4. Beam optics in circular accelerators

4.10 Chromaticity and its correction

4.11 Restriction of the dynamic aperture by sextupoles

4.12 Optical resonances

Chromaticity

spread in momentum \( \Rightarrow \) quadrupole gradient error

\[ \Rightarrow \text{ spread in tune} \]

Chromaticity (natural)

\[ \xi = \frac{\Delta V}{\delta} = -\frac{1}{4 \pi} \int \beta(s) K(s) ds < 0 \]

\( \Rightarrow \) can cause particle loss

\( \Rightarrow \) need compensations
• Chromaticity compensation with sextupole

\( \text{plan sextupole at location where closed orbit dispersion } \eta(s) \text{ is non-zero} \)

\( X_D(s) = \eta(s) \delta \) in sextupole

\( \text{energy dependent offset in sextupole leads to energy dependent quadrupole effect} \)

\( \text{Example: in focusing quadrupole} \)

\( \delta > 0 : \text{too weak focusing} \)

\( \delta < 0 : \text{too strong focusing} \)

\[ E \]

\( F_x \)

\( \delta > 0 \) for \( \delta > 0 \)

\[ x = x_D + x_\rho \]

\[ z = z_\rho \]

\( \Rightarrow \) for small quantites in \( x_\rho, z_\rho \): 

\[ B_x = \delta^' x^2 \approx \frac{1}{2} \delta' x_D^2 \]

\[ B_z = \delta^' (z^2 - x^2) \approx \frac{1}{2} \delta' x_D^2 + (\delta^' \eta(s) \delta) x_\rho \]

\( \text{causes deflection of closed dispersion orbit} \)
\[ \Delta K_{ext} = \frac{q}{p_0} \int \gamma \eta (s) \delta = m \eta \eta (s) \delta \]

Note: opposite effect in horizontal vs vertical plane!
- want \( \Delta K > 0 \) to increase tune for \( \delta > 0 \) case

\[ \bar{S} = -\frac{1}{4\pi} \int \left[ L(s) - m(s) \eta(s) \right] \beta(s) ds \]

Adjust for compensation
- place sextupole where \( \beta \) is large (i.e. next to focusing quadrupole)
- correct sign:
  \[ S_x = -\frac{1}{4\pi} \int \left[ L_x - m \eta \xi \beta_x \right] ds \]
  \[ S_z = +\frac{1}{4\pi} \int \left[ L_z - m \eta \xi \beta_z \right] ds \]
  
Note: focusing quadrupole: \( k > 0 \)
but - sextupole introduce non-linear fields and coupling between the horizontal and vertical motion

- non-harmonic betatron motion
- frequency of oscillation, tune dependent on amplitude of oscillation
- non-linear resonances, sudden particle loss above certain amplitude (chaotic particle motion)

4.11 Restriction of dynamic aperture by sextupoles

Sextupoles = non-linear particle dynamics

- particle loss for large betatron amplitudes
- limit in effective ("useable") aperture
- "dynamic aperture"
- study with numerical methods

Example: (particle tracking)

\[ \dot{X} = \begin{pmatrix} x' \\ y' \\ \frac{3}{2} \delta_x \\ \frac{3}{2} \delta_y \end{pmatrix} \text{ initial phase space} \]

\[ X = \begin{pmatrix} x \\ y \\ \frac{3}{2} \delta_x \\ \frac{3}{2} \delta_y \end{pmatrix} \text{ linear optics} \]

\[ X^* = M X \]

\[ B = \begin{pmatrix} \frac{1}{2} \delta^4 (x^2 - z^2) \\ 0 \end{pmatrix} \]
\[ \vec{F}' = g \sqrt{x' \times \vec{B}'} \text{ over effective length of sextupole} \]

\[ \Delta x_i' = -\frac{q}{P} \beta \frac{e}{c} \left( x_i - z_i \right) \]

\[ \Delta z_i' = \frac{q}{P} \beta x_i \left( m \ell \right) \]

\[ \vec{x}_2 = \begin{pmatrix} x_1 \\ x_1' + \Delta x_1' \\ z_2 \\ z_2' + \Delta z_2' \\ t_1' \end{pmatrix} \]

**Results:**

1. Particles with too large initial offset are lost:
   - Without sextupole
   - With sextupole
Chromaticity correction with large number of weak, distributed sextupoles gives larger dynamic aperture than with few, strong sextupoles!

4.12 Optical resonance

Step 1) Floquet transformation of the homogeneous equation of motion: \( u'' + \sqrt{\beta(s)} u = 0 \)

\[
\zeta(s) = \frac{u(s)}{\sqrt{\beta(s)}}
\]

define: \( \zeta(s) \) normalized particle trajectory amplitude

\[
\phi = \frac{\psi(s)}{\sqrt{\beta(s)}} = \frac{1}{\sqrt{\beta(s)}} \int_{s_0}^{s} \frac{1}{\sqrt{\beta(s)}} ds
\]

change by \( 2 \pi \) per turn

\[
\frac{d\zeta}{d\phi} = \frac{d\zeta}{ds} \frac{ds}{d\phi} = \frac{d}{ds} \left( \frac{x(s)}{\sqrt{\beta(s)}} \right) \sqrt{\beta(s)} // d\phi
\]

\[
= \left( \frac{x(s)}{\sqrt{\beta(s)}} x'(s) + \sqrt{\beta(s)} x'(s) \right) \sqrt{\beta(s)}
\]
\[
\frac{d^2 \eta}{d \phi^2} = \frac{d}{d \phi} \left( \frac{d \eta}{d \phi} \right) = \frac{d}{ds} \left( \frac{d \eta}{d \phi} \right) \nu \beta(s) \\
= \left\{ \beta^{3/2} \chi'' + \frac{\alpha^2}{\sqrt{\beta}} \chi + \alpha' \sqrt{\beta} \chi \right\} \nu^2 \\
\alpha' = \beta \chi'(s) - \chi \\
= \frac{d^2 \eta}{d \phi^2} = \left\{ \beta^{3/2} \chi'' + \frac{\alpha^2}{\sqrt{\beta}} \chi + \left[ \chi \beta - \frac{1 + \alpha^2}{\beta} \right] \chi \right\} \nu^2 \\
= \left\{ \beta^{3/2} \chi'' + \chi(s) \chi \right\} \nu^2 - \frac{\chi}{\sqrt{\beta}} \nu^2
\]
Step 2: Fuguet's transformation of the inhomogeneous equation of motion

Field error, non-linear beam optics \(\rightarrow\) inhom. eqn. of motion

For horizontal motion:

\[
\frac{q}{p} B_2 = \frac{q}{p} B_{2,0} + \frac{q}{p} g x + \frac{q}{p} dB(x_0, s)
\]

\[
= \frac{1}{s(s)} + \nu(s) x + \frac{q}{p} dB
\]

\[
\text{(linear beam optics)}
\]

\[
\Rightarrow \text{eqn. of motion: } x''(s) + \nu(s) x'(s) = -\frac{q}{p} dB
\]

\[
\Rightarrow \text{Fuguet's transformation: } \frac{d^2 \chi}{ds^2} + \nu^2 \chi = -\beta \nu^2 \frac{q}{p} dB
\]

\[
\Rightarrow \text{assume } t = 0, \text{ expand } dB \text{ in terms of multipoles:}
\]

\[
DB(x_0, s) = DB_0 + \frac{dDB}{dx} \int_{x_0}^x x + \frac{1}{2} \frac{d^2 DB}{dx^2} \bigg|_0 x^2 + ...
\]

\[
\Rightarrow \text{with } \frac{d}{d\chi} = \frac{d}{dx} \frac{dx}{d\chi} = \sqrt{\beta} \frac{d}{dx}
\]

\[
\Rightarrow DB(\chi) = DB_0 + \frac{dDB}{d\chi} \bigg|_0 \chi + \frac{1}{2} \frac{d^2 DB}{d\chi^2} \chi^2 + ...
\]

\[
\Rightarrow \frac{d^2 \chi}{d\phi^2} + \nu^2 \chi = -\beta \nu^2 \frac{q}{p} \left\{ DB_0 + \frac{dDB}{d\chi} \chi + \frac{1}{2} \frac{d^2 DB}{d\chi^2} \chi^2 + \ldots \right\}
\]