Lecture 33: 04/15/09

- Particle scattering II
  - Probability current

Recap:

IV$_6$ Energy-Time Uncertainty Principle

$$\Delta t \, \Delta E \geq \frac{\hbar}{2} \quad \text{with} \quad \Delta E \equiv \sigma_E \qquad \Delta t = \frac{\sigma_Q}{\left|\frac{d\langle Q\rangle}{dt}\right|}$$

$$\sigma_Q = \left|\frac{\langle dQ\rangle}{dt}\right| \Delta t$$

change = rate of change . time

Δt = amount of time it takes the expectation value of Q to change by one standard deviation

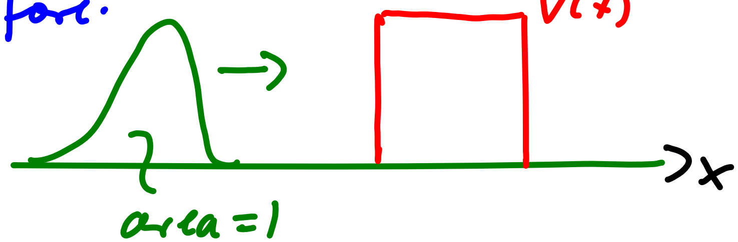
Note:

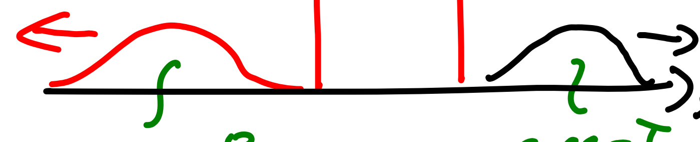
- Δt depends entirely on the choice of Q!
- if ΔE is small, then the rate of change of all observables must be very gradual
- if one observable changes rapidly, the uncertainty in energy must be large

Recap:

V_1 Particle Scattering: Generic Problem

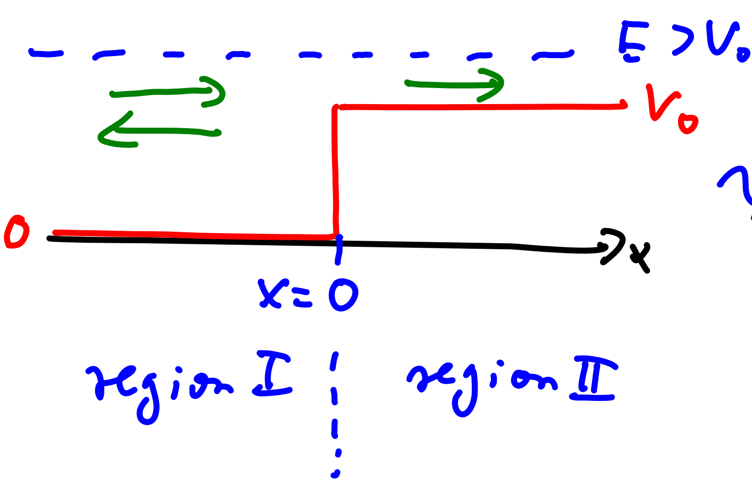
before: $\Longrightarrow$ after





$\underline{R+T=1}$

V_2 Example I: Step-up Potential



$$\psi_\omega(x) = \begin{cases} A_0\left[e^{ik_1x} + \dfrac{k_1-k_2}{k_1+k_2}e^{-ik_1x}\right] & \text{for } x \le 0 \\ A_0\dfrac{2k_1}{k_1+k_2}e^{ik_2x} & \text{for } x > 0 \end{cases}$$

$$k_1 = \frac{\sqrt{2mE}}{\hbar} \qquad k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$$

$\rightarrow$ region I : $x < 0, \quad V(x) = 0$

$$\psi_w(x) = \underbrace{A_0 e^{ik_1 x}}_{\text{"incomming" wave}} + \underbrace{A e^{-ik_1 x}}_{\text{"reflected" wave}} \quad \text{with } k_1 = \frac{\sqrt{2mE}}{\hbar}$$

$\rightarrow$ region II : $x > 0, \quad V(x) = V_0$

$$\psi_w(x) = \underbrace{B e^{ik_2 x}}_{\text{"transmitted" wave}} + \underbrace{C e^{-ik_2 x}}_{\Rightarrow\ \underline{C = 0 \text{ here}}} \quad \text{with } k_2 = \frac{2m(E - V_0)}{\hbar}$$

→ join wavefunctions in regions I and II at $x=0$

⇒ A, B in terms of A_0

- continuity of $\psi(x)$ at $x=0$

$$\Rightarrow \quad A_0 + A = B \qquad (1)$$

- continuity of $\left| \dfrac{d\psi}{dx} \right|_{x=0}$

$$\Rightarrow \quad ik_1 A_0 - ik_1 A = ik_2 B \Rightarrow k_1 A_0 - k_1 A = k_2 B \quad (2)$$

⇒ $(2) - k_2 (1)$: $k_1 A_0 - k_1 A - k_2 A_0 - k_2 A = 0$

$$\Rightarrow \quad (k_1 - k_2) A_0 = (k_1 + k_2) A \Rightarrow \underline{\underline{A}} = \frac{k_1 - k_2}{k_1 + k_2} A_0$$

⇒ $(2) + k_1 (1)$: $k_1 A_0 - k_1 A + k_1 A_0 + k_1 A = k_2 B + k_1 B$

$$\Rightarrow \quad 2k_1 A_0 = (k_1 + k_2) B \Rightarrow \underline{\underline{B}} = \frac{2k_1}{k_1 + k_2} A_0$$

⇒ final solution:

$$\psi_\omega(x) = \begin{cases} A_0 \left[e^{ik_1 x} + \frac{k_1 - k_2}{k_1 + k_2} e^{-ik_1 x} \right] & \text{for } x \le 0 \\ A_0 \frac{2k_1}{k_1 + k_2} e^{ik_2 x} & \text{for } x > 0 \end{cases}$$

Note: - use A_0 to normalize the final wave packet

- $\psi_\omega(x)$ for each $E = \hbar\omega > V_0$

⇒ continuous spectrum (eigenvalue E)

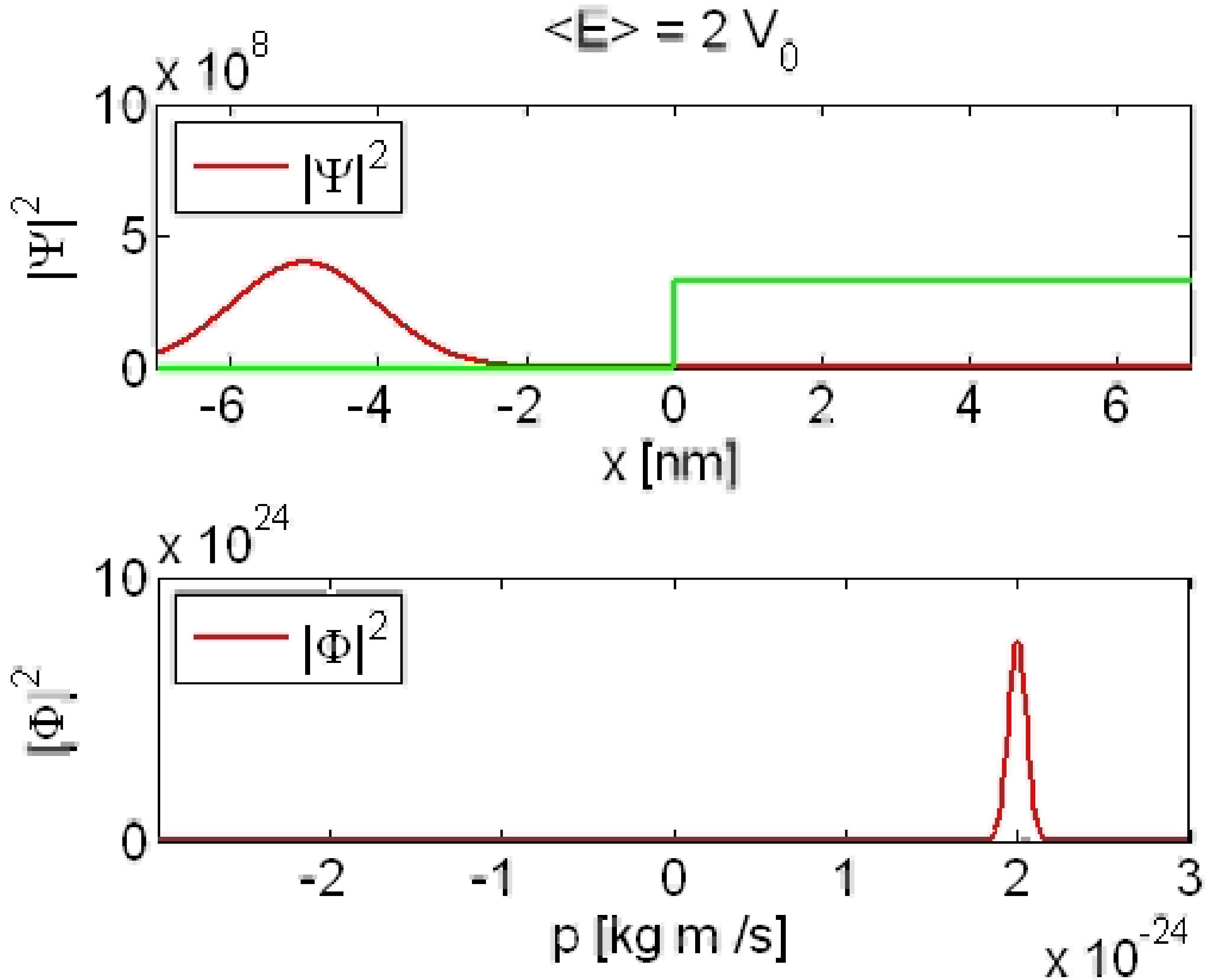
(this is not a bound state!)

⇒ general solution:

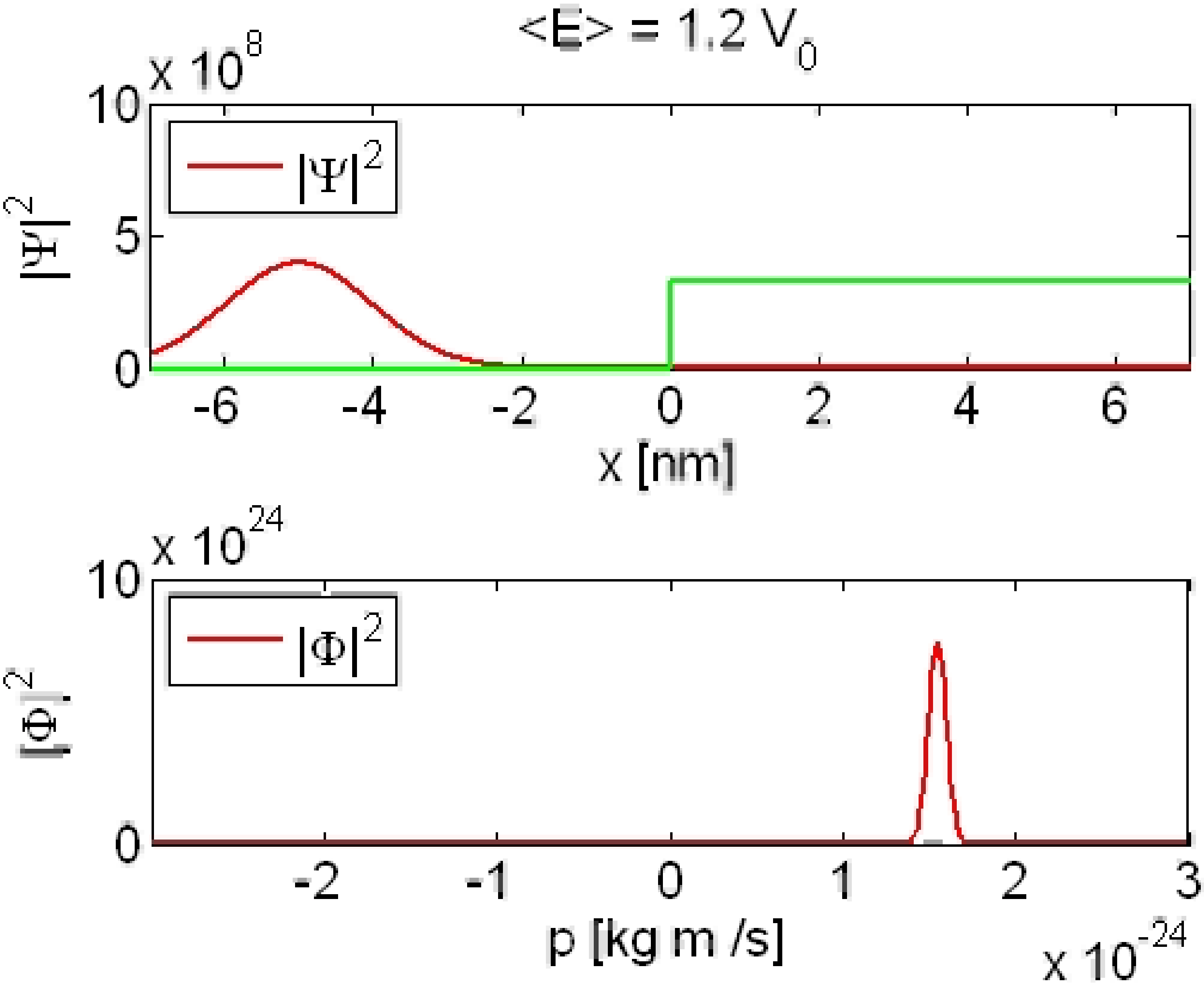
$$\Psi(x,t) = \sum_\omega c_\omega \psi_\omega(x) e^{-i\omega t}$$

- define c_ω to get wave packet at large $-x$ at $t=0$ with positive group velocity ($c_n = \langle \psi_n | \Psi(x, t=0) \rangle$)

- $\Psi(x,t)$ describes scattering from potential step



<E> = 2 V0
x 10^8
10
5
0
|Ψ|^2
-6
-4
-2
0
2
4
6
x [nm]
x 10^24
10
0
|Φ|^2
-2
-1
0
1
2
3
p [kg m /s]
x 10^-24


<E> = 1.2 V0
x 10^8
|Ψ|^2
x [nm]
x 10^24
|Φ|^2
p [kg m /s]
x 10^-24

→ wave packet has some probability ≥ 0 for transmission and reflection (even if $E > V_0$)

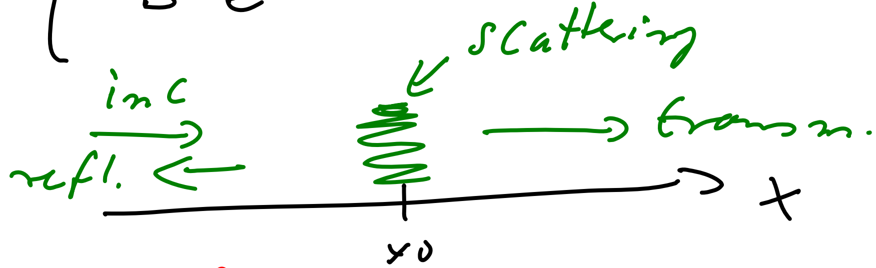
→ feature of QM: abrupt changes of the potential energy give reflection (analog to wave optics at abrupt changes of the index of refraction)

⇒ define reflection and transmission coefficients:

- wave packet: need weighted average over reflection and transmission probabilities of all stationary states in the superposition
- for wave packets with narrow range in energy: reflection and transmission coefficients ≈ const in that range ≈ coefficient for any stationary state in this range: σ_E small ⇒ $T_\omega \approx$ const, $R_W \approx$ const

→ define R,T for stationary state:

$$\text{if } \psi(x) = \begin{cases} A_0 e^{ik_1 x} + A e^{-ik_1 x} & \text{for } x \ll 0 \\ B e^{ik_2 x} & \text{for } x \gg 0 \end{cases}$$



⇒ Probability of reflection:

$$\boxed{R_\omega = \frac{|A|^2}{|A_0|^2} = \text{reflection coefficient}}$$

⇒ for step-up potential:

$$R_\omega = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2$$

Note:
- $R_\omega = 0$ if $k_1 = k_2$ (no step in $V(x)$)
- $R_\omega \to 0$ for $k_1 - k_2 \ll k_1 + k_2$, i.e. $E \gg V_0$

$\Rightarrow$ Probability of transmission:

$$\boxed{T_w = 1 - R_w = \text{transmission coefficient}}$$

$\Rightarrow$ for step-up potential:

$$T_w = 1 - \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 = \frac{4k_1k_2}{(k_1+k_2)^2} = \underbrace{\frac{k_2}{k_1}}_{\text{why this factor!?}} \cdot \frac{|B|^2}{|A_0|^2}$$

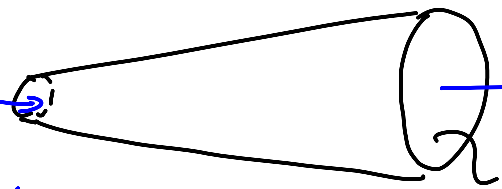
Note:

$$T_w \neq \frac{|B|^2}{|A_0|^2} \quad !$$

V_3 Probability Current / Flow

1) Key idea:

→ consider water pipe:



Flow rate $= F = \dfrac{dV}{dt} = A \cdot \dfrac{dx}{dt} = A \cdot$ velocity ($dV \leftarrow A\,dx$)

→ in QM : same idea: (for a plane wave here...)

probability current $\equiv J = \underbrace{|\Psi|^2}_{\text{probability density}} \cdot \underbrace{V_{group}}_{\text{"how fast it flows"}}$ } for plane waves only!

in 1-D:

$[J] = {}^{1}/_{sec}$

→ for step-up potential:

- prob. current for incident wave $= |A_0|^2 v_{g\,I}$ to the right $= \frac{\hbar k_1}{m}|A_0|^2$
- prob. current for reflected wave $= |A|^2 v_{g\,I}$ to the left $= \frac{\hbar k_1}{m}|A|^2$
- prob. current for transmitted wave $= |B|^2 v_{g\,II}$ to the right $= \frac{\hbar k_2}{m}|B|^2$

⇒ for stationary states / steady states (only!):

! probability current is conserved / constant ! !!

$$\Rightarrow \underbrace{k_1 |A_0|^2}_{\text{in}} = \underbrace{k_1 |A|^2}_{\text{reflected}} + \underbrace{k_2 |B|^2}_{\text{transmitted current} = \text{out}}$$

⇒ for step-up potential:

$$\boxed{R \equiv \frac{J_{reflected}}{J_{in}}} = \frac{k_1 |A|^2}{k_1 |A_0|^2} = \frac{|A|^2}{|A_0|^2} \text{ as before}$$

$$\boxed{T \equiv \frac{J_{transmitted}}{J_{in}}} = \frac{k_2 |B|^2}{\underbrace{k_1}_{} |A_0|^2} = 1 - R$$

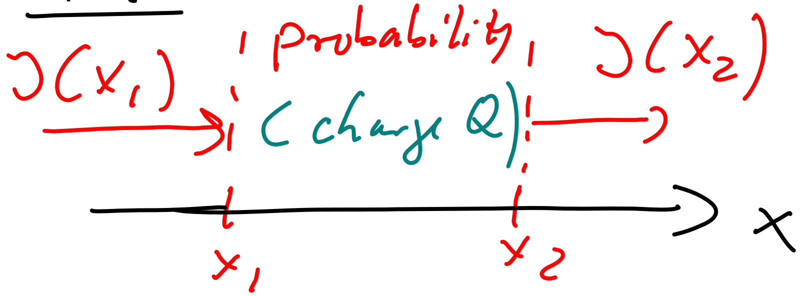
note factor k_2 / k_1 !

2) Rigious treatment: Probability current $J(x,t)$

→ analog to continuity equation in E&M:

recall: $$\frac{d}{dt} \rho = -\vec{\nabla} \cdot \vec{J} = \overbrace{-\frac{d}{dx} J(x)}^{\text{for 1-D}}$$

ρ: charge density; $\vec{J}$: charge current idea:



probability (charge) is conserved ⇒ any change in the total probability (total charge) in the region between x_1 and x_2 is due to current flowing in or out!

→ integral form:

$$\frac{d}{dt}\underbrace{\int_{x_1}^{x_2} \overbrace{|\Psi(x,t)|^2}^{\rho(x)} dx}_{\text{change in prob. in region}} = J(x_1) - J(x_2)$$

← prob. current (charge current)

= prob current in − current flowing out

⇑ integrate from x_1 to x_2

→ differential form:

$$\boxed{\frac{\partial}{\partial t}|\Psi(x,t)|^2 = -\frac{\partial}{\partial x} J(x)}$$