Lecture 18

5. RF Systems and Particle Acceleration

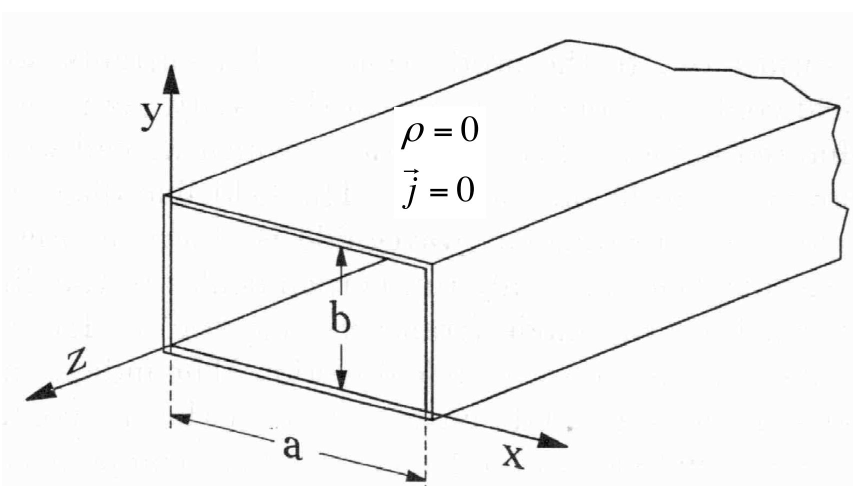
5.1 Waveguides

5.1.1 General; cut-off frequency

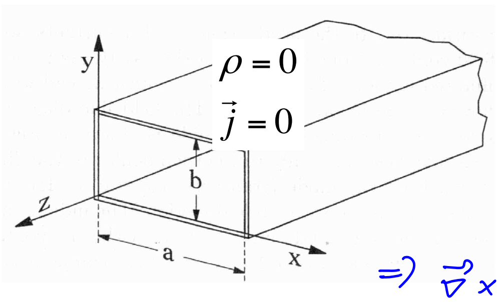
5.1.2 Rectangular Waveguides

5.1.3 Cylindrical Waveguides

5.1 Waveguides



5.1.1 General; cut-off Frequency



- Maxwell:

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\partial_t \vec{E} + \mu_0 \vec{j}$$

- inside waveguide: $\rho = 0$, $\vec{j} = 0$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{c^2}\partial_t^2 \vec{E}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\frac{1}{c^2}\partial_t^2 \vec{B}$$

$$\Rightarrow \text{with: } \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \underbrace{\vec{\nabla}(\vec{\nabla} \cdot \vec{E})}_{=0,\text{ since } \rho = 0 \text{ here}} - \vec{\nabla}^2 \vec{E}$$

$$\Rightarrow \vec{\nabla}^2 \vec{E} = \frac{1}{c^2}\partial_t^2 \vec{E}$$

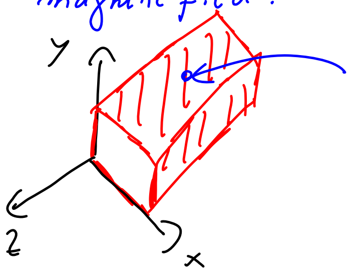
$$\vec{\nabla}^2 \vec{B} = \frac{1}{c^2}\partial_t^2 \vec{B}$$

wave equation

(a) Boundary conditions at conducting walls:

electric field: $\perp$ to conducting wall $\Rightarrow E_{\parallel} = 0$

magnetic field: $\parallel$ to wall $\Rightarrow \vec{B}_{\perp} = 0$



$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\partial_t \vec{E}$$

gives:

$$\vec{e}_x \overbrace{(\partial_y B_z - \partial_z B_y)}^{=0} + \vec{e}_y(\partial_z B_x - \partial_x B_z) + \vec{e}_z(\partial_x B_y - \partial_y B_x) = \frac{1}{c^2}\partial_t \vec{E}_y$$

$$\Rightarrow \boxed{\partial_{\perp} B_z = 0} \quad \text{on walls}$$

(b) Waveguide modes:

- TEM : Transverse electric and magnetic ($E_z = 0$ and $B_z = 0$) can not exit

$$\left.\begin{array}{l}\vec{\nabla}\times\vec{E} = -\partial_t \vec{B} \\ \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\partial_t \vec{E}\end{array}\right\} \quad \begin{array}{l}\vec{\nabla}_\perp \times \vec{E}_\perp = -\partial_t \vec{B}_z \\ \vec{\nabla}_\perp \times \vec{B}_\perp = \frac{1}{c^2}\partial_t \vec{E}_z\end{array}$$

$$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{E} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0\end{array}\right\} \quad \begin{array}{l}\vec{\nabla}_\perp \cdot \vec{E}_\perp + \partial_z E_z = 0 \\ \vec{\nabla}_\perp \cdot \vec{B}_\perp + \partial_z B_z = 0\end{array}$$

if $E_z = 0$ and $B_z = 0$ $\Rightarrow \vec{E}_\perp = \text{const}$ and $\vec{B}_\perp = \text{const}$

- TE modes: transverse electric ; have $E_z = 0$
- TM modes: transverse magnetic; have $B_z = 0$
- due to different boundary conditions, E_z and B_z can not simultaneously be non-zero.

(c) Solving the wave equation with boundary conditions

$$\vec{\nabla}^2 \vec{E} = \frac{1}{c^2}\partial_t^2 \vec{E} \qquad \vec{\nabla}^2 \vec{B} = \frac{1}{c^2}\partial_t^2 \vec{B}$$

- separate out harmonic time dependence:

$$\vec{E} = \vec{E}(\vec{r})\, e^{\pm i\omega t}$$

$$\vec{B} = \vec{B}(\vec{r})\, e^{\pm i\omega t}$$

$$\Rightarrow \quad \vec{\nabla}^2 \vec{E} = -\left(\frac{\omega}{c}\right)^2 \vec{E} = -k^2 \vec{E}$$

$$\vec{\nabla}^2 \vec{B} = -\left(\frac{\omega}{c}\right)^2 \vec{B} = -k^2 \vec{B}$$

with wave number $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$

$\Rightarrow$ focus on z-components (for now):

$$\partial_x^2 E_z + \partial_y^2 E_z + \partial_z^2 E_z = -k^2 E_z$$

$\Rightarrow$ Ansatz: $E_z(x,y,z) = f_x(x) f_y(y) f_z(z)$

$$\Rightarrow \quad \frac{\partial_x^2 f_x}{f_x} + \frac{\partial_y^2 f_y}{f_y} + \frac{\partial_z^2 f_z}{f_z} = -k^2 = \text{const}$$

$\Rightarrow$ define:

$$k_x^2 \equiv -\frac{\partial_x^2 f_x}{f_x} \qquad k_y^2 \equiv -\frac{\partial_y^2 f_y}{f_y} \qquad k_z^2 \equiv -\frac{\partial_z^2 f_z}{f_z}$$

$$\Rightarrow \quad \boxed{k_x^2 + k_y^2 + k_z^2 = k^2 = \left(\frac{\omega}{c}\right)^2}$$

$$\Rightarrow \quad \partial_z^2 f_z = -k_z^2 f_z \quad \Rightarrow \quad \partial_z^2 E_z = -k_z^2 E_z$$

solution: $E_z = E_0(x,y)\, e^{ik_z z}$ for TM mode

$B_z = B_0(x,y)\, e^{ik_z z}$ for TE mode

$\Rightarrow$ x,y-dependence:

$$\left.\begin{array}{l} \partial_x^2 f_x + k_x f_x = 0 \\ \partial_y^2 f_y + k_y f_y = 0 \end{array}\right\}$$

combined:

$$(\partial_x^2 + \partial_y^2) E_z = -(k_x^2 + k_y^2) E_z$$

define $\boxed{k_c^2 = k_x^2 + k_y^2 = k^2 - k_z^2}$

$$\Rightarrow (\partial_x^2 + \partial_y^2) E_z = -[k^2 - k_z^2] E_z \quad \text{for TM modes}$$

$$(\partial_x^2 + \partial_y^2) B_z = -[k^2 - k_z^2] B_z \quad \text{for TE modes}$$

$\Rightarrow$ Boundary conditions $\Rightarrow$ solutions only for discrete set of eigenvalues

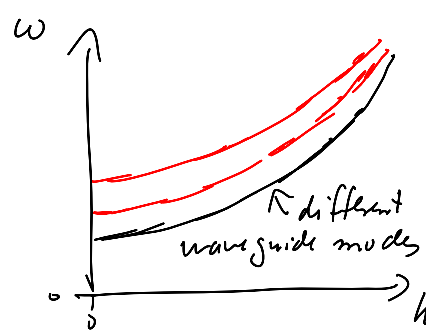
$$k_c^2 = k_x^2 + k_y^2 = k^2 - k_z^2$$

$\Rightarrow$ k_x, k_y define "waveguide mode"

(d) dispersion relation:

$$\omega(k_z) = kc = c\sqrt{k_c^2 + k_z^2}$$

discrete value, depends on mode



- phase velocity:

$$v_{ph} = \frac{\omega}{k_z} = c\sqrt{1 + \left(\frac{k_c}{k_z}\right)^2} > c$$

- group velocity:

$$v_{gr} = \frac{d\omega}{dk_z} = \frac{c}{\sqrt{1 + \left(\frac{k_c}{k_z}\right)^2}} < c$$

(e) wave propagation along z in waveguide:

for excitation frequency ω:

$\Rightarrow$ wave propagates according to $e^{ik_z z}$

$\Rightarrow k_z = \sqrt{k - k_c^2} = \begin{cases} \text{complex if } k_c^2 > k^2 \Rightarrow \underline{\text{damping}} \\ \text{real if } k_c^2 < k^2 \Rightarrow \underline{\text{propagation}} \end{cases}$

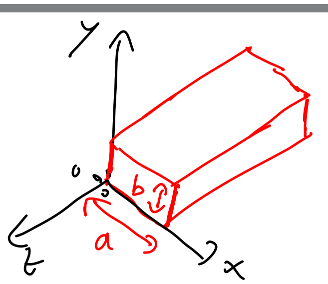
$\Rightarrow$ cut-off frequency: $\boxed{\omega_c = c k_c}$

determined by waveguide mode

$\Rightarrow$ for $\omega > \omega_c$: wave transport $\Rightarrow$ propagate RF power

for $\omega < \omega_c$: damping

5.1.2 Rectangular Waveguide



- TM mode: $B_z = 0$

$E_z = f_x(x)\, f_y(y)\, f_z(z)$

with $\underline{E_z = 0}$ at wall

and $\partial_x^2 f_x + k_x^2 f_x = 0$

$\partial_y^2 f_y + k_y^2 f_y = 0$

$\Rightarrow$ general solution:

$f_x(x) = A \sin(k_x x) + B \cos(k_x \cdot x)$

$f_y(y) = C \sin(k_y y) + D \cos(k_y \cdot y)$

$\Rightarrow$ boundary conditions: $B = 0$, $D = 0$

$k_x a = n\pi$

$k_y b = m\pi$

with m, n = integer

$\Rightarrow TM_{n,m}$ mode

$$\Rightarrow E_z = E_0 \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)e^{ik_z z}$$

$\Rightarrow$ cut-off-frequency:

$$\omega_c = c\, k_c = c\sqrt{k_x^2 + k_y^2}$$

$$= c\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

- TE modes: $E_z = 0$

$B_z = f_x(x)\, f_y(y)\, f_z(z)$ with $\partial_\perp B_z = 0$ at wall

and $\partial_x^2 f_x + k_x^2 f_x = 0$

$\partial_y^2 f_y + k_y^2 f_y = 0$

$\Rightarrow$ general solution:

$f_x(x) = A\sin(k_x x) + B\cos(k_x x)$

$f_y(y) = C\sin(k_y y) + D\cos(k_y y)$

$\Rightarrow$ from boundary conditions: $A = 0$, $C = 0$

$k_x a = n\pi$

$k_y b = m\pi$

with n, m = integers

(same as for TM modes in rectangular waveguides)

$$\Rightarrow B_z = B_0 \cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right)e^{ik_z z}$$

TE_{nm} modes

$\Rightarrow$ cut-off frequency

$$\omega_c = c\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

Rectangular Wave Guide

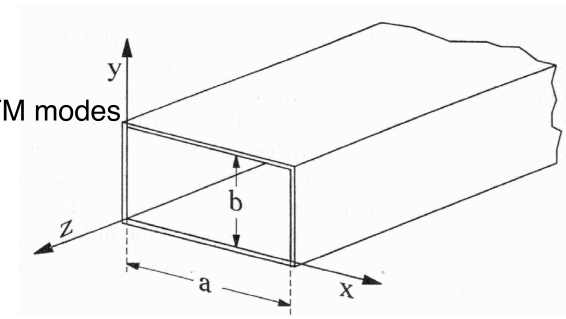
Boundary conditions:

$$E_z(\vec{x}_0) = 0 \quad \vec{\nabla}_\perp^2 E_z = [k_z^2 - (\tfrac{\omega}{c})^2]E_z$$

$$\boxed{E_z(\vec{x}) = E_0 \sin(\tfrac{n\pi}{a}x)\sin(\tfrac{m\pi}{b}y)}$$ for TM modes

$$(\tfrac{\omega}{c})^2 - k_z^2 = k_{nm}^{(B)2} = \left(\tfrac{n\pi}{a}\right)^2 + \left(\tfrac{m\pi}{b}\right)^2$$

$$\partial_\perp B_z(\vec{x}_0) = 0 \quad \vec{\nabla}_\perp^2 B_z = [k_z^2 - (\tfrac{\omega}{c})^2]B_z$$



$$\boxed{B_z(\vec{x}) = B_0 \cos(\tfrac{n\pi}{a}x)\cos(\tfrac{m\pi}{b}y)}$$ for TE modes

$$(\tfrac{\omega}{c})^2 - k_z^2 = k_{nm}^{(E)2} = \left(\tfrac{n\pi}{a}\right)^2 + \left(\tfrac{m\pi}{b}\right)^2$$

TE and TM modes happen to have the same eigenvalues.

For simplicity one still looks at TE and TM modes separately.

Transverse fields

TE-modes: Once B_z is know, the full field can be found

$$\vec{\nabla}_\perp \times \vec{E}_z + ik_z\vec{e}_z \times \vec{E}_\perp = i\omega\vec{B}_\perp$$

$$\vec{\nabla}_\perp \times \vec{B}_z + ik_z\vec{e}_z \times \vec{B}_\perp = -i\omega\tfrac{1}{c^2}\vec{E}_\perp$$

$$k_z\vec{E}_\perp + \omega\vec{e}_z \times \vec{B}_\perp = \vec{e}_z \times (\vec{\nabla}_\perp \times \vec{E}_z) = -i\vec{\nabla}_\perp E_z$$

$$\omega\tfrac{1}{c^2}\vec{E}_\perp + k_z\vec{e}_z \times \vec{B}_\perp = i\vec{\nabla}_\perp \times \vec{B}_z$$

$$(\tfrac{\omega^2}{c^2} - k_z^2)\vec{E}_\perp = i(k_z\vec{\nabla}_\perp E_z + \omega\vec{\nabla}_\perp \times \vec{B}_z)$$

$$(\tfrac{\omega^2}{c^2} - k_z^2)\vec{e}_z \times \vec{B}_\perp = -i(\tfrac{\omega}{c^2}\vec{\nabla}_\perp E_z + k_z\vec{\nabla}_\perp \times \vec{B}_z)$$

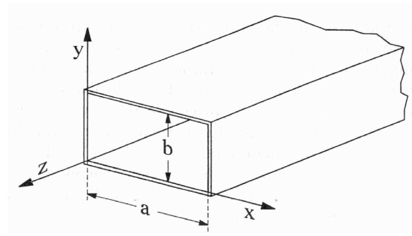
$$\boxed{\begin{aligned}\vec{E}_\perp &= \tfrac{i}{\frac{\omega^2}{c^2} - k_z^2}(k_z\vec{\nabla}_\perp E_z + \omega\vec{\nabla}_\perp \times \vec{B}_z)\\ \vec{B}_\perp &= \tfrac{i}{\frac{\omega^2}{c^2} - k_z^2}(k_z\vec{\nabla}_\perp B_z - \tfrac{\omega}{c^2}\vec{\nabla}_\perp \times \vec{E}_z)\end{aligned}}$$

Rectangular TE Modes

$$E_z(\vec{x}) = 0\ , \quad B_z(\vec{x}) = B_0 \cos(\tfrac{n\pi}{a}x)\cos(\tfrac{m\pi}{b}y)$$

$$\vec{E}_\perp = \frac{i}{\frac{\omega^2}{c^2}-k_z^2}(k_z\vec{\nabla}_\perp E_z + \omega\vec{\nabla}_\perp \times \vec{B}_z)$$

$$\vec{B}_\perp = \frac{i}{\frac{\omega^2}{c^2}-k_z^2}(k_z\vec{\nabla}_\perp B_z - \tfrac{\omega}{c^2}\vec{\nabla}_\perp \times E_z)$$

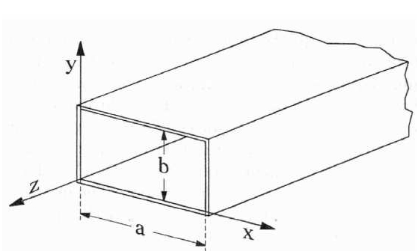


$$\vec{E}(\vec{x}) = \frac{\omega}{k_{nm}^{(E)2}}B_0\begin{pmatrix} \frac{m\pi}{b}\cos(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ -\frac{n\pi}{a}\sin(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ 0 \end{pmatrix}$$

$$\vec{B}(\vec{x}) = \frac{k_z}{k_{nm}^{(E)2}}B_0\begin{pmatrix} \frac{n\pi}{a}\sin(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\ \sin(k_z z-\omega t) \\ \frac{m\pi}{b}\cos(\frac{n\pi}{a}x)\ \sin(\frac{m\pi}{b}y)\ \sin(k_z z-\omega t) \\ \frac{k_{nm}^{(E)2}}{k_z}\cos(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\cos(k_z z-\omega t) \end{pmatrix}$$

Example: Rectangular TE_{01} Mode

Notation: TE_{nm} Mode

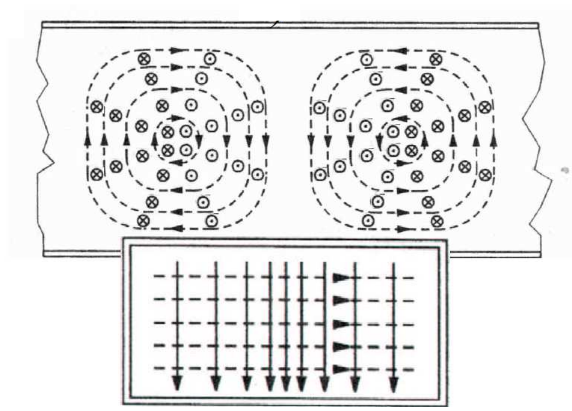


$\vec{E}$ ⟶

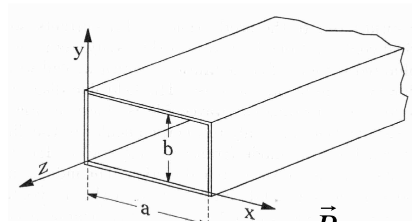
$\vec{B}$ - - →

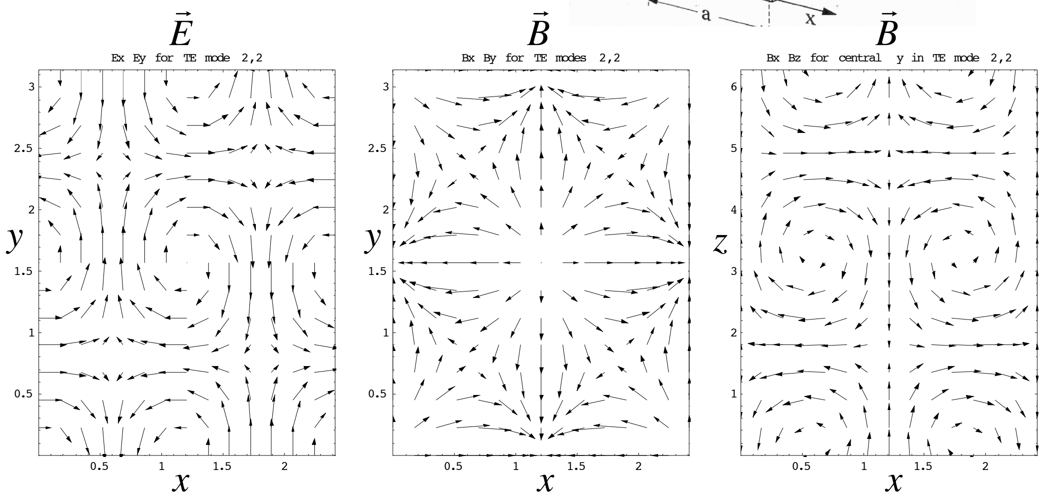
$n=1$

$m=0$



Example: Rectangular TE_{22} Mode



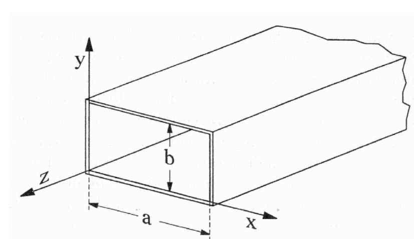


Rectangular TM Modes

$$E_z(\vec{x}) = E_0 \sin(\tfrac{n\pi}{a}x)\sin(\tfrac{m\pi}{b}y)\,, \quad B_z = 0$$

$$\vec{E}_\perp = \frac{i}{\frac{\omega^2}{c^2}-k_z^2}(k_z\vec{\nabla}_\perp E_z + \omega\vec{\nabla}_\perp \times \vec{B}_z)$$

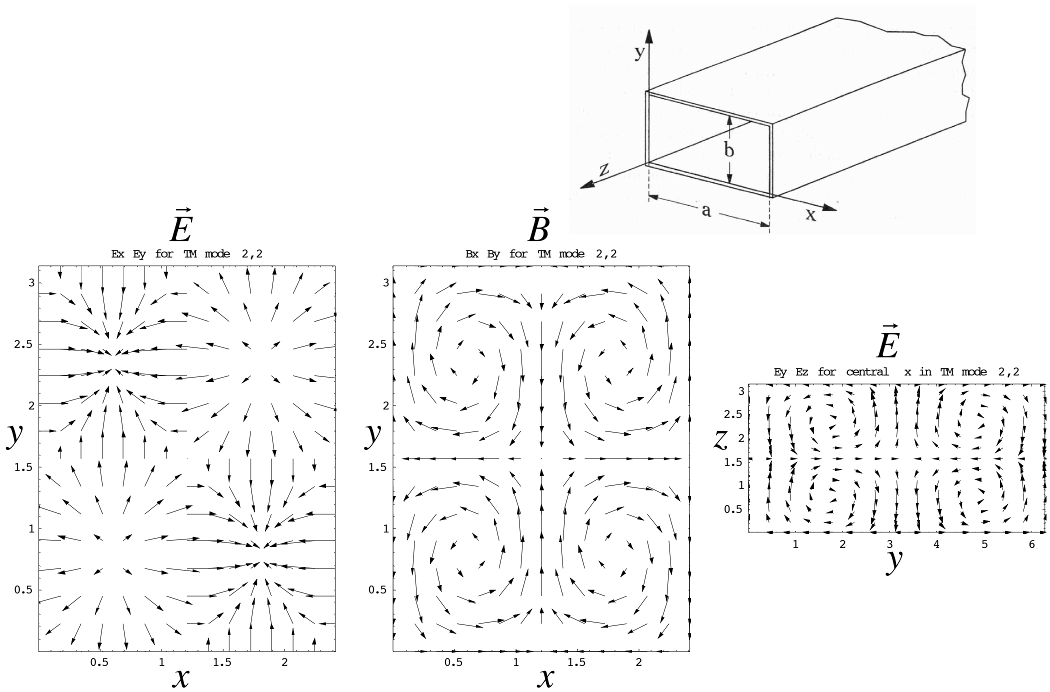
$$\vec{B}_\perp = \frac{i}{\frac{\omega^2}{c^2}-k_z^2}(k_z\vec{\nabla}_\perp B_z - \tfrac{\omega}{c^2}\vec{\nabla}_\perp \times E_z)$$



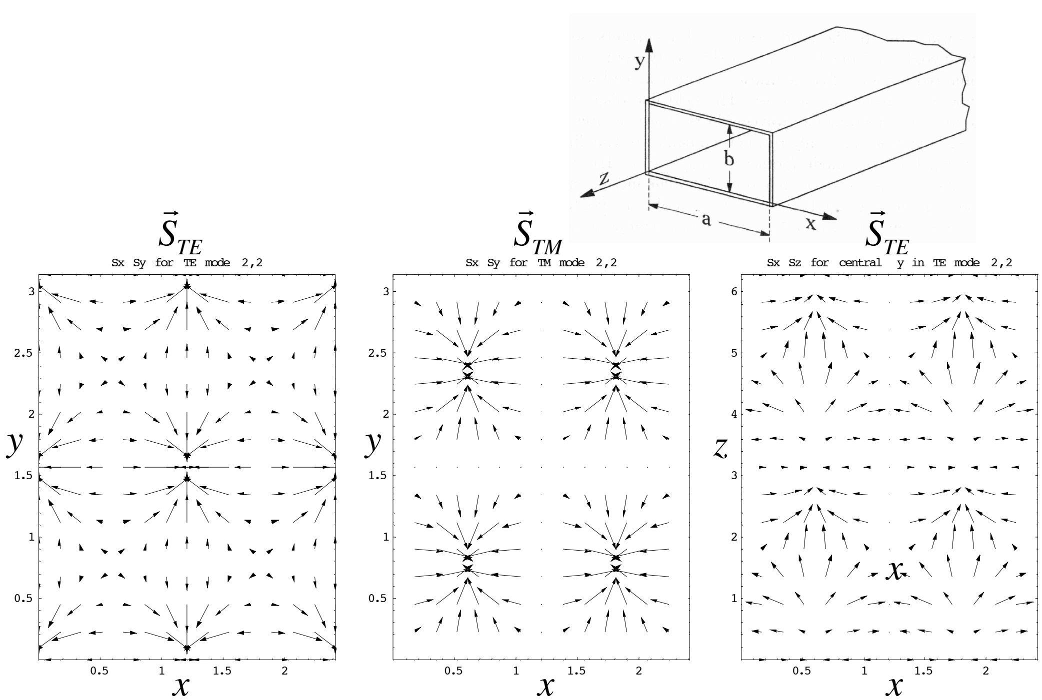
$$\vec{E}(\vec{x}) = \frac{k_z}{k_{nm}^{(E)2}}E_0 \begin{pmatrix} -\frac{n\pi}{a}\cos(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ -\frac{m\pi}{b}\sin(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ \frac{k_{nm}^{(E)2}}{k_z}\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\cos(k_z z-\omega t) \end{pmatrix}$$

$$\vec{B}(\vec{x}) = \frac{\omega}{c^2 k_{nm}^{(E)2}}E_0 \begin{pmatrix} \frac{m\pi}{b}\sin(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\ \sin(k_z z-\omega t) \\ -\frac{n\pi}{a}\cos(\frac{n\pi}{a}x)\ \sin(\frac{m\pi}{b}y)\ \sin(k_z z-\omega t) \\ 0 \end{pmatrix}$$

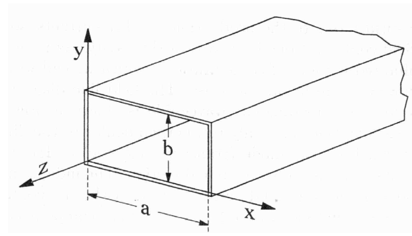
Example: Rectangular TM_{22} Mode

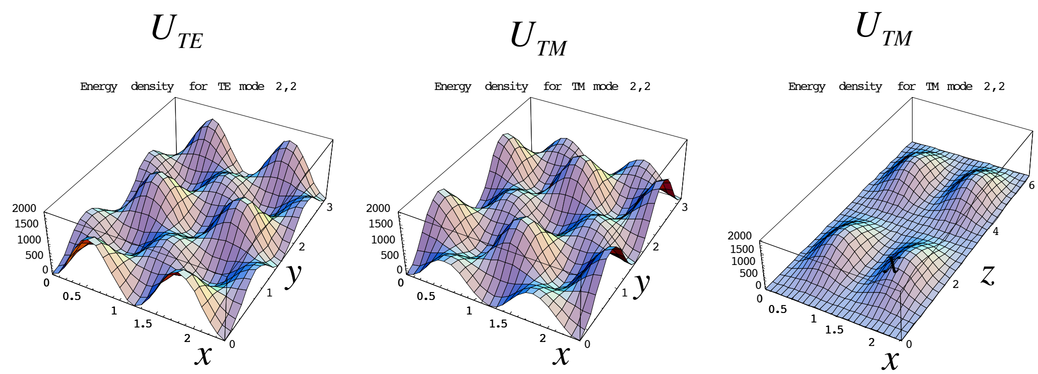


Pointing Vector

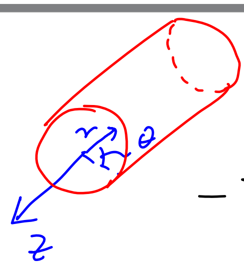


Energy Density





5.1.3 Circular Waveguide



use cylindrical coordinate system: r, θ, z

R = radius of waveguide

– TM modes: $B_z = 0$

$$\vec{\nabla}^2 \vec{E} = \frac{1}{c^2} \partial_t^2 E$$

$$\Rightarrow \left(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \partial_z^2 \right) E_z = \frac{1}{c^2} \partial_t^2 E_z$$

Try: $E_z = E_0(r)\, e^{in\theta}\, e^{ik_z z}\, e^{-i\omega t}$

with $\underline{\underline{n = \text{integer}}}$

$$\Rightarrow \left\{ \partial_r^2 + \frac{1}{r} \partial_r + \left(\left(\frac{\omega}{c}\right)^2 - \frac{n^2}{r^2} - k_z^2 \right) \right\} E_z = 0$$