• Formalism III
  - Eigenfunctions of a hermitian operator
  - Statistical interpretation
  - Dirac notation
III₃ Operators and Observables:

- Observables are represented by hermitian operators.

\[ \langle Q \rangle = \langle \hat{Q} \rangle = \langle \hat{Q} f | f \rangle \]

Also \( \langle f | \hat{Q} g \rangle = \langle \hat{Q} f | g \rangle \) for all \( f(x), g(x) \)

Recap

III₄ Eigenfunctions of a hermitian operator:

- Determinate states are eigenfunctions of the hermitian operator \( \hat{Q} \).

\[ \hat{Q} \psi = \lambda \psi \]  \text{ eigenvalue equation }

For discrete spectra:
- Eigenvalues \( \lambda \) are real
- Eigenfunctions are orthogonal:
  \[ \langle f_n | f_m \rangle = \delta_{nm} \]
- Eigenfunctions are complete

\( \Rightarrow \) can expand any wave function in terms of the (basis) functions:
• For degenerated spectrum \( \mathcal{Q}' = \mathcal{Q} \)
  
  =) Can construct orthonormal eigenfunctions within each degenerated subspace
  
  =) Eigenfunctions can be chosen to be orthogonal

C) • Axiom: The set of eigenfunctions of an observable operator \( \hat{Q} \) is complete.

  =) Any function in Hilbert space can be expressed as a linear combination of the eigenfunctions!
Case II  continuous spectra:

1. Eigenfunctions cannot be normalized.
2. But still: the three essential properties (reality, orthogonality, and completeness) hold.

\[ \Sigma \rightarrow \int dx \; \delta_{nm} \rightarrow \delta(x) \]

Kronecker delta \( \leftrightarrow \) Dirac delta function.

Example: Position operator \( \hat{x} = x \)

Eigenvalue equation:

\[ \hat{x} g_x(x) = \lambda g_x(x) \]

Solutions: Eigenfunctions:

\[ g_x(x) = A \delta(x-\gamma) \]

Dirac delta functions, zero except for one point \( x = \gamma \).

These functions are not square-integrable.

Do not represent a physical particle.

Cannot be normalized.

But still: Eigenvalues are real (property 3).
The Dirac delta function:

- **Kronecker delta:** \( \delta_{ik} = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases} \Rightarrow \delta_k = \sum_i f_i \delta_{ik} \)

- Similar for continuous variable:
  \[
  f(y) = \int_{-\infty}^{\infty} f(x) \delta(x-y) \, dx
  \]
  with the Dirac delta function:
  \[
  \delta(x-y) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{(x-y)^2 + \epsilon^2}
  \]

- Some useful equations:
  \[
  \delta(ax) = \frac{1}{|a|} \delta(x)
  \]
  \[
  \int_{-\infty}^{\infty} \delta(x) \, dx = 1
  \]
  \[
  \int_{-\infty}^{\infty} \delta(x-y) \delta(y-x') \, dy = \delta(x-x')
  \]
  \[
  \delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} \, dk
  \]
\[ \langle g_y, (x) | g_y, (x) \rangle = \int g_y^*(x) g_y(x) \, dx \]

\[ = l A l^2 \int \delta(x - y') \delta(x - y) \, dx = l A l^2 \delta(y - y') \]

\[ = ) \text{ if we pick } A = 1 \]

\[ = ) \quad g_y(x) = \delta(x - y) \quad \text{\{eigenfunction with real eigenvalue\}} \]

\[ = ) \quad \langle g_y | g_y \rangle = \delta(y - y') \quad \text{\{"Dirac orthonormal\}} \]

\[ \text{Dirac delta function} \]
Also: Set of eigenfunctions of $\hat{X}$ is complete:

\[ f(x) = \int_c^\infty \langle \gamma | \psi(x) \rangle \psi(y) \, dy = \int_c^\infty \langle \gamma | \delta(x - \gamma) \rangle \, dy = C(x) \]

with $C(y) = f(y) = \langle \gamma \psi | f(x) \rangle$:

\[ = \int \psi(\gamma, t) = \langle \gamma | \psi \rangle \psi \]

wave function projection amplitude into position space
III. Generalized Statistical Interpretation:

Found that set of eigenfunctions \( \{ f_n \} \) of an operator representing an observable \( \hat{Q} \) is orthonormal and complete.

\[ \Rightarrow \] can expand any wave function in terms of these base functions: Note: different operator \( \hat{Q}' \).

\[ \Psi(x,t) = \sum_{n} C_n(t) f_n(x) \] (discrete spectrum)

\[ = \int C(f,q,t) f_q(x) \, dq \] (continuous spectrum)

with: \( C_n(t) = \langle f_n | \Psi(x,t) \rangle = \) projection amplitude; quantum amp.

Note: coefficients include the time dependence.

\( C_n \) tells you “how much \( f_n \) is contained in \( \Psi(x,t) \)"
Statistical Interpretation:

If one measures an observable \( \hat{Q} \) on a particle in the state \( \Psi(x,t) \), one is certain to get one of the eigenvalues of the corresponding hermitian operator \( \hat{Q} \).

If: I: spectrum of \( \hat{Q} \) is discrete:

- Probability of getting the eigenvalue \( q_n \) associated with the eigenfunction \( f_n(x) \) is \( |C_n|^2 \), where \( C_n = \langle f_n | \Psi \rangle \).

If: II: continuous spectrum:

- Eigenvalues \( q \), eigenfunctions \( f_q(x) \).
- Probability of getting a result in the interval \( [q, q + dq] \) is \( |C(q)|^2 dq \), where \( C(q) = \langle f_q | \Psi \rangle \).
Note: Upon measurement, the wave function "collapses" to the corresponding eigenstate!

- Total probability of getting a result $b = 1$
  \[ \sum_n |C_n|^2 = 1 \]

Proof:
\[
1 = \langle \Psi | \Psi \rangle = \sum_n \langle f_n | \sum_n C_n^* C_n | f_n \rangle = \sum_n |C_n|^2
\]

- Expectation value:
  recall: for energy $\langle H \rangle = \langle E \rangle = \sum_n E_n |C_n|^2$
  \[ \langle Q \rangle = \sum_n q_n |C_n|^2 \] possible outcomes of measurement

Example: for position:
\[
\langle x \rangle = \int_{-\infty}^{\infty} x |C(y)|^2 dy \quad \text{with} \quad C(y) = \langle y \psi | \Psi \rangle
\]
\[
|C(y)|^2 = |\psi(y, t)|^2 \quad \text{prob of measuring value $y$ as position}
\]
\[
= \int_{-\infty}^{\infty} \delta(x-y) \psi(x, t) dx = \psi(y, t)
\]
**III\textsubscript{6} Dirac Notation:**

\[ \text{Dirac: chop bracket notation for inner product in two pieces:} \]

- **bra:** \( \langle \alpha \rangle \)
- **ket:** \( | \beta \rangle \)

\( | \beta \rangle \) represents the state of the system/particle

- in position space: \( | \beta \rangle \) is represented by a function \( \psi (x,t) \)
- in vector space: state \( | \beta \rangle \) is represented by a state vector: \( | \beta \rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \)

"all that can be known"