Lecture 8: 02/04/09

- Wave Properties of Particles
  - Standing Waves in Bohr Atom
  - Superposition of Particle Waves and Group and Phase Velocity

Louis de Broglie (1892 – 1987):
Nobel Prize for Physics (1929)

Recap:

$I_{2,2}$ The Bohr Atom (1913)

Quantized Energy, Radii, Angular Momentum ($L = n\hbar$)

+ good prediction for spectra of 1-electron atoms
- arbitrary quantization
- can not be expanded to atoms with more than one e^-

I_3 Particle Waves

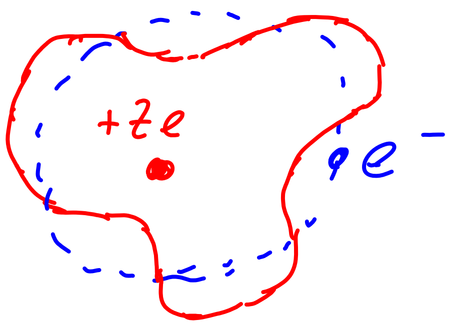
$I_{3,1}$ De Broglie Hypothesis:

Particle: Energy E $\longleftrightarrow$ associated wave
Momentum p

$\lambda = \frac{h}{p}$ $\qquad \nu = \frac{E}{h}$

phase velocity: $v_{phase} = \frac{c^2}{v}$

$I_{3,2}$ Back to the Bohr Atom: More insight with $\lambda = h/p$?



— try following model:
- $\lambda = h/p$ for orbiting electron
- assume that "particle waves" of e^- in orbits are standing waves with integer number of wavelengths around the orbital circumference
- no further assumptions!

$\Rightarrow 2\pi r_n = n\lambda = n\,h/p = n\,h/mv \quad n = 1, 2, 3, \ldots$

$\Rightarrow \dfrac{nh}{2\pi} = m v r_n = L_n$: angular momentum $\Rightarrow L_n = \hbar n$

$\Rightarrow$ get Bohr's quantized angular momentum for e^-

$\Rightarrow$ " " " energy levels " "

(Note: this is still not a rigorous QM picture! but not too far from modern models, at least in spirit)

$I_{3,3}$ Superposition of Particle Waves

– Back to de Broglie's Particle wave:

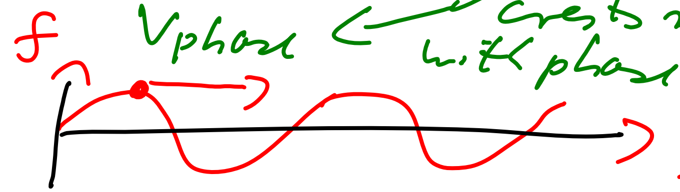
particle $\longrightarrow$ associated wave

$$f(x,t) = A \sin(kx - \omega t) = A \sin\{k(x - v_{phase}\, t)\}$$

move in +x direction

crests move with phase vel.

with
$$\boxed{\begin{aligned} k &= \frac{2\pi}{\lambda} = \frac{2\pi}{h}\, p \\ \omega &= 2\pi\nu = 2\pi\, E/h \end{aligned}}$$



phase velocity: $v_{phase} = \dfrac{\omega}{k} = \dfrac{E}{p} = \dfrac{c^2}{v} \geq c$ ← particle speed

⇒ Explains diffraction, interference experiments with particles!

- Conceptual Problems:
  1) plane wave, extended over all space $\overset{?}{\longleftrightarrow}$ localized particle
  2) particle with speed v $\longleftrightarrow$ wave with phase velocity $v_{phase} = \dfrac{c^2}{v} \geq c$ and not $\underline{v}$

Also: What is the significance of the wave amplitude?

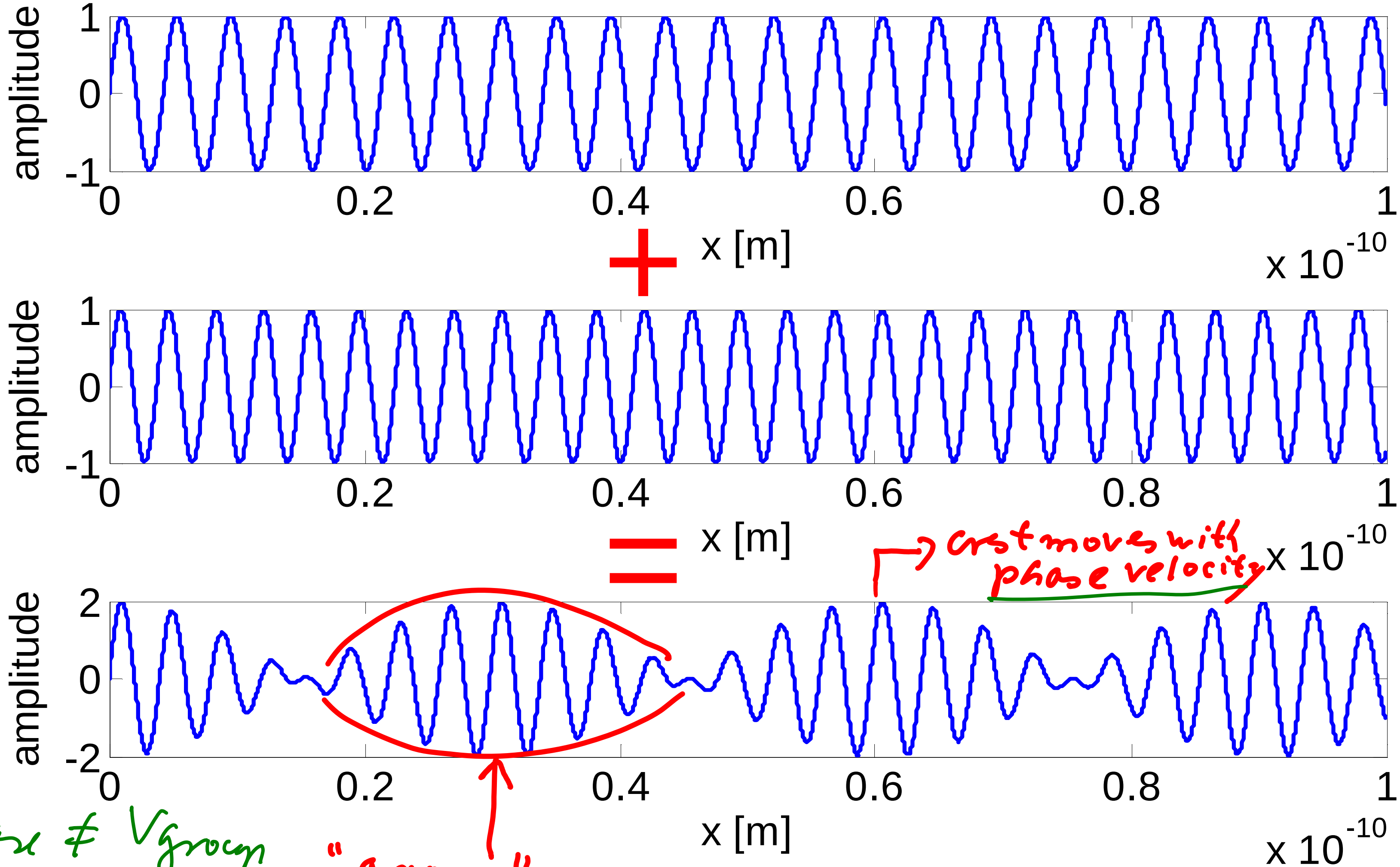
The superposition of two sine waves of slightly different wavelengths results in …

A. A beating pattern

B. Another sine wave

C. Something else

Superposition of two Waves: Beats

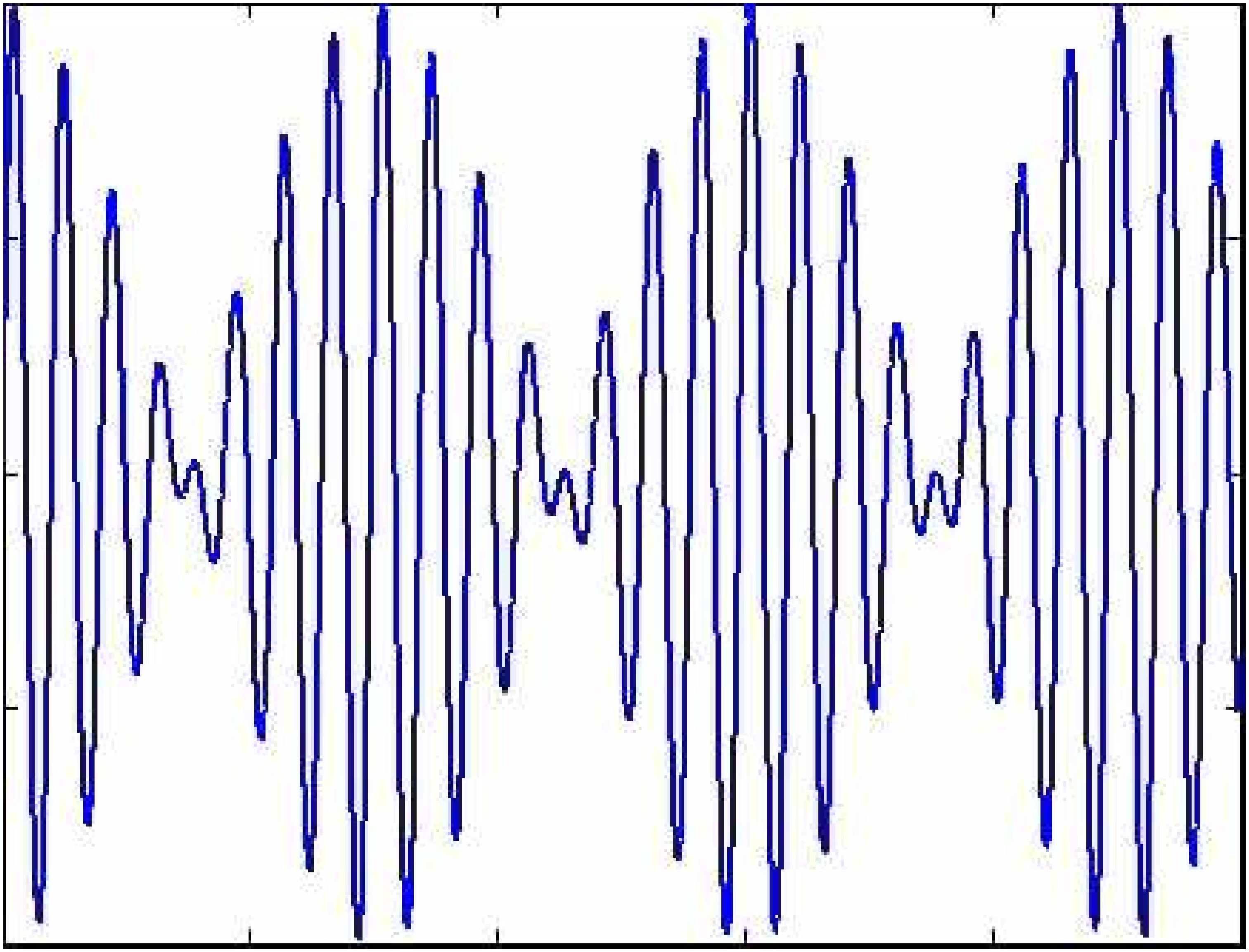


→ crest moves with phase velocity

"group"; moves with group velocity, not phase velocity

$V_{phase} \neq V_{group}$ in general!

Example: Superposition of two Waves



– Superposition of two sine waves (with $k_1 \approx k_2$, $\omega_1 \approx \omega_2$)

$$f_1(x,t) = A \sin(kx - \omega t)$$

$$f_2(x,t) = A \sin\{(k+\Delta k)x - (\omega+\Delta\omega)t\}$$

$\frac{\Delta k}{k} \ll 1$

$\frac{\Delta \omega}{\omega} \ll 1$

$\Rightarrow$ Sum:

$$f(x,t) = f_1 + f_2 = A\left[\sin(kx-\omega t) + \sin\{(k+\Delta k)x - (\omega+\Delta\omega)t\}\right]$$

use: $\sin A + \sin B = 2 \sin\frac{A+B}{2} \cos\frac{A-B}{2}$

$$\Rightarrow f(x,t) = 2A \sin\left\{\left(k+\frac{\Delta k}{2}\right)x - \left(\omega+\frac{\Delta\omega}{2}\right)t\right\} \cdot \cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$

plane wave, crests move with phase velocity

$$V_{phase} = \frac{\omega + \Delta\omega/2}{k + \Delta k/2} \approx \frac{\omega}{k}$$

envelope function (modulates amplitudes)

$$\cos\left(\frac{\Delta k}{2}\left(x - \frac{\Delta\omega}{\Delta k}t\right)\right)$$

moves with group velocity

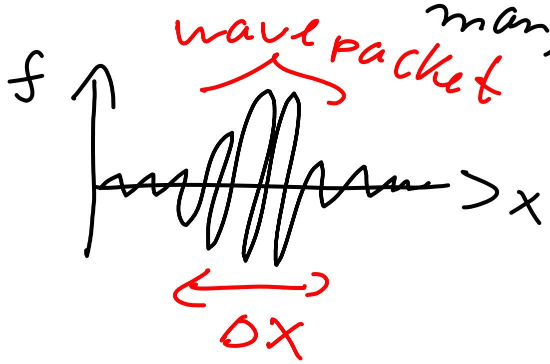
$$V_{group} = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} \neq V_{phase}$$

for light in vacuum (only!)

$V_{phase} = c \Rightarrow \omega = ck \Rightarrow V_{gr} = \frac{d\omega}{dk} = c$

– wave packets: ("localized" waves of localized particles)

• key idea: superposition of many plane waves ⟶ "localized wave"



• describe by sum/integral of plane waves

$$f(x,t) = \mathrm{Re}\left\{ \int \underbrace{\phi(k-k_0)}_{\text{wave amplitude function, centered around } k_0} e^{i(kx-\omega(k)t)}\, dk \right\}$$

Note: 1) Destructive/constructive interference of waves gives localized wave

2) small Δx ⟶ need large range of k (Δk)

$\Rightarrow$ large $\Delta\lambda$ ⟶ large Δp $\left(k = \frac{2\pi}{\lambda} = \frac{2\pi}{h} p = \frac{p}{\hbar} \right)$

$$\frac{1}{\Delta x} \sim \Delta k \sim \Delta p \Rightarrow \Delta p \cdot \Delta x \approx \text{const}!$$

$\Rightarrow$ Heisenberg Position-Momentum uncertainty Relation: $\sigma_x \cdot \sigma_{p_x} \geq \hbar/2$

Conclusion: Localized particle has uncertainty/range of momentum p!

- wave packets:

$$f(x,t) = \mathrm{Re}\left\{ \int \underbrace{\phi(k-k_0)} e^{i(kx-\omega(k)t)} dk \right.$$

• assume that ϕ is nonzero only over some small range k around some k_0

$\Rightarrow$ Taylor expand $\omega(k)$ about k_0:

$$\omega(k) = \omega(k_0) + \left.\frac{d\omega}{dk}\right|_{k_0} (k-k_0) + \text{ignore higher order terms (short times)}$$

$$\Rightarrow f(x,t) = \mathrm{Re}\left\{ \int \phi(k-k_0) e^{i[(k-k_0)x + k_0 x - \omega(k_0)t - \left.\frac{d\omega}{dk}\right|_{k_0}(k-k_0)t]} dk \right\}$$

$$= \mathrm{Re}\left\{ e^{i[k_0 x - \omega(k_0)t]} \int \phi(k-k_0) e^{i[(k-k_0)x - \left.\frac{d\omega}{dk}\right|_{k_0}(k-k_0)t]} dk \right\}$$

change variables: $s = k - k_0$

$$f(x,t) = \mathrm{Re}\left\{ \underbrace{e^{i[k_0 x - \omega(k_0)t]}}_{\text{infinite plane wave}} \underbrace{\int \phi(s)\, e^{is\left(x - \frac{d\omega}{dk}\big|_{k_0} t\right)} ds}_{\text{envelope function}} \right\}$$

infinite plane wave

crests move at

$$\underline{v_{phase}} = \frac{\omega(k_0)}{k_0}$$

(<u>phase velocity</u>)

<u>envelope function</u>

$$\tilde{f}\left(x - \frac{d\omega}{dk}\Big|_{k_0} t\right)$$

- modulates the amplitude of plane wave
- travels at <u>group velocity</u>

$$v_{group} = \frac{d\omega}{dk}\Big|_{k_0}$$