Lecture 5

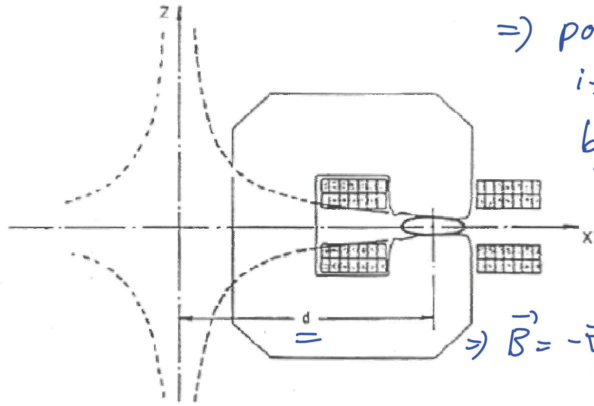
2. Charged particles in magnetic fields

- 2.2 Magnets (synchrotron magnet)
- 2.3 Multipole expansion
- 2.4 Superconducting magents

2.2 Magnets

combined function: synchrotron magnet

Combined function magnet: synchrotron magnet



⇒ port of quadrupole which is shifted horizontally by $d = B_0/g$ from the axis

potential: $\psi(\tilde{x}, z) = -g z (\tilde{x} + d)$

$$= -B_0 z - g \tilde{x} z$$

$$\Rightarrow \vec{B} = -\vec{\nabla}\psi = \begin{pmatrix} g z \\ B_0 + g\tilde{x} \end{pmatrix}$$

} quadrupole + vertical dipole

⇒ "field index" n

$$n = -\frac{R}{B_0}\left(\frac{\partial B_z}{\partial \tilde{x}}\right)_{\tilde{x}=R} = -\frac{R}{B_0} g = -k R^2$$

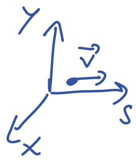
$\frac{1}{R} = \frac{q}{p} B_0$

$k = \frac{q}{p} g$

2.3 Multipole expansion

General multipole expansion

- assume $\vec{B}$ has transverse components (B_x, B_y) only (neglect fringe fields at magnet ends)



- $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow$ vector potential $\vec{A}$ exists, such that

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Note: $\vec{A} = \begin{pmatrix} 0 \\ 0 \\ A_s \end{pmatrix}$ since $\vec{B} = \begin{pmatrix} B_x \\ B_y \\ 0 \end{pmatrix}$

- in vacuum (inside magnet)

$\vec{\nabla} \times \vec{B} = 0 \Rightarrow$ scalar potential ψ exists, such that

$$\vec{B} = -\vec{\nabla} \psi$$

- combine:

$$B_x = -\frac{\partial \psi}{\partial x} = \frac{\partial A_s}{\partial y} \qquad B_y = -\frac{\partial \psi}{\partial y} = -\frac{\partial A_s}{\partial x}$$

$\Rightarrow$ These are the Cauchy-Riemann conditions for the real and imaginary part of an analytic function!

General multipole expansion: Cylindrical coordinate representation (I)

$\Rightarrow$ define complex function of $z = x + iy$

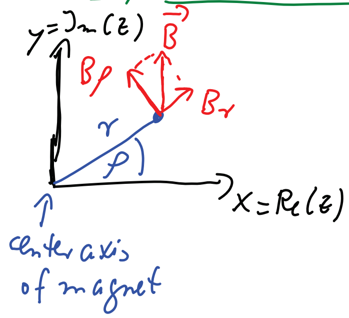
$$\tilde{A}(z) = A_s(x,y) + i\,\psi(x,y)$$ analytic function!

$\Rightarrow$ can be expanded into a power series:

$$\tilde{A}(z) = \sum_{n=0}^{\infty} \kappa_n z^n \quad \text{with } \kappa_n = \lambda_n + i\mu_n$$

(λ_n, μ_n: real constants)

- cylindrical coordinates:



$$x = r\cos\varphi \qquad y = r\sin\varphi$$

$$\Rightarrow z^n = r^n e^{in\varphi} = r^n(\cos n\varphi + i \sin n\varphi)$$

General multipole expansion: Cylindrical coordinate representation (II)

⇒ for scalar potential:

$$\Psi(r,\varphi) = \mathrm{Im}(\tilde{A}) = \sum_{n=0}^{\infty} (\mu_n \cos n\varphi + \lambda_n \sin n\varphi) r^n$$

⇒ vector potential:

$$A_s(r,\varphi) = \mathrm{Re}(\tilde{A}) = \sum_{n=0}^{\infty} (\lambda_n \cos n\varphi - \mu_n \sin n\varphi) r^n$$

⇒ from $\vec{B} = -\vec{\nabla}\Psi$

$$B_\varphi = -\frac{1}{r}\frac{\partial \Psi}{\partial \varphi} = -\sum_{n=1}^{\infty} n(\lambda_n \cos n\varphi - \mu_n \sin n\varphi) r^{n-1}$$

$$B_r = -\frac{\partial \Psi}{\partial r} = -\sum_{n=1}^{\infty} n(\mu_n \cos n\varphi + \lambda_n \sin n\varphi) r^{n-1}$$

General multipole expansion: Cylindrical coordinate representation (III)

define:
- reference radius $= r_0$ (e.g. beam pipe radius)
- (magnitude of main field component of magnet) $= B_{main}$
- "normal" multipole coefficients:

$$b_n = -\frac{n\lambda_n}{B_{main}} r_0^{n-1}$$

- "skew" multipole coefficients:

$$a_n = +\frac{n\mu_n}{B_{main}} r_0^{n-1}$$

⇒ multipole expansion:

$$\Psi(r,\varphi) = -B_{main} r_0 \sum_{n=1}^{\infty} \left(-\frac{a_n}{n}\cos n\varphi + \frac{b_n}{n}\sin n\varphi\right)\left(\frac{r}{r_0}\right)^n$$

$$A_s(r,\varphi) = -B_{main} r_0 \sum_{n=1}^{\infty} \left(\frac{b_n}{n}\cos n\varphi + \frac{a_n}{n}\sin n\varphi\right)\left(\frac{r}{r_0}\right)^n$$

General multipole expansion: Cylindrical coordinate representation (IV)

$$B_\varphi(r,\varphi) = B_{main} \sum_{n=1}^{\infty} (b_n \cos n\varphi + a_n \sin n\varphi)\left(\frac{r}{r_0}\right)^{n-1}$$

$$B_r(r,\varphi) = B_{main} \sum_{n=1}^{\infty} (-a_n \cos n\varphi + b_n \sin n\varphi)\left(\frac{r}{r_0}\right)^{n-1}$$

⇒ for ideal "normal" $2m$-pole magnet:

$b_n = 1$ for $m = n$, $a_n, b_n = 0$ else

$m=1$	Dipole	$m=4$	Octupole
$m=2$	Quadrupole	$m=5$	Decapole
$m=3$	Sextupole	$m=6$	Dodecapole, 12-pole

⇒ consider: $B_\varphi + i B_r$

$$B_\varphi + i B_r = B_{main} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \left[b_n (\cos n\varphi + i \sin n\varphi) - i a_n (\cos n\varphi + i \sin n\varphi) \right]$$

$$= B_{main} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (b_n - i a_n) e^{in\varphi}$$

⇒ Thus:

$$|B_n| = \left(\sqrt{B_r^2 + B_\varphi^2}\right)_n = B_{main} \left(\frac{r}{r_0}\right)^{n-1} \sqrt{a_n^2 + b_n^2}$$

⇒ magnitude of $2n$-pole field does not depend on φ and scales r^{n-1}

⇒ at reference radius $r = r_0$: multipole coefficients a_n and b_n are the relative field contributions of the nth multipole

⇒ cylindrical coordinate representation

- useful for air-core magnet design (rel. S.C. magnet)
- measurement of multipole components with a rotating coil in the field:

  nth Fourier component of the induced voltage $\propto \sqrt{a_n^2+b_n^2}$, phase related to a_n/b_n

  good magnet: unwanted multipole coefficients $\lesssim 10^{-4}$
- symmetry:
  - quadrupole with perfect constructional symmetry → only <u>odd</u> harmonics of 4-pole are allowed
  - 2n-pole magnet → transforms into a skew 2n-pole magnet if rotated by $\pi/2n$

General multipole expansion: Cartesian coordinate representation (I)

$$\tilde{A}(x,y) = A_s(x,y) + i\,\Psi(x,y) = \sum_{n=0}^{\infty} \mathcal{K}_n z^n$$

$$= \sum_{n=0}^{\infty} (\lambda_n + i\mu_n)(x+iy)^n$$

⇒ scalar potential:

$$\Psi(x,y) = \mathrm{Im}(\tilde{A}) = B_{main}\Big[a_1 x - b_1 y + \frac{a_2}{2r_0}(x^2-y^2) - \frac{b_2}{r_0}xy$$

$$+ \frac{a_3}{3r_0^2}(x^3-3xy^2) - \frac{b_3}{3r_0^2}(3x^2y-y^3)$$

$$+ \frac{a_4}{4r_0^3}(x^4-6x^2y^2+y^4) - \frac{b_4}{r_0^3}(x^3y-xy^3) \pm \dots\Big]$$

⇒ vector potential:

$$A_s(x,y) = \mathrm{Re}(\tilde{A}) = -B_{main}\Big[b_1 x + a_1 y + \frac{b_2}{2r_0}(x^2-y^2) + \frac{a_2}{r_0}xy$$

$$+ \frac{b_3}{3r_0^2}(x^3-3xy^2) + \frac{a_3}{3r_0^2}(3x^2y-y^3) + \frac{b_4}{4r_0^3}(x^4-6x^2y^2+y^4)$$

$$+ \frac{a_4}{r_0^3}(x^3y-xy^3) \pm \dots\Big]$$

General multipole expansion: Cartesian coordinate representation (II)

⇒ from $\vec{B} = -\vec{\nabla}\psi$

$$B_x(x,y) = -\frac{\partial\psi}{\partial x} = B_{main}\Big[-a_1 + \frac{b_2}{r_0}y - \frac{a_2}{r_0}x - \frac{a_3}{r_0^2}(x^2-y^2) + \frac{b_3}{r_0^2}2xy - \frac{a_4}{r_0^3}(x^3-3xy^2) + \frac{b_4}{r_0^3}(3x^2y-y^3) \pm \dots\Big]$$

$$B_y(x,y) = -\frac{\partial\psi}{\partial x} = B_{main}\Big[b_1 + \frac{a_2}{r_0}y + \frac{b_2}{r_0}x + \frac{a_3}{r_0^2}2xy + \frac{b_3}{r_0^2}(x^2-y^2) + \frac{a_4}{r_0^3}(3x^2y-y^3) + \frac{b_4}{r_0^3}(x^3-3xy^2) \pm \dots\Big]$$

⇒ Cartesian representation of multipoles:

- useful if particle motion is described in Cartesian coord.
- $\psi(x,y) = \text{const}$ gives contour of the iron pole shoes in conventional magnets

Field distribution of the first few multipoles

Normal dipole: $n=1$ $\quad b_1 B_{main} = B_{vert}$ (horiz. bending)

$B_\varphi(r,\varphi) = B_{vert}\cos\varphi \qquad B_r(r,\varphi) = B_{vert}\sin\varphi$

$B_x(x,y) = 0 \qquad B_y(x,y) = B_{vert} = \frac{\mu_0 n I}{a}$

Skew dipole: $n=1$ $\quad a_1 B_{main} = B_{horiz}$ (vertically bending)

$B_\varphi(r,\varphi) = B_{horiz}\sin\varphi \qquad B_r(r,\varphi) = -B_{horiz}\cos\varphi$

$B_x(x,y) = -B_{horiz} \qquad B_y(x,y) = 0 \qquad B_{horiz} = \frac{\mu_0 n I}{a}$

Normal quadrupole: $n=2$ $\quad b_2 B_{main} = -g r_0$

$B_\varphi(r,\varphi) = -g r\cos 2\varphi \qquad B_r = -g r\sin 2\varphi$

$B_x(x,y) = -g y \qquad B_y(x,y) = -g x$

$g = \text{gradient} = \frac{2\mu_0 n I}{a^2} \qquad a$: pole radius

Field distribution of the first few multipoles

Skew quadrupole: $n=2$ $\quad a_2 B_{main} = -g r_0$ (rotated by 45°)

$$B_\varphi(r,\varphi) = -g r \sin 2\varphi \qquad B_r(r,\varphi) = g r \cos 2\varphi$$

$$B_x(x,y) = g x \qquad B_y(x,y) = -g y$$

$$g = \frac{2\mu_0 n I}{a^2}$$

normal sextupole: $n=3$ $\quad b_3 B_{main} = \frac{1}{2} g' r_0^2$

$$B_\varphi(r,\varphi) = \frac{1}{2} g' r^2 \cos 3\varphi \qquad B_r(r,\varphi) = \frac{1}{2} g' r^2 \sin 3\varphi$$

$$B_x(x,y) = g' xy \qquad B_y(x,y) = \frac{1}{2} g' (x^2 - y^2)$$

$$g' = 6\mu_0 n I / a^3$$

normal 2n pole: $\quad b_n \cdot B_{main} = \frac{1}{(n-1)!} g_n r_0^{n-1}$

$$B_\varphi(r,\varphi) = \frac{1}{(n-1)!} g_n r^{n-1} \cos n\varphi \qquad B_r(r,\varphi) = \frac{1}{(n-1)!} g_n r^{n-1} \sin n\varphi$$

$$g_n = \frac{\mu_0 n I}{a^n} (n-1)!\, n \qquad \Psi_n(r,\varphi) = -\frac{\mu_0 m I}{a^n} r^n \sin n\varphi$$

m = # of turns in coil

Higher order Multipoles

Multipole strength: $$k_n = \frac{q}{p} \partial_x^{n-1} B_y \Big|_{x,y=0}$$ units: $\frac{1}{m^{n+1}}$

Names: dipole, quadrupole, sextupole, octupole, decapole, duodecapole, …

Higher order multipoles come from

- Field errors in magnets
- Magnetized materials
- From multipole magnets that compensate such erroneous fields
- To compensate nonlinear effects of other magnets
- To stabilize the motion of many particle systems
- To stabilize the nonlinear motion of individual particles

2.4 Superconducting magnets

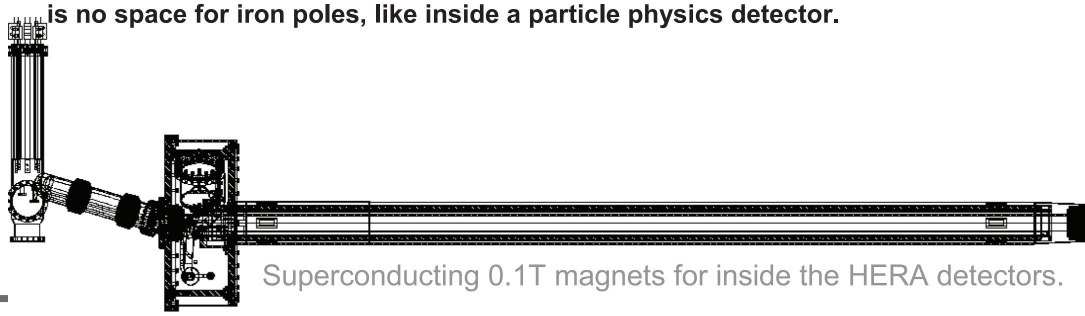
Superconducting Magnets

Conventional iron core magnets: Above 1.5 T the field from the bare coils dominate over the magnetization of the iron.
But Cu wires cannot create much filed without iron poles:
5T at 5cm distance from a 3cm wire would require a current density of

$$j = \frac{I}{d^2} = \frac{1}{d^2}\frac{2\pi\, r\, B}{\mu_0} = 1389\frac{\mathrm{A}}{\mathrm{mm}^2}$$

Normal-conducting Cu can only support about 100A/mm^2.
Solution: Superconducting cables!

- **Superconducting cables routinely allow current densities of 1500A/mm^2 at 4.6 K and 6T. Materials used are usually Nb alloys, e.g. NbTi ($T_c \approx$ 10K), Nb_3Ti or Nb_3Sn.**
- **Superconducting: dipole fields >6T, quadrupole gradients >100 T/m**
- **Superconducting magnets are not only used for strong fields but also when there is no space for iron poles, like inside a particle physics detector.**



Superconducting 0.1T magnets for inside the HERA detectors.

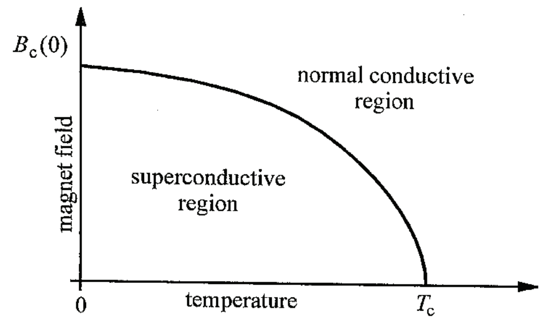
Superconducting Magnets

Problems:

- Superconductivity brakes down for too large fields (quenching) Critical field:

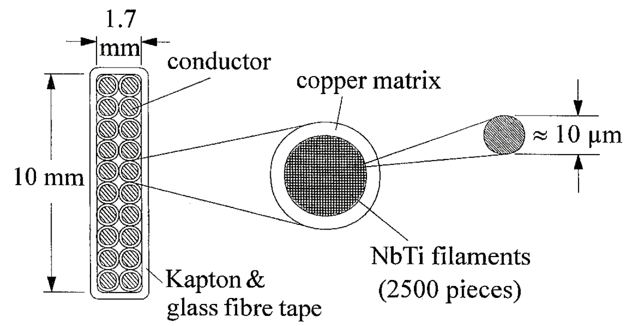
$$B_c(T) = B_c(0)\left[1-\left(\frac{T}{T_c}\right)^2\right]$$

- Due to the Meissner-Ochsenfeld effect magnetic field is pushed out of superconductor.
  -> current can only flow in a thin surface layer (~ 50 nm deep).



Remedy:

- Superconducting cable consists of many very thin filaments (about 10µm) to maximize surface area.



Air-coil Multipoles (I)

Straight wire at the origin: $\vec{\nabla}\times\vec{B} = \mu_0\,\vec{j} \quad \Rightarrow \quad \vec{B}(r) = \frac{\mu_0\, I}{2\pi\, r}\vec{e}_\varphi = \frac{\mu_0\, I}{2\pi\, r}\begin{pmatrix} -y \\ x \end{pmatrix}$

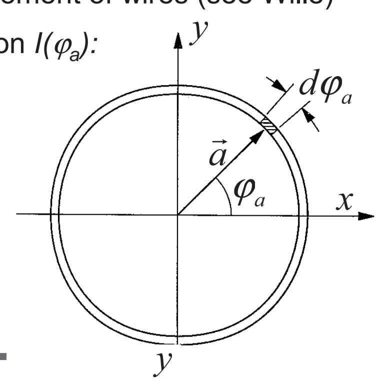
Wire at $\vec{a}$: $\vec{B}(x,y) = \frac{\mu_0\, I}{2\pi\,(\vec{r}-\vec{a})^2}\begin{pmatrix} -[y-a_y] \\ x-a_x \end{pmatrix}$

Potential: $\vec{B}(x,y) = -\vec{\nabla}\Psi$ $\rightarrow$ potential of single wire

Use principle of superposition to get potential of arrangement of wires (see Wille)

For cylindrical current shell with given current distribution $I(\varphi_a)$:

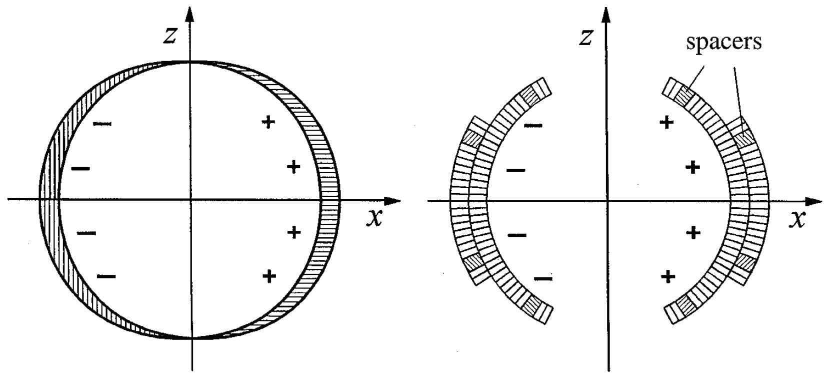
$$\Psi(r,\varphi) = \int_0^{2\pi} \Psi_{wire}\frac{dI}{d\varphi_a}\,d\varphi_a$$



Air-coil Multipoles (II)

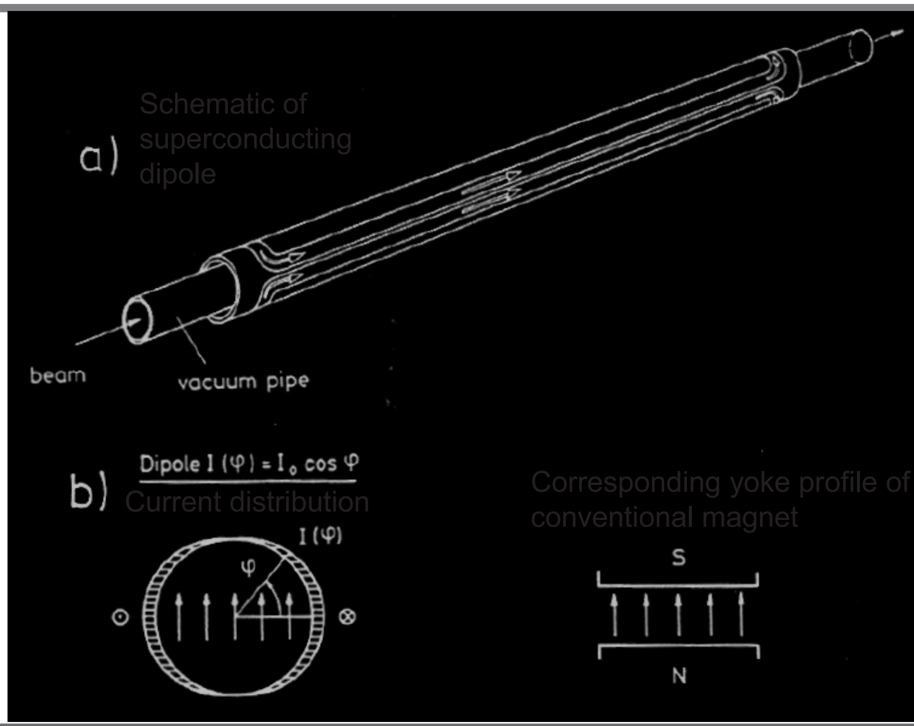
Can create a given multipole by an arrangement of wires:

$$\Psi_n(r,\varphi) = -\frac{\mu_0 \hat{I}}{a^n} r^n \sin(n\varphi) \qquad \text{if} \qquad \boxed{I(\varphi_a) = \hat{I}\cos n\varphi_a}$$



Ideal multipole (diople) Approximate multipole

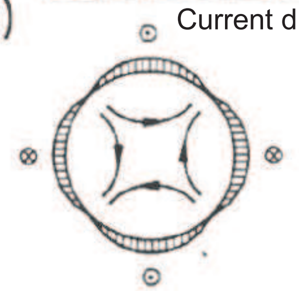
Current distribution for an ideal air-core magnet



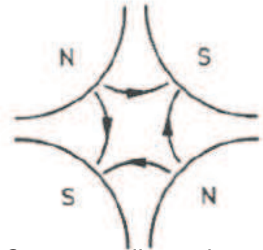
Current distribution for an ideal air-core magnet

c) Quadrupole $I(\varphi) = I_0 \cos 2\varphi$

Current distribution

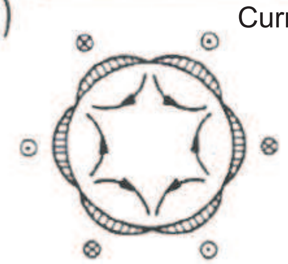


Corresponding yoke profile of conventional magnet

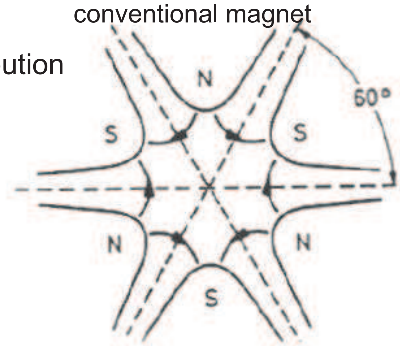


d) Sextupole $I(\varphi) = I_0 \cos 3\varphi$

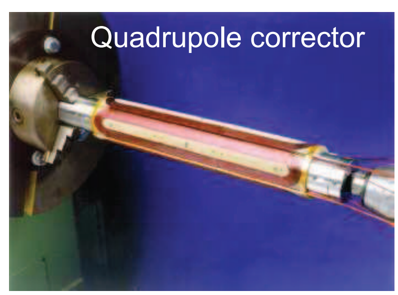
Current distribution

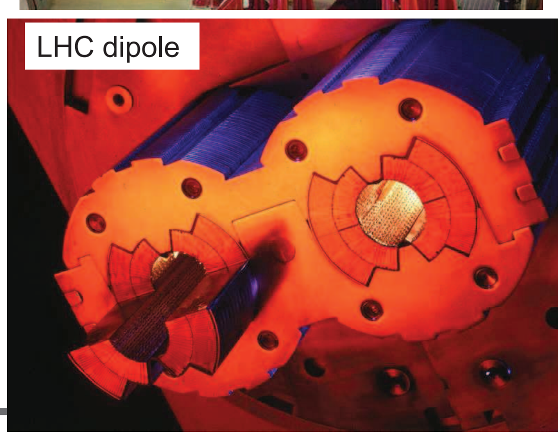


Corresponding yoke profile of conventional magnet



Real Air-coil Multipoles

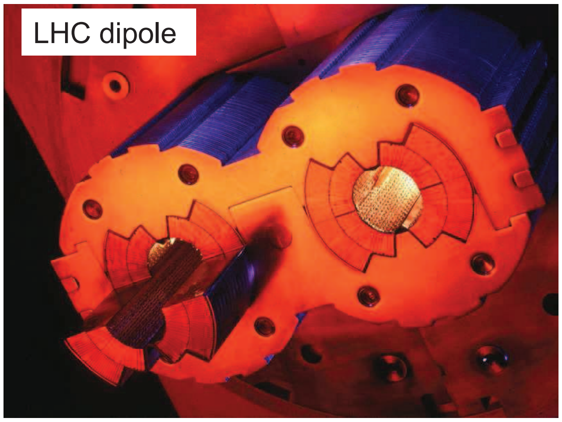


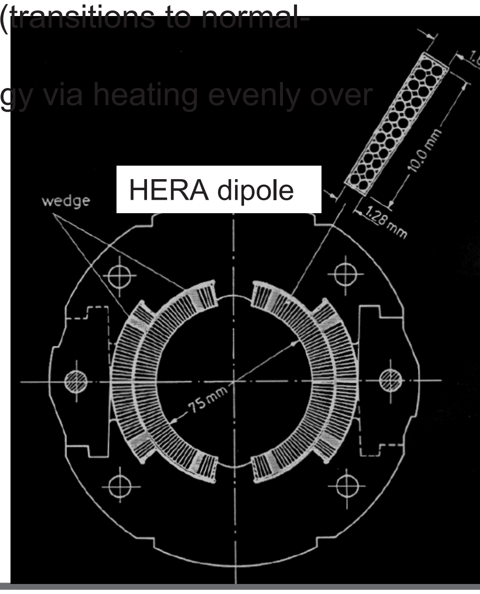


- Large magnetic fields >2T required for protons/ ions at high energies >100 GeV (RHIC, HERA, LHC)

Real Air-coil Multipoles

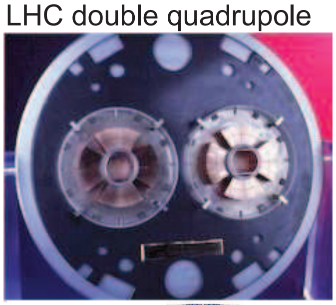
- Forces of >10^6 N/m between conductors!
  -> copper sleeve around cords; aluminum clamps hold current shells together
  - If field too high: Magnet quenches (transitions to normal-conducting)
  -> Important to distribute stored energy via heating evenly over entire coil!





Special SC Air-coil Magnets

LHC double quadrupole



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Accuracy