2. Charged particles in magnetic fields

2.2 Magnets (synchrotron magnet)

2.3 Multipole expansion

2.4 Superconducting magnets

2.2 Magnets

combined function: synchrotron magnet
Combined function magnet: synchrotron magnet

\[ \Rightarrow \text{portion of quadrupole which} \]
\[ \Rightarrow \text{shifted horizontally} \]
\[ \Rightarrow d = \frac{B_0}{g} \text{ from the axis} \]
\[ \text{potential: } \psi(x, z) = -g z (x + d) \]
\[ = -B_0 z - g x z \]
\[ \Rightarrow B = -\nabla \psi = \left( \frac{g^2}{B_0 + g x} \right)^{\text{quadrupole}} + \text{vertical dipole} \]
\[ \Rightarrow \text{field in } dx \approx \]
\[ n = -\frac{R}{B_0} \left( \frac{\partial B_0}{\partial x} \right) \bigg|_{x = R} = -\frac{R}{B_0} g = -\frac{k R^2}{\frac{g}{p}} \]
\[ k = \frac{g}{p} \]

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2.3 Multipole expansion

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General multipole expansion

- Assume \( \vec{B} \) has transverse components (\( B_x, B_y \)) only
- Neglect fringe fields at magnet ends
- \( \vec{B} \cdot \vec{A} = 0 \) \( \Rightarrow \) vector potential \( \vec{A} \) exist, such that
  \[ \vec{B} = \vec{\nabla} \times \vec{A} \]

Note: \( \vec{A} = \left( \begin{array}{c} A_x \\ A_y \\ A_z \\ \end{array} \right) \) \( \Rightarrow \) \( \vec{B} = \left( \begin{array}{c} B_x \\ B_y \\ 0 \end{array} \right) \)

- In vacuum (inside magnet)
  \( \vec{\nabla} \times \vec{B} = 0 \) \( \Rightarrow \) scalar potential \( \psi \) exists, such that
  \[ \vec{B} = -\vec{\nabla} \psi \]

- Combines:
  \( B_x = -\frac{\partial \psi}{\partial x} = \frac{\partial A_z}{\partial y} \)
  \( B_y = -\frac{\partial \psi}{\partial y} = -\frac{\partial A_x}{\partial x} \)

\( \Rightarrow \) There are the Cauchy–Riemann conditions for the real and imaginary part of an analytic function \( \psi \)

General multipole expansion: Cylindrical coordinate representation (I)

\( \Rightarrow \) Define complex function of \( z = x + iy \)
  \[ \hat{A}(z) = A_x(x, y) + i \psi(x, y) \] \( \Rightarrow \) analytic function!

\( \Rightarrow \) Can be expanded into a power series:
  \[ \hat{A}(z) = \sum_{n=0}^{\infty} A_n z^n \]
  \( \text{with} \quad A_n = A_n(r, \phi) \) \( \text{real constants} \)

- Cylindrical coordinates:
  \[ x = r \cos \phi \quad y = r \sin \phi \]

\( \Rightarrow \) \( z^n \rightarrow r^n e^{in\phi} = r^n (\cos(n\phi) + i\sin(n\phi)) \)
General multipole expansion: Cylindrical coordinate representation (II)

$$= \text{for scalar potential:}$$
$$\psi (r, \theta) = (\hat{\lambda}) = \sum_{n=0}^{\infty} \left( \frac{\mu_n \cos n\theta + \lambda_n \sin n\theta}{r^n} \right)$$

$$= \text{vector potential:}$$
$$A_s (r, \theta) = \mathbf{e}_r (\hat{A}) = \sum_{n=1}^{\infty} \left( \frac{\lambda_n \cos n\theta - \mu_n \sin n\theta}{r^{n-1}} \right)$$

$$B = -\nabla \psi$$
$$B_\rho = -\frac{1}{r} \frac{\partial \psi}{\partial \rho} = -\sum_{n=1}^{\infty} \left( \frac{\lambda_n \cos n\theta - \mu_n \sin n\theta}{r^{n-1}} \right)$$
$$B_\theta = -\frac{\partial \psi}{\partial \theta} = -\sum_{n=1}^{\infty} \left( \frac{\mu_n \cos n\theta + \lambda_n \sin n\theta}{r^{n-1}} \right)$$

General multipole expansion: Cylindrical coordinate representation (III)

define:
- reference radius: \( r_0 \) (e.g., beam pipe radius)
- (magnetic field of main field) = \( B_{\text{main}} \)
- "normal" multipole coefficients:
  \( b_n = -\frac{n \lambda_n}{B_{\text{main}}} r_0^{n-1} \)
- "skew" multipole coefficients:
  \( a_n = \frac{n \mu_n}{B_{\text{main}}} r_0^{n-1} \)

$$= \text{multipole expansion:}$$
$$\psi (r, \theta) = -B_{\text{main}} r_0 \sum_{n=0}^{\infty} \left( \frac{a_n}{n} \cos n\theta + \frac{b_n}{n} \sin n\theta \right) \left( \frac{r}{r_0} \right)^n$$
$$A_s (r, \theta) = -B_{\text{main}} r_0 \sum_{n=1}^{\infty} \left( \frac{b_n}{n} \cos n\theta + \frac{a_n}{n} \sin n\theta \right) \left( \frac{r}{r_0} \right)^n$$
General multipole expansion: Cylindrical coordinate representation (IV)

\[ B_\phi (\gamma, \rho) = B_{\text{main}} \sum_{n=1}^{\infty} \left( b_n \cos n \phi + a_n \sin n \phi \right) \left( \frac{\rho}{\rho_0} \right)^{n-1} \]

\[ B_\rho (\gamma, \rho) = B_{\text{main}} \sum_{n=1}^{\infty} \left( -a_n \cos n \phi + b_n \sin n \phi \right) \left( \frac{\rho}{\rho_0} \right)^{n-1} \]

\[ = \) for ideal "normal" 2m - pole magnet:

\[ b_n = 1 \text{ for } m = n , \quad a_n, b_n = 0 \text{ else} \]

\[ m = 1 \text{ Dipole} \quad m = 4 \text{ Octupole} \]

\[ m = 2 \text{ Quadrupole} \quad m = 5 \text{ Decapole} \]

\[ m = 3 \text{ Sextupole} \quad m = 6 \text{ Dodecapole, 12-pole} \]

\[ = \) consider: \[ B_\phi + i B_\rho \]

\[ B_\phi + i B_\rho = B_{\text{main}} \sum_{n=1}^{\infty} \left( \frac{\rho}{\rho_0} \right)^{n-1} \left[ b_n (\cos n \phi + i \sin n \phi) - i a_n (\cos n \phi + i \sin n \phi) \right] \]

\[ = B_{\text{main}} \sum_{n=1}^{\infty} \left( \frac{\rho}{\rho_0} \right)^{n-1} \left( b_n - i a_n \right) e^{in\phi} \]

\[ = \) Then:

\[ |B_n| = \sqrt{\left( B_\phi^2 + B_\rho^2 \right)} = B_{\text{main}} \left( \frac{\rho}{\rho_0} \right)^{n-1} \sqrt{a_n^2 + b_n^2} \]

\[ = \) magnitude of 2m - pole field does not depend on \( \phi \) and scales \( \rho^{n-1} \)

\[ = \) at reference radius \( \rho = \rho_0 \) : multipole coefficients \( a_n \) and \( b_n \) are the relative field contributions of the \( n \)th multipole
Cylindrical coordinate representation

- Useful for air-core magnet design (see, e.g., megnet)
- Measurement of multipole components with a rotating coil in the field:
  - $n$th Fourier component of the induced voltage $\propto \sqrt{a_n^2 + b_n^2}$, phase related to $a_n/b_n$
  - Good magnet: unwanted multipole coefficients $\leq 10^{-4}$
- Symmetry:
  - $q$-pole with perfect constructive $\Rightarrow$ only odd harmonics of $q$-pole are allowed
  - $2n$-pole magnet $\Rightarrow$ transforms into a show $2n$-pole magnet if rotated by $\pi/2n$

General multipole expansion: Cartesian coordinate representation (I)

$$\hat{A}(x,y) = A_0(x,y) + i\Psi(x,y) = \sum_{n=0}^{\infty} \hat{A}_n \hat{Z}^n$$

$$= \sum_{n=0}^{\infty} (\lambda_n + i\mu_n) (x + iy)^n$$

=) Scalar pole trial:
$$\Psi(x,y) = J_\gamma (\sqrt{A}) = B_{\text{main}} \left[ a \sqrt{1 - b^2} \gamma + \frac{a_y}{r_0} (x^2 - y^2) - \frac{b_y}{r_0^2} (3x^2 - y^2) \right]$$
$$+ \frac{a_2}{r_0^3} (x^2 - 3x y^2) - \frac{b_2}{r_0^3} (3x^2 - 3y^2) - \frac{a_4}{r_0^5} (x^4 - 6x^2 y^2 + y^4) - \frac{b_4}{r_0^5} (x^2 y^2 - y^4) + \ldots$$

=) Vector pole trial:
$$A_0(x,y) = \text{Re} (\hat{A}) = -B_{\text{main}} \left[ b \hat{z} + a \hat{x} + \frac{b_2}{2r_0} (x z - y z^2) + \frac{a_2}{2r_0} (3x z^2 - y z^3) \right]$$
$$+ \frac{b}{3r_0^3} (x^2 z^2 - 3x y z^2) + \frac{a}{3r_0^3} (3x^2 y^2 - y^3) + \frac{b y}{r_0^5} (x^6 y^4 - 6x^4 y^4 + x^2 y^6)$$
$$+ \frac{a y}{r_0^5} (x^4 y^2 - x y^4) + \ldots$$
General multipole expansion: Cartesian coordinate representation (II)

\[ B_x(x,y) = -\frac{\partial \Psi}{\partial x} = B_{\text{main}} \left[ -a + \frac{b^2}{r_0} y - \frac{a^2}{r_0} x - \frac{a^2}{r_0^2} (x^2 + y^2) \right. \\
+ \left. \frac{b^3}{r_0^2} 2x y + \frac{a^2 y}{r_0^3} (x^3 - 3x y^2) + \frac{b y}{r_0^2} (x^3 - 3x y^2) \right] \]

\[ B_y(x,y) = -\frac{\partial \Psi}{\partial y} = B_{\text{main}} \left[ b + \frac{a^2}{r_0} y + \frac{b^2}{r_0} x + \frac{a^2}{r_0^2} 2x y \\
+ \frac{b^3}{r_0^2} (x^2 + y^2) + \frac{a y}{r_0^3} (x^3 - 3x y^2) + \frac{b y}{r_0^2} (x^3 - 3x y^2) \right] \]

=) Cartesian representation of multipoles:
- Useful if particle motion is described in Cartesian coordinates.
- \( \Psi(x,y) = \text{const} \) gives contours of the iron pole
- \( \text{shoes in conventional magnets} \)

---

Field distribution of the first few multipoles

**Normal dipole**: \( n = 1 \)
\[ b, B_{\text{main}} = B_{\text{vert}} \text{ (horizontal)} \]
\[ B_x(y) = B_{\text{vert}} \cos \varphi \]
\[ B_y(x,y) = 0 \]

**Skew dipole**: \( n = 1 \)
\[ a, B_{\text{main}} = B_{\text{horiz}} \text{ (vertically horizontal)} \]
\[ B_y(y) = B_{\text{horiz}} \sin \varphi \]
\[ B_x(x,y) = -B_{\text{horiz}} \]

**Normal quadrupole**: \( n = 2 \)
\[ b_2 B_{\text{main}} = -g r_0 \]
\[ B_x(y) = -g y \cos 2\varphi \]
\[ B_y(x,y) = -g y \]
\[ g = \frac{2 \mu_0 n I}{a^2} \text{ (pole radius)} \]
Field distribution of the first few multipoles

<table>
<thead>
<tr>
<th>Multipole</th>
<th>Polynomial</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skew quadrupole</td>
<td>$n=2$</td>
<td>$B_y (r, \phi) = -g r \sin 2\phi$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$B_x (r, \phi) = g x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$g = \frac{2 \mu_0 n I}{a^3}$</td>
</tr>
<tr>
<td>Normal 2nd multipole</td>
<td>$n=3$</td>
<td>$B_y (r, \phi) = \frac{1}{2} g' r^2 \cos 3\phi$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$B_x (r, \phi) = g' x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$g' = \frac{6 \mu_0 n I}{a^3}$</td>
</tr>
<tr>
<td>Normal 1st multipole</td>
<td>$n=4$</td>
<td>$\mathbf{B} \cdot \mathbf{B_{main}} = \frac{1}{(n-1)!} \sum_{k=1}^{n} g_{2k} r^{-k}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$B_y (r, \phi) = \frac{1}{(n-1)!} g_{2n} r^{-n} \cos n\phi$</td>
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<td></td>
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<td>$B_x (r, \phi) = \frac{1}{(n-1)!} g_{2n} r^{-n} \sin n\phi$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$g_{2n} = \frac{\mu_0 n I (n-1)!}{a^{n+1}}$</td>
</tr>
</tbody>
</table>

Higher order Multipoles

Multipole strength: $k_n = \frac{q}{p} \frac{\partial^{n-1}}{x^{n-1}} B_y \bigg|_{x,y=0}$

Names: dipole, quadrupole, sextupole, octupole, decapole, duodecapole, ...

Higher order multipoles come from
- Field errors in magnets
- Magnetized materials
- From multipole magnets that compensate such erroneous fields
- To compensate nonlinear effects of other magnets
- To stabilize the motion of many particle systems
- To stabilize the nonlinear motion of individual particles
2.4 Superconducting magnets

Superconducting Magnets

Conventional iron core magnets: Above 1.5 T the field from the bare coils dominate over the magnetization of the iron. But Cu wires cannot create much field without iron poles: 5T at 5cm distance from a 3cm wire would require a current density of

\[ j = \frac{I}{d^2} = \frac{1}{d^2} \frac{2\pi r B}{\mu_0} = 1389 \frac{A}{mm^2} \]

Normal-conducting Cu can only support about 100A/mm².
Solution: Superconducting cables!
• Superconducting cables routinely allow current densities of 1500A/mm² at 4.6 K and 6T. Materials used are usually Nb alloys, e.g. NbTi (T_c = 10K), Nb₃Ti or Nb₃Sn.
• Superconducting: dipole fields >6T, quadrupole gradients >100 T/m
• Superconducting magnets are not only used for strong fields but also when there is no space for iron poles, like inside a particle physics detector.

Superconducting 0.1T magnets for inside the HERA detectors.
Superconducting Magnets

Problems:
- Superconductivity brakes down for too large fields (quenching)

Critical field:
\[ B_c(T) = B_c(0) \left[ 1 - \left( \frac{T}{T_c} \right)^2 \right] \]
- Due to the Meissner-Ochsenfeld effect magnetic field is pushed out of superconductor.
  \( \Rightarrow \) current can only flow in a thin surface layer (~ 50 nm deep).

Remedy:
- Superconducting cable consists of many very thin filaments (about 10µm) to maximize surface area.

Air-coil Multipoles (I)

Straight wire at the origin:
\[ \vec{V} \times \vec{B} = \mu_0 \vec{j} \quad \Rightarrow \quad \vec{B}(r) = \frac{\mu_0 I}{2\pi r} \hat{\phi} = \frac{\mu_0 I}{2\pi r} \begin{pmatrix} -y \\ x \end{pmatrix} \]

Wire at \( \vec{a} \):
\[ \vec{B}(x, y) = \frac{\mu_0 I}{2\pi (\vec{r} - \vec{a})^2} \begin{pmatrix} -[y - a_y] \\ x - a_x \end{pmatrix} \]

Potential:
\[ \vec{B}(x, y) = -\vec{\nabla} \Psi \quad \Rightarrow \quad \text{potential of single wire} \]

Use principle of superposition to get potential of arrangement of wires (see Wille)
For cylindrical current shell with given current distribution \( I(\phi_a) \):
\[ \Psi(r, \phi) = \int_0^{2\pi} \Psi_{\text{wire}} \frac{dI}{d\phi_a} \ d\phi_a \]
Air-coil Multipoles (II)

Can create a given multipole by an arrangement of wires:

\[ \Psi_n(r, \varphi) = -\frac{\mu_0 I}{a^n} r^n \sin(n \varphi) \quad \text{if} \quad I(\varphi_a) = \hat{I} \cos n \varphi_a \]

Ideal multipole (diople)  Approximate multipole

Current distribution for an ideal air-core magnet

Schematic of superconducting dipole

Dipole \( I(\varphi) = I_x \cos \varphi \)

Corresponding yoke profile of conventional magnet
Current distribution for an ideal air-core magnet

Corresponding yoke profile of conventional magnet

Corresponding yoke profile of conventional magnet

Real Air-coil Multipoles

• Large magnetic fields >2T required for protons/ions at high energies >100 GeV (RHIC, HERA, LHC)
Real Air-coil Multipoles

- Forces of >10^6 N/m between conductors!
  -> copper sleeve around cords; aluminum clamps hold current shells together
- If field too high: Magnet quenches (transitions to normal conducting)
  -> Important to distribute stored energy via heating evenly over entire coil!

LHC dipole

HERA dipole

Special SC Air-coil Magnets

LHC double quadrupole

1e-5 Accuracy

RHIC Siberian Snake dipole (2 pi rotation)