3. Linear transverse beam optics

3.2 General solution of the equation of motion

3.3 Building blocks for beam transport lines

Matrix Formulation

\[ \tilde{z}_s = M(s) \tilde{z}_0 \]

Matrix solution of the starting condition \( \tilde{z}(0) = \tilde{z}_0 \)

\[ \tilde{z} = M_{\text{bend}}(L_4) M_{\text{drift}}(L_3) M_{\text{quad}}(L_2) M_{\text{drift}}(L_1) \tilde{z}_0 \]

\[ \tilde{z} = M_{\text{drift}}(L_3) M_{\text{quad}}(L_2) M_{\text{drift}}(L_1) \tilde{z}_0 \]

\[ \tilde{z} = M_{\text{quad}}(L_2) M_{\text{drift}}(L_1) \tilde{z}_0 \]

\[ \tilde{z} = M_{\text{drift}}(L_1) \tilde{z}_0 \]
Properties of the transformation matrix (I)

Consider a linear homogeneous differential equation of second order:
\[ u'' + V(s) u' + w(s) u = 0 \]

\( \Rightarrow \) rules describing the properties of the solution

(a) there is only one solution which meets the initial conditions \( u(s_0) = u_0, \ u'(s_0) = u'_0 \) at \( s = s_0 \)

(b) \( u(s) \) is also a solution if \( u(s) \) is a solution and \( c \cdot u(s) \)

(c) if \( u_1(s) \) and \( u_2(s) \) are two linearly independent solutions, any linear combination of these two is also a solution.

Properties of the transformation matrix (II)

(d) for two linearly independent solutions \( u_1(s) \) and \( u_2(s) \)

Wronskian determinant:
\[ W = \begin{vmatrix} u_1(s) & u_2(s) \\ u_1'(s) & u_2'(s) \end{vmatrix} = u_1 u_2' - u_2 u_1' \neq 0 \]

with \( u_1'' + V(s) u_1' + w(s) u_1 = 0 \) \( \Leftrightarrow u_1 \)
\( u_2'' + V(s) u_2' + w(s) u_2 = 0 \) \( \Leftrightarrow u_2 \)

combine: \( (u_1 u_2'' - u_2 u_1'') + V(s) (u_1 u_2' - u_2 u_1') = 0 \)

\[ \Rightarrow \frac{dw}{ds} = u_1 u_2'' - u_2 u_1'' \Leftrightarrow V(s) W \]

integrate: \( W(s) = W_0 e^{\int_{s_0}^{s} V(s) \ ds} \)
Properties of the transformation matrix (III)

Conclusion: 
\[ W(s) = W_0 = \text{const} \]  
\( \left( \frac{dW}{ds} = 0 \right) \)

From initial conditions:
\[ \zeta(0) = 1 \quad \zeta'(0) = 0 \quad s(0) = 0 \quad s'(0) = 1 \]

\[ W = \left| \begin{array}{cc} \zeta'(s) & s(s) \\ \zeta''(s) & s'(s) \end{array} \right| = W_0 = c_0 s' - c_0' s = 1 
\]
\( (\text{if } V(s) = 0) \)

Solution of the inhomogeneous equation of motion (I)

Equation of motion: 
\[ u'' + R(s) u = p(s) \]

General solution:
\[ u(s) = \zeta'(s) a + s(s) b + P(s) \]

- General solution of the homogeneous equation with parameters determined by initial parameters
- A particular solution of the inhomogeneous equation

- Particular solution \( P(s) \)

Can be found from 
\[ P(s) = \int p(s) G(s, s') ds' \]

Suitable Green's function can be constructed from the principle solutions of the homogeneous equation.
Solution of the inhomogeneous equation of motion (II)

\[ G(s, \tilde{s}) = s'(s) C'(\tilde{s}) - C'(s) S'(\tilde{s}) \]

\[ \Rightarrow \text{ for the particular solution:} \]

\[ P(s) = s'(s) \int_{0}^{s} p(s') C'(s') ds' - C'(s) \int_{0}^{s} p(s') S(s') ds' \]

**Proof:**

\[ P'(s) = \int_{0}^{s} p(s') C'(s') ds' + s'(s) C'(s) p(s) \]

\[ - C'(s) \int_{0}^{s} \tilde{p}(s') S'(s') ds' - C'(s) S(s) p(s) \]

\[ \Rightarrow \]

\[ P''(s) = \int_{0}^{s} p(s') C'(s') ds' + \int_{0}^{s} \tilde{p}(s') C'(s') ds' - C'(s) \int_{0}^{s} \tilde{p}(s') S'(s') ds' \]

\[ - C'(s) S(s) p(s) = \int_{0}^{s} \tilde{p}(s') C'(s') ds' - C'(s) \int_{0}^{s} \tilde{p}(s') S'(s') ds' \]

\[ \omega = \frac{c}{s^2 \omega^2} \Rightarrow C' = 1 \]

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Solution of the inhomogeneous equation of motion (III)

with \( s'' + \omega s = 0 \quad C'' + \omega C = 0 \)

\[ \Rightarrow \]

\[ P''(s) + \omega p(s) P(s) = p(s) \]

\[ \Rightarrow P(s) \text{ is a particular solution of the inhomogeneous equation} \]
Dispersion Function (I)

most important example: perturbation term from
momentum error: \( \delta = 1 - \frac{P}{P_0} \ll 1 \)
( chromatic error )

\[ u'' + \lambda(s) u = \frac{l}{s(s)} \delta \]

\( \Rightarrow \) deviation in path from path of particle
with nominal energy

\[ u(s) = G'(s) a + \int^s s'(s) b + P(s) \]

now, since \( P(s) \propto \delta \Rightarrow P(s) \propto \delta \)

\( \Rightarrow \) define normalized dispersion function:

\[ D(s) = \frac{P(s)}{\delta} \]

Dispersion Function (II)

which is a particular solution of the homogeneous

equation:

\[ D''(s) + \lambda(s) D(s) = \frac{1}{P(s)} \]

From above:

\[ D(s) = \int^{s} \frac{1}{s(s)} G'(s) \frac{d\tilde{s}}{l(s)} - \int^{s} \frac{1}{s(s)} s'(s) \frac{d\tilde{s}}{l(s)} \]
General solution with chromatic correction

\[ u(s) = \zeta(s) u(s_0) + \phi(s) u'(s_0) + \delta S'(s) \]
\[ u'(s) = \zeta'(s) u(s) + \phi'(s) u'(s_0) + \delta D(s) \]

With initial conditions at \( s = s_0 \):
\[ \zeta(s) = 1 \quad \zeta'(s_0) = 0 \quad D(s) = 0 \quad D'(s_0) = 0 \]
\[ u'(s) = 0 \quad u'(s_0) = 0 \]

\[ \begin{bmatrix} u(s) \\ u'(s) \\ \delta \end{bmatrix} = \begin{bmatrix} \zeta(s) & \phi(s) & D(s) \\ \zeta'(s) & \phi'(s) & D'(s) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u(s_0) \\ u'(s_0) \\ \delta \end{bmatrix} \]

3.3 Building blocks for beam transport lines

general focusing

dispersion

first order achromat

first order isochronous
(a) General focusing ($\delta=0$)

$$f = - \frac{u_0}{u'(s)}$$

$u_0 > 0, u'_0 > 0$

Principal plane

Focus point

$$u'(s) = \zeta'(s) u_0, \quad u(s) = \zeta(s) u_0$$

$\Rightarrow \text{at focal point: } u(S_f) = 0 \Rightarrow \zeta'(S_f) = 0$

$\Rightarrow \text{focal length of focusing: } f^{-1} = \zeta'(s)$

$$\Rightarrow u(S) = \zeta(s) u_0$$

More on general focusing ($\delta=0$)

$\Rightarrow$ Magnification of focusing system:

$$\zeta'(S_f) = 0$$

$$S(S_i) = 0$$

$$H_i$$ at image point: $H_i = \{\zeta(S_i)\} H_0$

$$u(S_i) = \zeta(S_i) H_0$$

$$= \text{Magnification: } M = |\zeta'(S_i)|$$

$$(\text{large } \zeta(S_i) \Rightarrow \text{large magnification})$$
(b) Zero Dispersion point: $D(s)=0$ at $s=s_d$

\[ D(s) = S(s) \left( \int_0^s \frac{1}{S(s')} C'(s') ds' - \int_0^s \frac{1}{S(s')} S'(s') ds' \right) \]

\[ \Rightarrow \text{ only dipole fields can change dispersion} \]

\[ \begin{align*}
  I_c (s) &= \int_0^s \frac{1}{S(s')} C'(s') ds' \\
  I_s (s) &= \int_0^s \frac{1}{S(s')} S'(s') ds'
\end{align*} \]

\[ \Rightarrow \text{ for } D(s) = 0 \quad \text{at} \quad s = s_d, \quad \text{need:} \]

\[ \frac{S'(s_d)}{C'(s_d)} = \frac{I_s (s_d)}{I_c (s_d)} \quad \text{adjust focusing structure accordingly} \]

(c) First order achromatic lattice

\[ D(s_d) = S'(s_d) I_c (s_d) - C'(s_d) I_s (s_d) = 0 \]

\[ D'(s_d) = S''(s_d) I_c (s_d) - C''(s_d) I_s (s_d) = 0 \]

\[ \Rightarrow \text{ solve for } I_c \text{ and } I_s: \]

\[ \begin{align*}
  [C'(s_d) S'(s_d) - C'(s_d) S'(s_d)] I_c (s_d) &= 0 \\
  [C'(s_d) S'(s_d) - C'(s_d) S'(s_d)] I_s (s_d) &= 0
\end{align*} \]

\[ = 1 \quad \text{since} \quad |W| = 1 \]
First order achromat

\[\Rightarrow\] for \( D(s_d) = 0 \) and \( D'(s_d) = 0 \)

Conditions:

\[ I_c(s_d) = \int_{0}^{s_d} \frac{1}{p(s_\tilde{\xi})} G'(s_\tilde{\xi}) d\tilde{\xi} = 0 \] for first order

\[ I_s(s_d) = \int_{0}^{s_d} \frac{1}{p(s_\tilde{\xi})} S'(s_\tilde{\xi}) d\tilde{\xi} = 0 \] achromat