Lecture 32: **04/10/09**

- Energy-time uncertainty principle
- Particle scattering I

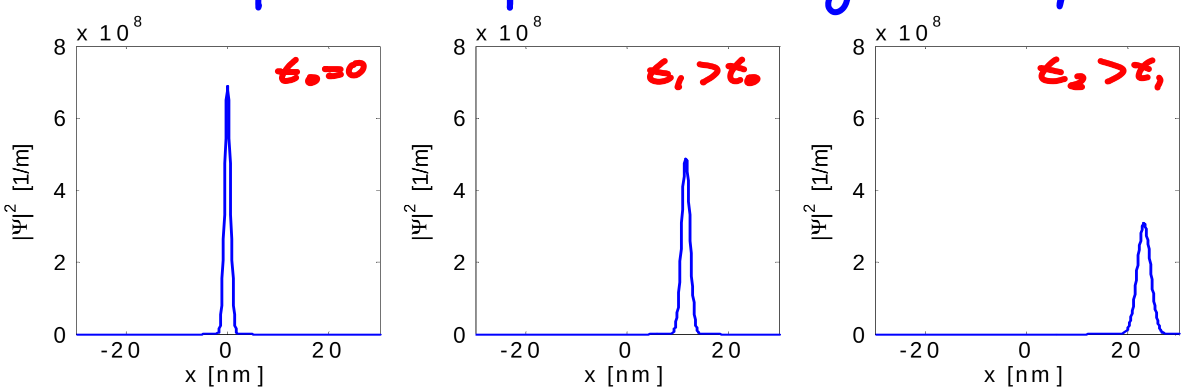
Recap:

IV$_5$ The Free Particle II: Particle Dynamics:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Phi(p)\, e^{ipx/\hbar}\, \underbrace{e^{-i\frac{p^2}{2m}\frac{t}{\hbar}}}_{\text{time dependence of state of definite momentum}} dp$$

→ if $\Phi(p)$ is centered around some value p_0: wave packet moves with v_{group}

→ width of wave packet changes: dispersion



IV$_6$ Energy-Time Uncertainty Principle

$$\Delta t \, \Delta E \geq \frac{\hbar}{2}$$

$$\boxed{\Rightarrow \frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle \underbrace{[\hat{H},\hat{Q}]}_{\text{commutator of } \hat{Q} \text{ and } \hat{H}}\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle}$$

→ typically, $\hat{Q}$ is <u>not</u> explicitly time dependent

$$\Rightarrow \left\langle \left(\frac{\partial \hat{Q}}{\partial t}\right)\right\rangle = 0$$

⇒ for this case, if $\hat{Q}$ and $\hat{H}$ <u>do</u> commute, then $\langle Q\rangle$ is constant

⇒ Q is a conserved quantity!

$$\boxed{\frac{d}{dt}\langle Q\rangle = 0 \quad \text{if} \quad [\hat{H},\hat{Q}] = 0}$$

→ assume: $A = \hat{H}$, and $\vec{B} = \hat{Q}$ in the generalized uncertainty principle:

$$\Rightarrow \sigma_H^2 \sigma_Q^2 \geq \left(\frac{1}{2i}\langle[\hat{H},\hat{Q}]\rangle\right)^2$$

$$= \left(\frac{1}{2i}\frac{\hbar}{i}\frac{d\langle Q\rangle}{dt}\right)^2 = \left(\frac{\hbar}{2}\right)^2\left(\frac{d\langle\hat{Q}\rangle}{dt}\right)^2$$

for $\left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle = 0$

$$\Rightarrow \sigma_H \sigma_Q \geq \frac{\hbar}{2}\left|\frac{d\langle Q\rangle}{dt}\right|$$

define:

$$\boxed{\begin{aligned} \Delta E &\equiv \sigma_H \\ \Delta t &\equiv \sigma_Q \Big/ \left|\frac{d\langle Q\rangle}{dt}\right| \end{aligned} \quad \Rightarrow \Delta E\,\Delta t \geq \frac{\hbar}{2}}$$

energy-time uncertainty principle!

$$\sigma_Q = \underbrace{\left|\frac{d\langle Q\rangle}{dt}\right|}_{\text{rate of change}}\Delta t$$

change = rate of change . time

Δt = { amount of time it takes the expectation value of Q to change by one standard deviation σ_Q }

Note:

$\Delta E \Delta t \geq \frac{\hbar}{2}$ does not mean that energy is not conserved in QM!

- Δt depends entirely on the choice of Q!
- if ΔE is small, then the rate of change of all observables must be very gradual
- if one observable changes rapidly, the uncertainty in energy must be large
- $\Delta t \approx$ time it takes the system to undergo "substantial change"

• examples for Δt:

- periode of oscillation of some observable
- time it takes a wavepacket of a traveling particle to move by σ_x
- lifetime of an unstable particle
- decay time of excited states

  $\Rightarrow$ line width of spectral lines of excited atoms (see F&T, 8.11)

V Particle Scattering and Barrier Penetration (tunneling)

V$_1$ Introduction: Generic Problem

moving wave packet → encounters jump in potential energy ⟺ conservative forces

→ momentum of particle changes

⇒ what is $\Psi(x,t) = ?$

$|\Psi(x,t)|^2 = ?$

→ <u>classically:</u> "ball" keeps going in same direction → "transmission"

or: is reflected → "reflection"

but: for given $V(x)$ and E_{ball}, always the same happens, if experiment is repeated

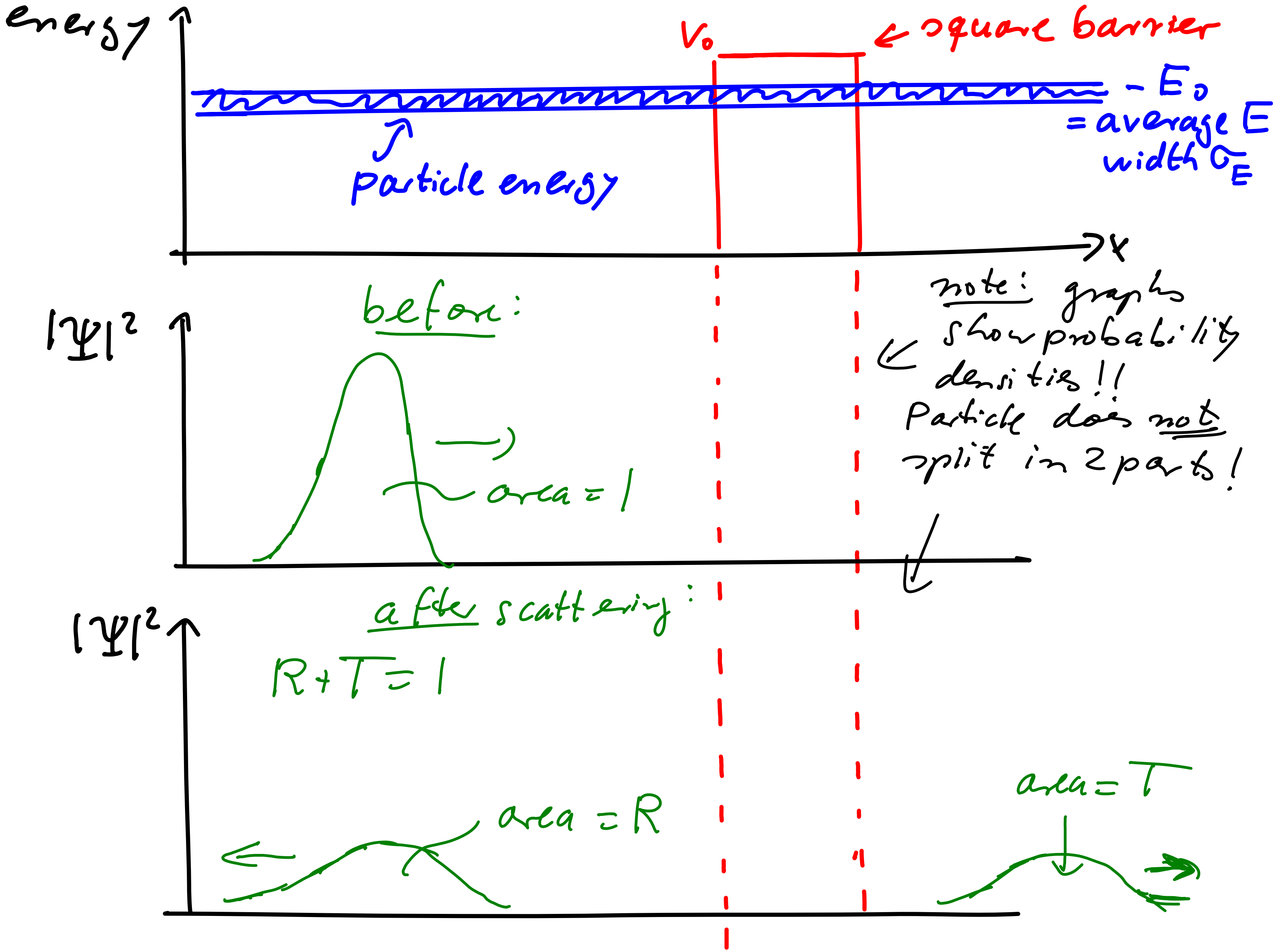
→ <u>in quantum mechanics:</u>

for single particle, given $E_{particle}$ and $V(x)$ <u>either</u> one can happen!

⇒ probability of reflection $= R$

⇒ probability of transmission $= T$

$$\boxed{R + T = 1}$$


energy
V_0
square barrier
E_0 = average E
width σ_E
particle energy
x
$|\Psi|^2$
before:
area = 1
note: graphs show probability densities!!
Particle does not split in 2 parts!
after scattering:
$|\Psi|^2$
R+T=1
area = R
area = T

· How to calculate $\Psi(x,t)$? (general solution of the time dep. S.E. for given $V(x)$)

Key idea: (as before)

① solve time-indep. S.E. first → stationary states

② then add up the stationary solutions to make time-dependent general solution

⇒ general solution:

$$\Psi(x,t) = \sum_n c_n \underbrace{\psi_n(x)\, e^{-i E_n/\hbar \cdot t}}_{\text{stationary states of definite energy}}$$

projection amplitudes (c_n)

$$c_n = \langle \psi_n | \Psi(x, t=0) \rangle = \int_{-\infty}^{+\infty} \psi_n^*(x)\, \Psi(x, t=0)\, dx$$

stationary states of definite energy ⇒ solutions of eigenvalue equation

$$\hat{H}\, \psi_n(x) = E_n\, \psi_n(x)$$

(time indep. S.E.)

$\Rightarrow$ find basis states $\psi_n(x) \rightarrow \Psi(x,t)$

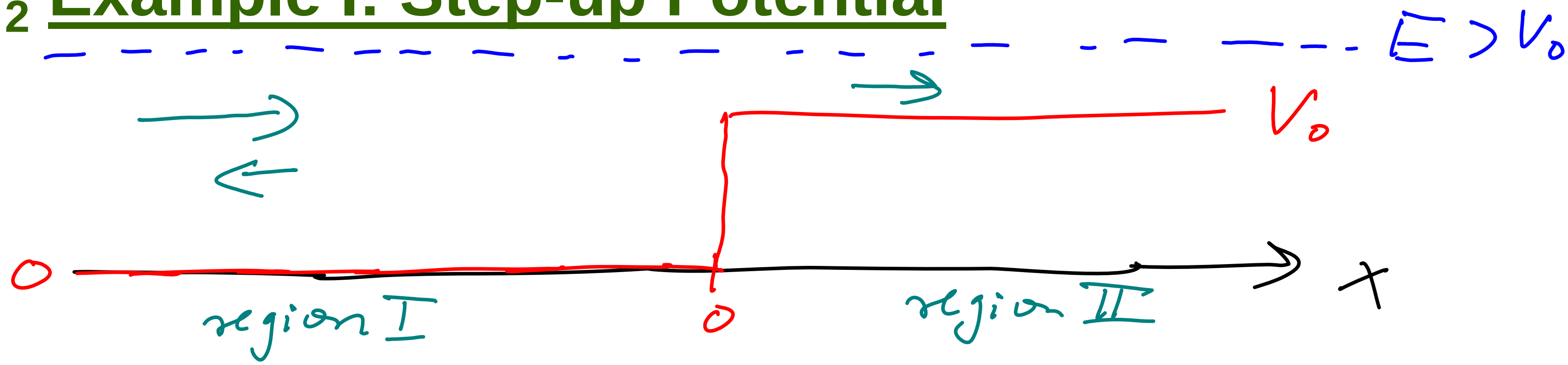
$\Rightarrow$ main challenge is to find ψ_n

Note:

- $\psi_n(x)$ will not be normalizable here (only normalizable for bound states!) but superposition of them can be normalizable (wave packet)
- $\{\psi_n(x)\}$ are Dirac-orthonormal:

$$\langle \psi_n | \psi_m \rangle = \delta(E_m - E_n)$$

V_2 Example I: Step-up Potential



- assume particle energy $E_0 > V_0$ for now
- find stationary states with $E_0 > V_0$

$\Rightarrow$ solve the time-indep. S.E. for regions with $V(x) = \text{const}$

$\Rightarrow$ join at boundary $x=0$; make $\psi(x)$ and $\frac{d\,\psi(x)}{dx}$ continuous!

→ **region I** : $x < 0$, $V(x) = 0$

⇒ time indep. S.E. $\quad -\frac{\hbar^2}{2m}\frac{d^2\psi_\omega(x)}{dx^2} = E\,\psi_\omega(x)$

solution for $E = \hbar\omega$

⇒ solution:

$$\psi_\omega(x) = \underbrace{A_0\, e^{ik_1 x}}_{\text{corresponds to}} + \underbrace{A\, e^{-ik_1 x}}_{\text{corresponds to}} \quad \text{with } k_1 = \frac{\sqrt{2mE}}{\hbar}$$

corresponds to to right going wave

$e^{i(kx-\omega t)}$

"incomming wave"

corresponds to left going wave

$e^{-i(kx+\omega t)}$

"reflected wave"

→ <u>region II</u>: $x > 0$, $V(x) = V_0$

⇒ time-indep. S.E: $-\frac{\hbar^2}{2m}\frac{d^2\psi_w(x)}{dx^2} + V_0\psi_w(x) = E\psi_w(x)$

⇒ Solution:

$$\psi_w(x) = \underbrace{B e^{ik_2 x}}_{\text{"transmitted wave" moves in } +x \text{ direction}} + \underbrace{C e^{-ik_2 x}}_{\text{correspond to left going wave in region II}} \quad \text{with } k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$$

"transmitted wave" moves in +x direction

correspond to left going wave in region II (particle incomming from the right ⇒ $\underline{\underline{C = 0}}$ here)