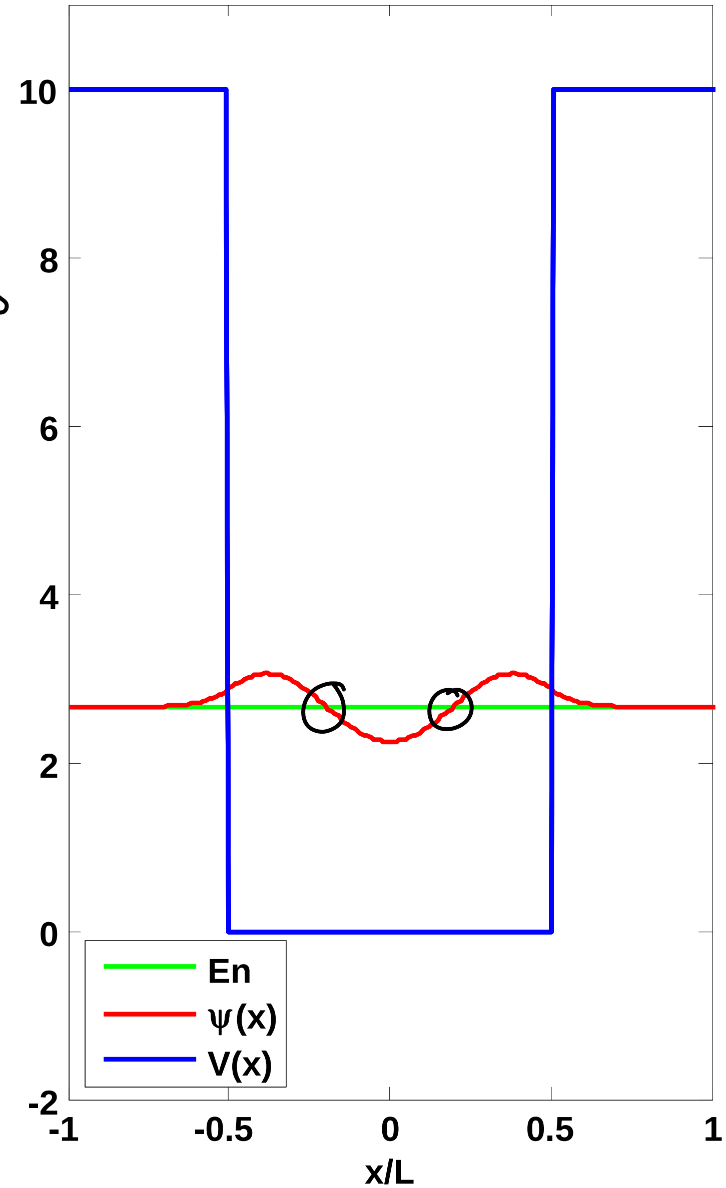


A particle in an finite square well is associated with the stationary state wave function shown below.

What state is it in?

n^{th} energy level	# of nodes
1	0
2	1
3	2
4	3
5	4

- **A.** Ground state
- **B.** 1st excited state
- **C.** 2nd excited state
- **D.** 3rd excited state
- **E.** 4th excited state



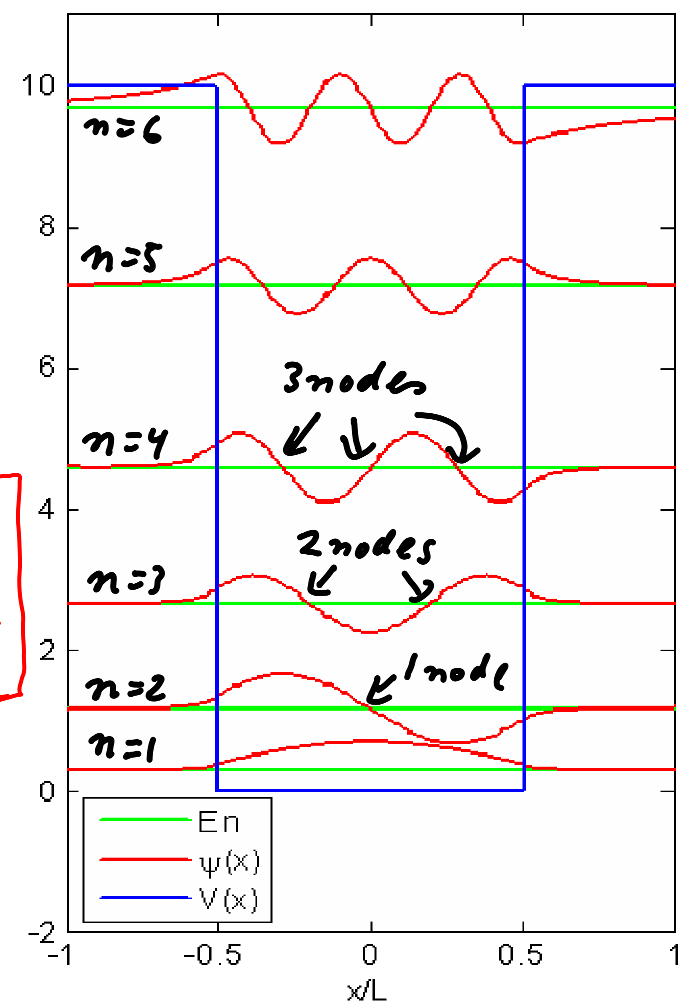
2) # of nodes in stationary state (node = zero crossing)

as energy increases =) shorter wavelength

=) more oscillations

=) one more zero crossing in $\psi(x)$ for each new level

$$\boxed{\Rightarrow \# \text{ of nodes} = n-1 \quad \text{with } n = \text{energy level number}}$$



$n=1$: ground state

$n=2$: 2^{nd} energy state (1^{st} excited state)

$n=3$ 3^{rd} energy state ...

A particle of energy *E* is bound in a stationary state in the potential shown below.

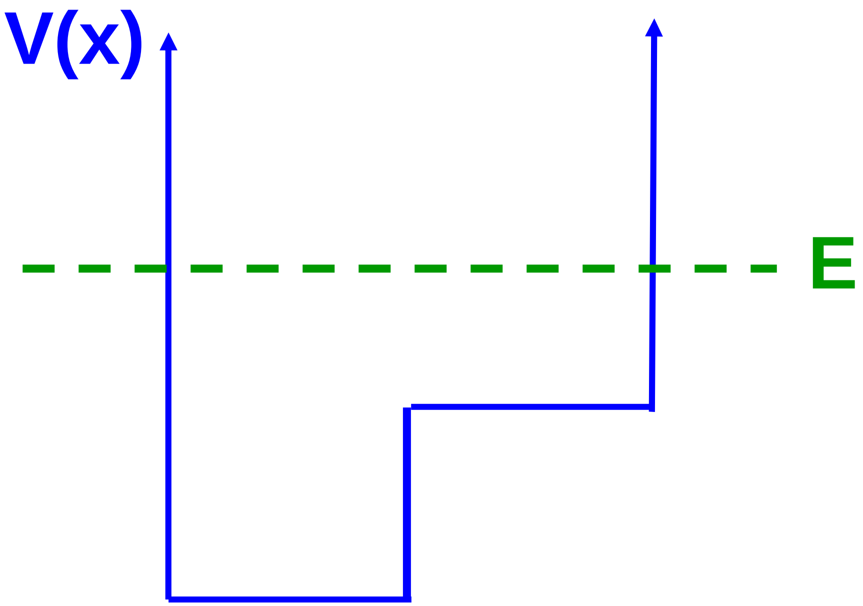
Where is it more likely to find the particle if its position would be measured?

A. Deep part of the well

B. Shallow part of the well

C. Both the same

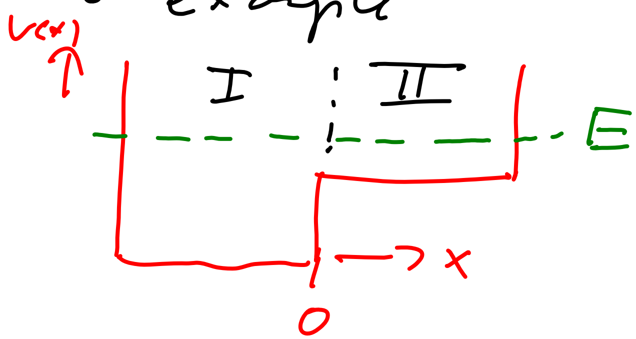
D. No idea



3) Amplitudes of wavefunction for $E > V(x)$

$\Rightarrow$ inside of well

- example



i) classical: when $E - V = KE$ small

$\Rightarrow$ small velocity

$\Rightarrow$ probability of finding classical particle $P(x) \propto \frac{1}{\text{velocity}}$

large if velocity small

ii) by correspondence principle

expect: $P(x) = |\psi(x)|^2$ to be large when $E - V$ is small!

iii) Proof:

- region I: $\psi(x) = A_1 \sin(k_1 x + \delta_1)$ $\quad k_i = \frac{\sqrt{2m(E-V_i)}}{\hbar}$

region II: $\psi(x) = A_2 \sin(k_2 x + \delta_2)$

- at boundary $(x=0)$: $\psi(x)$, $d\psi/dx$ continuous!

$\Rightarrow \psi(0) = A_i \sin(\delta_i)$

$$\Rightarrow \gamma_{\text{slope}} = d\psi/dx\big|_{x=0} = A_i k_i \cos(k_i x + \delta_i)\big|_{x=0}$$
$$= A_i k_i \cos(\delta_i)$$

$\Rightarrow$ (1) $\psi(0) = A_1 \sin(\delta_1) = A_2 \sin(\delta_2)$

(2) $\dfrac{\gamma}{k_i} = A_i \cos(\delta_i) \qquad i = 1, 2$

$\Rightarrow$ square and add:

$$(\psi(0))^2 + \left(\frac{\gamma}{k_i}\right)^2 = A_i^2$$

$$\Rightarrow \underline{\underline{A_i}} = \sqrt{(\psi(0))^2 + \left(\frac{\gamma}{k_i}\right)^2} = \sqrt{(\psi(0))^2 + \frac{\gamma^2 \hbar^2}{2m(E - V_i)}}$$

$\Rightarrow$ large $E - V$ gives small oscillation amplitude A } for

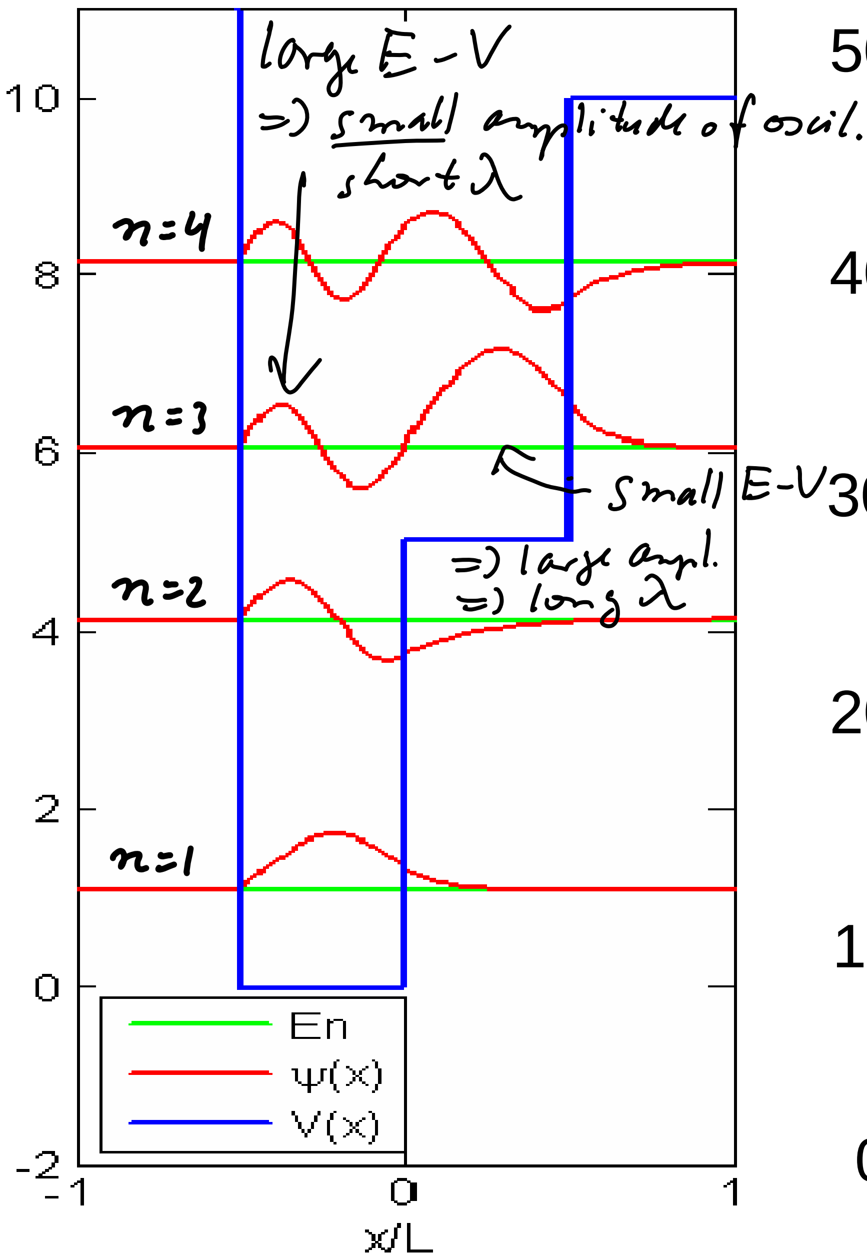
$\Rightarrow$ small $E - V$ gives large " " A } $\gamma \neq 0$!

Note: – A is small in deep part of well!

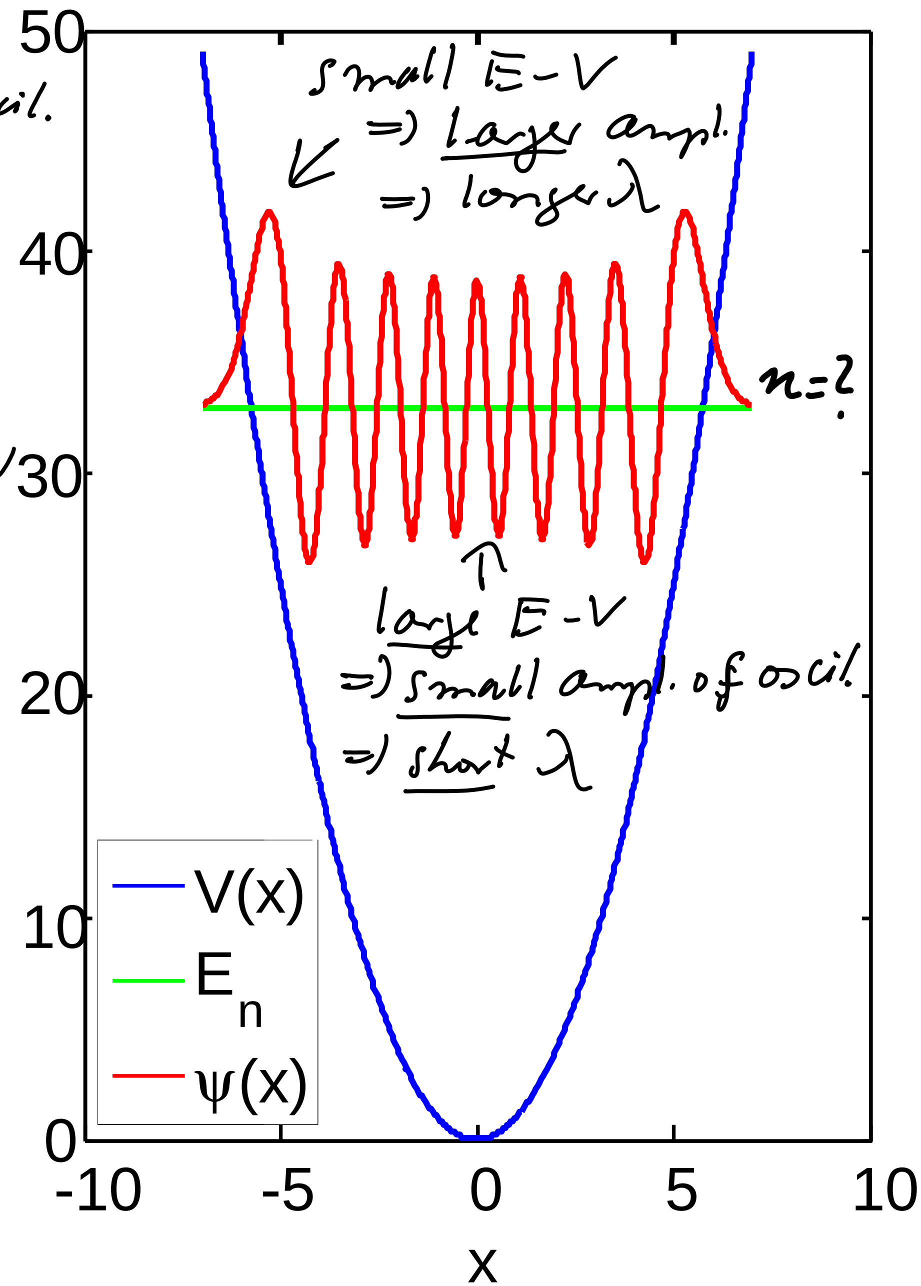
– also true for continuously varying potentials

$\Rightarrow$ small $E - V$: large amplitudes, longer "local wavelengths" (no longer sinusoidal!)

Examples: Step - potential



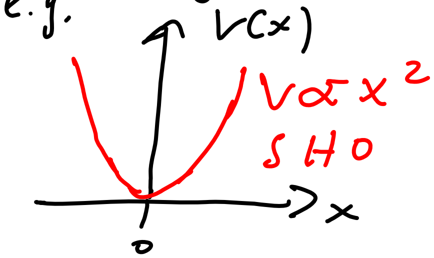
Continuously varying potential

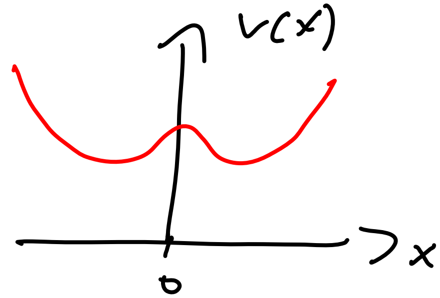


4) Symmetry:

symmetric potentials (about origin $x=0$)

e.g.





$\Rightarrow$ Probability of finding particle in stationary state:

$$P(x) = P(-x)$$

$$\Rightarrow |\psi(x)|^2 = |\psi(-x)|^2$$

$$\Rightarrow \psi(x) = \pm \psi(-x) \Rightarrow \text{2 types:}$$

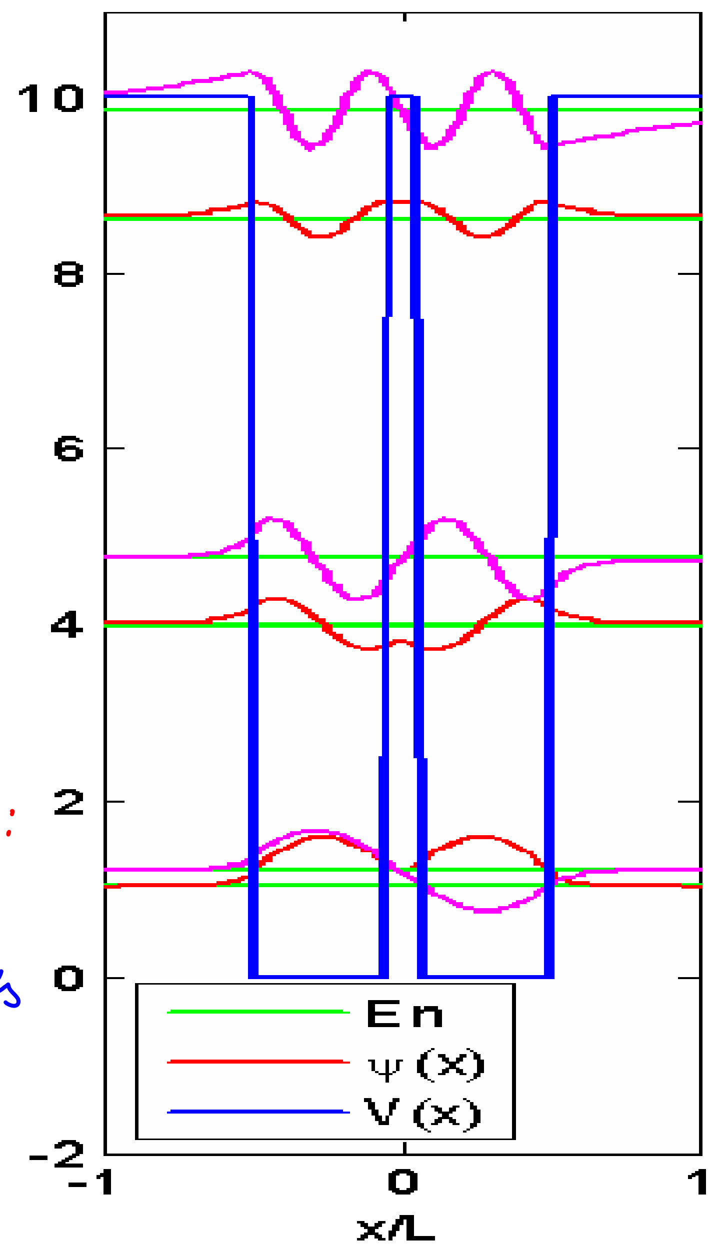
- even functions: $\psi(x) = \psi(-x)$

$\Rightarrow$ $d\psi/dx = 0$ at $x=0$ $\Rightarrow$ even # of nodes

- odd functions: $\psi(x) = -\psi(-x)$

$\Rightarrow$ $\psi(0) = 0$; $d\psi/dx|_{x=0} \neq 0$

$\rightarrow$ odd # of nodes



- **Summary: Guidelines for qualitative plots of $\psi(x)$**

1) The amount of curvature (rate of oscillation in space or decay) increases with increasing $|V-E|$

   1a) $\psi(x)$ oscillates when $E > V$ (curvature towards x-axis)

   1b) $\psi(x)$ has curvature away from x-axis, when $E < V$

2) The n^{th} energy level has $n-1$ zero crossings

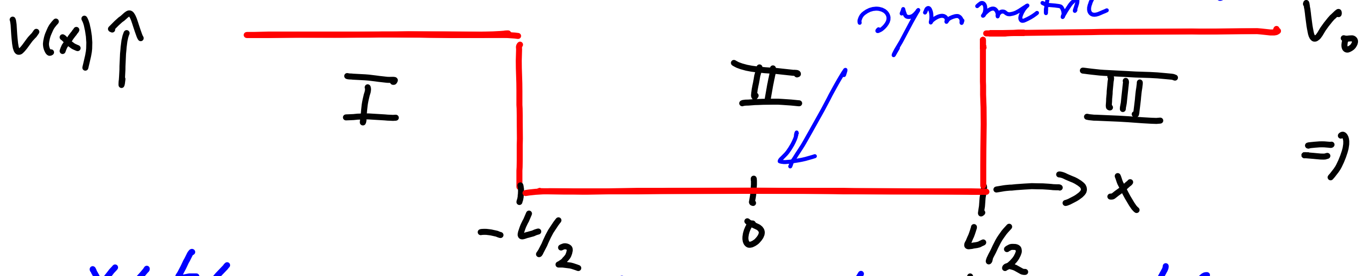
3) When $E > V$, larger $E-V$ gives smaller wave amplitudes

4) If the potential $V(x)$ is symmetric, then $\psi(x)$ is either symmetric or antisymmetric (alternating)

5) $\psi(x) \xrightarrow[x \to \pm\infty]{} 0$ for allowed bound states

$II_{2,4}$ Square Well of Finite Depth, part II :

symmetric well ⇒ place $x=0$ at centre of well



⇒ solve SE in 3 sections

$x < -L/2$

$$\psi(x) = C\, e^{\alpha x}$$

$$\alpha = \frac{\sqrt{2m(V_0-E)}}{\hbar}$$

$$\psi(x) \xrightarrow[x \to -\infty]{} 0$$

$-L/2 \le x \le L/2$

$$\psi(x) = A \sin(kx) + B\cos(kx)$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

spatial oscillating wave

$x > L/2$

$$\psi(x) = D\, e^{-\alpha x} \leftarrow \text{decaying exponential wave}$$

$$\alpha = \frac{\sqrt{2m(V_0-E)}}{\hbar}$$

$$\psi(x) \xrightarrow[x \to \infty]{} 0$$

symmetric well:

⇒ even solutions

$$\psi(x) = D\, e^{\alpha x} \qquad \psi = B\cos(kx) \qquad \psi(x) = D\, e^{-\alpha x} \; : \; \psi(x) = \psi(-x)$$

⇒ odd solutions:

$$\psi(x) = -D\, e^{\alpha x} \qquad \psi = A\sin(kx) \qquad \psi(x) = D\, e^{-\alpha x} \; : \; \psi(x) = -\psi(-x)$$

⇒ find $A\,(B)$, D, energy E ($\Rightarrow k, \alpha$) } from boundary conditions and normalization

- **Even functions/states:**

- inside well: $-\frac{L}{2} \le x \le \frac{L}{2}$: $\psi(x) = B\cos(kx)$

$$\frac{d\psi}{dx} = -Bk\sin(kx)$$

- outside well: $x > \frac{L}{2}$: $\psi(x) = De^{-\alpha x}$ $\quad \frac{d\psi}{dx} = -\alpha De^{-\alpha x}$

- continuity of $\frac{d\psi}{dx}$ at $x = L/2$: $Bk\sin(k\frac{L}{2}) = \alpha De^{-\alpha L/2}$

- continuity of $\psi(x)$ at $x = L/2$: $B\cos(k\frac{L}{2}) = De^{-\alpha L/2}$

$\Rightarrow$ divide these equ. $\quad \Rightarrow k\tan\left(\frac{kL}{2}\right) = \alpha$

$$\Rightarrow \tan\left(\frac{kL}{2}\right) = \frac{\alpha}{k}$$

$\Rightarrow$ insert equ. for α, k

$$\tan\left(\frac{L}{2\hbar}\sqrt{2mE}\right) = \frac{\sqrt{2m(V_0 - E)}}{\sqrt{2mE}} = \sqrt{\frac{V_0 - E}{E}} = \sqrt{\frac{V_0}{E} - 1}$$