5. RF Systems and Particle Acceleration

5.4 Phase focusing and longitudinal (synchrotron) beam oscillation

**(a) Linear**

\[ T_b : \text{bunch spacing} \]

\( \rightarrow \)

\( \text{accelerating voltage } V(t) = V \sin(\omega t \cdot t) \)

\( \Rightarrow \text{for synchronization, need:} \)

1. \[ f_{RF} = \frac{1}{T_b} = h f_b \]

2. \text{phase focusing (for } v < c \text{ particle)}
(b) \textit{for circular accelerators}

\[ V = \sqrt{2} \sin (\omega_{RF} t) \]

\[ f_0 = \frac{1}{T_0} \]

\[ \delta E = q \sqrt{2} \sin \phi \]

\[ \phi : \text{RF accelerating phase} \]

\[ \phi = \omega_{\text{tang}} \]

\[ f_{RF} = h \cdot f_0 \]

\[ \text{integer, harmonic number} \]

\[ \text{for proton in synchrotron: } V_{CC} = f_0 \text{ increases with energy over time } \Rightarrow f_{RF} \text{ needs to increase during energy ramp} \]
2) correct accelerating phase $\phi_0$ for reference particle with $\delta = 0$ (same phase for each turn) =) phase focusing

\[ V(t) \]

\[ \phi_0 \]

ideal (reference) particle $\Rightarrow$ passes at nominal phase $\phi_0$

particle with $\delta \neq 0$ $\Rightarrow$ phase focusing results in longitudinal particle oscillation about nominal phase $\phi_0$

$\Rightarrow$ synchrotron oscillations

\[ \text{revolution time } T = \frac{C}{v} \]

$\Rightarrow$ depends on circumference $C$ and particle speed $v$

$\Rightarrow$ both depend on particle momentum!

$\Rightarrow$ for small changes $\Delta C, \Delta v$

\[ \frac{\Delta T}{T} = \frac{\Delta C}{C} - \frac{\Delta v}{v} \]

recall momentum compaction factor

\[ \frac{\Delta v}{v} = \frac{1}{2} \frac{\Delta p}{p} \quad p < p_{\text{syn}} \]

\[ \Delta C/C = \int \frac{D(s)}{P(s)} \, ds > 0 \]
\[ \frac{\Delta T}{T} = \left( \alpha - \frac{1}{\gamma^2} \right) \frac{\Delta p}{p} \]

\Rightarrow \text{define } \textit{transition energy} \ E_r = \gamma_r m_0 c^2 \ \text{with } \frac{1}{\gamma_r^2} = \alpha

\Rightarrow \frac{\Delta T}{T} = 0 \ at \ E = E_r

\gamma_r \ is \ defined \ by \ the \ accelerator \ optics

\[
\frac{\Delta T}{T} = \left( \frac{1}{\gamma_r^2} - \frac{1}{\gamma^2} \right) \frac{\Delta p}{p}
\]

\Rightarrow \text{two distinct regions:}

\( E < E_r \Rightarrow \gamma < \gamma_r \Rightarrow \Delta T < 0 \ \text{for } \Delta p > 0 \)

\Rightarrow \text{increase in particle energy reduces revolution time, since increase in particle speed dominates an increase in circumference/particle length}

\( E > E_r \Rightarrow \gamma > \gamma_r \Rightarrow \Delta T > 0 \ \text{for } \Delta p > 0 \)

\Rightarrow \text{increase in particle energy increases revolution time since increase in particle length dominates an increase in particle speed}
Note: - for electrons: $\nu > c \implies E > E_{\text{r}}$ always
  - for protons/ions in synchrotron:
    - often: $E_{\text{min}} < E_{\text{r}} < E_{\text{final}}$
    - energy passes through $E_{\text{r}}$ during ramp up of energy
    - depending on region, different nominal accelerating phase $\phi_0$ of ideal particle is required for stable operation

\[ E < E_{\text{r}}: \quad \Delta T < 0 \quad \text{for } \Delta \phi > 0 \quad \text{for } \Delta E > 0 \]

\[ \Delta E = \frac{\nu}{N} \]

\[ (k-1)^{1/4} \text{turn} \]

\[ \Rightarrow \] restoring force for $0 < \phi_0 < \frac{\pi}{2}$

\[ \Rightarrow \] synchrotron oscillations about $\phi = \phi_0$

for particle with $\delta \neq 0$
\[ \delta E = \frac{\delta E}{\delta t} = E_k - E_{k-1} = \frac{1}{2} m V^2 \sin \phi \]

\[ \delta E = \delta E_0 + \delta \phi \sin \phi_0 \]

\[ \delta E - \delta E_0 = \frac{1}{2} m V^2 (\sin \phi - \sin \phi_0) = \Delta E_k - \Delta E_{k-1} \]
- Assume that change of DE per turn is small
- Rate of change of DE

\[
\frac{d}{dt}(\Delta E) \approx \frac{\Delta E_n - \Delta E_{n-1}}{T_0 = \text{revolution time}} \quad \frac{1}{T_0} = \frac{2\pi}{\omega_0}
\]

\[
\frac{d}{dt}(\Delta E) = \frac{q V_0}{2\pi} (\sin \phi - \sin \phi_0) \quad (1)
\]

\[
\phi_k - \phi_{k-1} = \omega_{RF} \cdot \Delta T = h \omega_0 \Delta T = \Delta \phi = \text{const}
\]

\[
\Delta T = T - T_0
\]

\[
= \text{revolution time difference compared to } T_0 \text{ of ideal particle}
\]

- Rate of change of accel. phase:

\[
\frac{\Delta \phi_{\text{rpm}}}{T_0} = \frac{\phi_k - \phi_{k-1}}{T_0} \approx \frac{d \phi}{dt}
\]

\[
\Rightarrow \frac{d \phi}{dt} \approx h \omega_0 \frac{\Delta T}{T_0} = h \omega_0 \eta \frac{DP}{\rho}
\]

\[
\text{with } \eta = \left( \frac{1}{\gamma_0^2} - \frac{1}{\gamma^2} \right)
\]

\[
\Rightarrow \sin \alpha \frac{dE}{dp} = \beta^2 \frac{E}{\rho}
\]

\[
\frac{d \phi}{dt} \approx h \omega_0 \eta \frac{1}{\rho_0^2} \frac{\Delta E}{E_0} \quad (2) \quad \left| \frac{d}{dt} \right|
\]
with (1) for \( \frac{d\phi}{dt} \)

\[
\frac{E_0 \beta_0^2}{w_0^2 \hbar \eta} \frac{d^2 \phi}{dt^2} = \frac{qV}{2\pi} \left( \sin \phi - \sin \phi_0 \right)
\]

Diff. eqn. for synchronous oscillations \( \phi(t) \)
in adiabatic approximation: assume that \( E_0, w_0, \beta_0, q \)
are changing only slowly, i.e. are \( \approx \) const with the time scale \( T_0 \).

\( = \) for small deviations \( \delta \phi = \phi - \phi_0 \) from ideal phase \( \phi_0 \): 

\[
\sin \phi = \sin (\phi_0 + \delta \phi) = \sin \phi_0 \cos \delta \phi + \cos \phi_0 \sin \delta \phi
\]

also:

\[
\frac{d^2(\delta \phi)}{dt^2} = \frac{d^2 \phi}{dt^2}, \quad \sin \phi_0 = \text{const}
\]

\( \Rightarrow \) this gives for small \( \delta \phi \):

\[
\frac{d^2(\delta \phi)}{dt^2} + \Omega^2 \delta \phi = 0 \quad \text{with} \quad \Omega^2 = -\gamma \cos \phi \frac{qV w_0^2 \hbar}{2\pi E_0 \beta_0^2}
\]
for $\Omega^2 > 0$: simple harmonic oscillations with frequency $\Omega$

- if $\Omega^2 < 0$: exponential solution $\Rightarrow$ instability
- need $\Omega^2 > 0$ for stable operation
  - $E < E_w$ case: $\eta = \left(\frac{1}{\Omega_{11}^2} - \frac{1}{\Omega^2}\right) < 0$
    - need $\cos \phi_0 > 0$

$V(t) = V_{t_0} e^{\left(\frac{1}{\Omega_{11}^2} - \frac{1}{\Omega^2}\right) t}$

- need $0 < \phi_0 < \frac{\pi}{2}$ for stability during acceleration
- $0 < \phi_0 < \pi$ during deceleration ($\frac{\pi}{2} \pi < \phi_0 < 2\pi$)