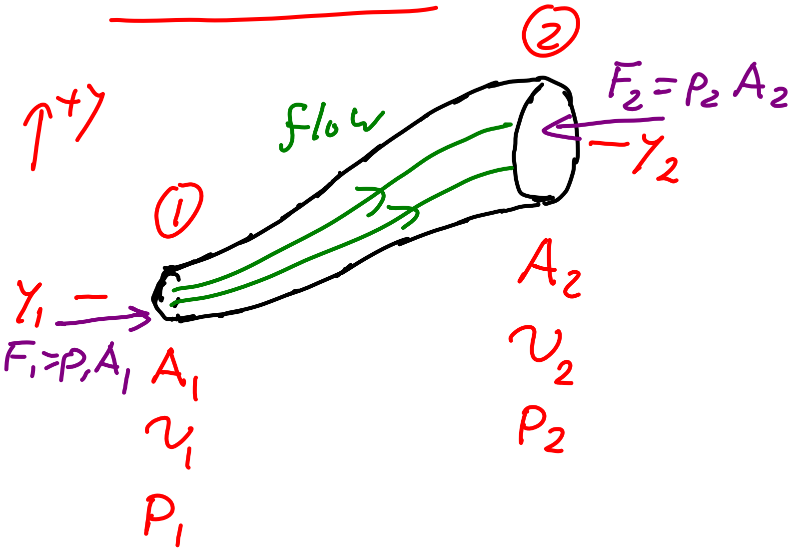
Recap: Ideal Fluids in Motion

Lecture 16

- **Ideal Fluid:** no fluid friction; $\rho = \text{const}$; laminar flow
- **Ideal Flow:**



- Volume flow rate:

$$R = \frac{\Delta \text{Volume passing}}{\Delta t} = \overset{\text{x-sectional area}}{A}\,\underset{\text{speed of flow}}{v}$$

- Continuity:

$$R_1 = R_2 \Rightarrow A_1 v_1 = A_2 v_2$$

- Bernoulli's equation for ideal flow

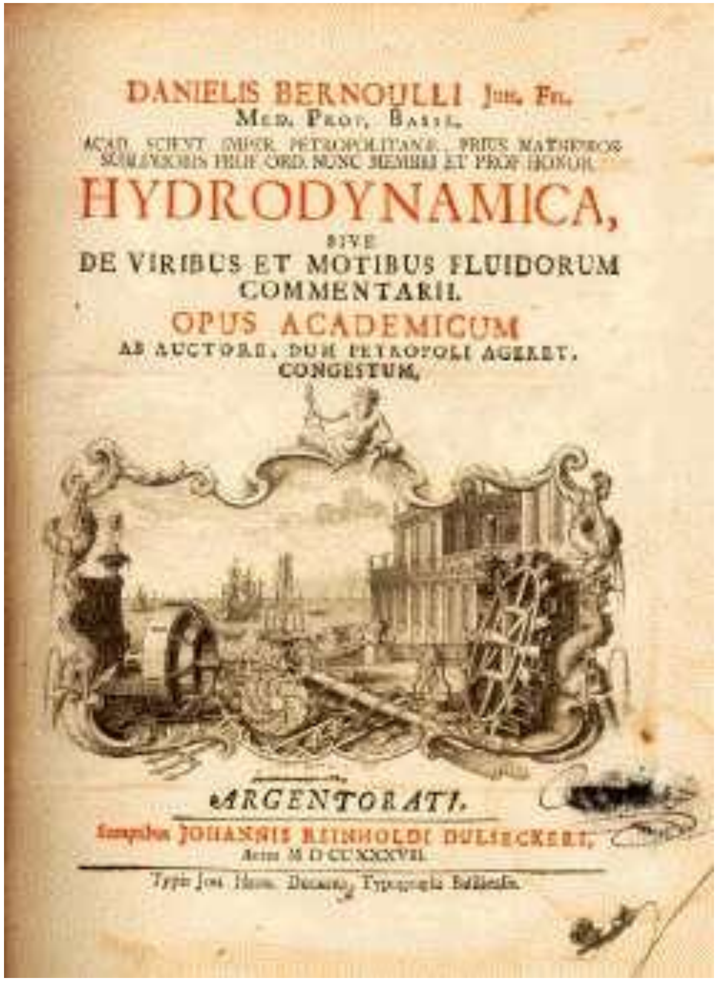
$$\underset{\text{by } \Delta p}{W_{\text{on fluid}}} = \Delta U_g + \Delta K$$

$$\Rightarrow \underbrace{p_1}_{\frac{\text{work by } F_1}{\text{volume}}} + \underbrace{\rho g y_1}_{U_g/\Delta V} + \underbrace{\frac{1}{2}\rho v_1^2}_{K/\Delta V} = p_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2$$

$\Rightarrow$ for flow at const. height: where v is big, p is small!

Today:

- Fluid friction
- Viscous flow though tube
- Viscous and turbulent drag
- Terminal speed

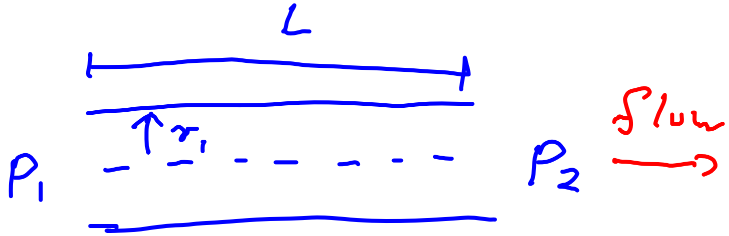

DANIELIS BERNOULLI Joh. Fil.
Med. Prof. Basil.
ACAD. SCIENT. IMPER. PETROPOLITANAE, PRIUS MATHESEOS
SUBLIMIORIS PROF. ORD. NUNC MEMBRI ET PROF. HONOR.
HYDRODYNAMICA,
SIVE
DE VIRIBUS ET MOTIBUS FLUIDORUM
COMMENTARII.
OPUS ACADEMICUM
AB AUCTORE, DUM PETROPOLI AGERET,
CONGESTUM.

ARGENTORATI,
Sumptibus JOHANNIS REINHOLDI DULSECKERI,
Anno M D CC XXXVIII.
Typis Joh. Henr. Deckeri, Typographi Basiliensis.

When blood flows though a blood vessel, it experiences **viscous drag, i.e. friction forces**.

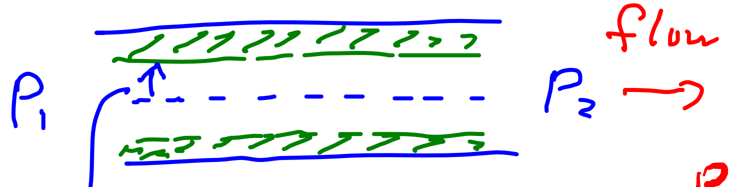
Two **blood vessels** of **equal length** are connected in parallel so that they have **same pressure drop Δp**. The radius of the first vessel is $\mathbf{r_1}$. Due to formation of atheriosclerotic plaques, the radius $\mathbf{r_2}$ of the second vessel has been reduced by 50%.

What is the ratio $\mathbf{R_2/R_1}$ of the **volume flow rates** in the two vessels?



volume flow rate:

$$R = \underbrace{\frac{\Delta P}{L}\frac{\pi}{8\eta}}_{const} r^4$$



$r_2 = \frac{1}{2} r_1$

$$\Rightarrow R \propto r^4$$

$$\Rightarrow \frac{R_2}{R_1} = \left(\frac{r_2}{r_1}\right)^4 = \left(\frac{1}{2}\right)^4$$

R_2 / R_1 = ?

A. 1

B. 1/2

C. 1/4

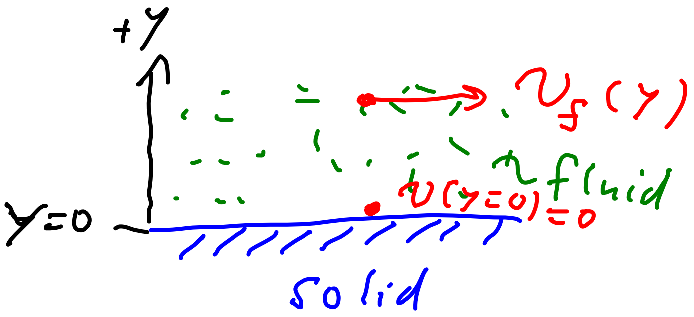
D. 1/8

[✓ E. 1/16]

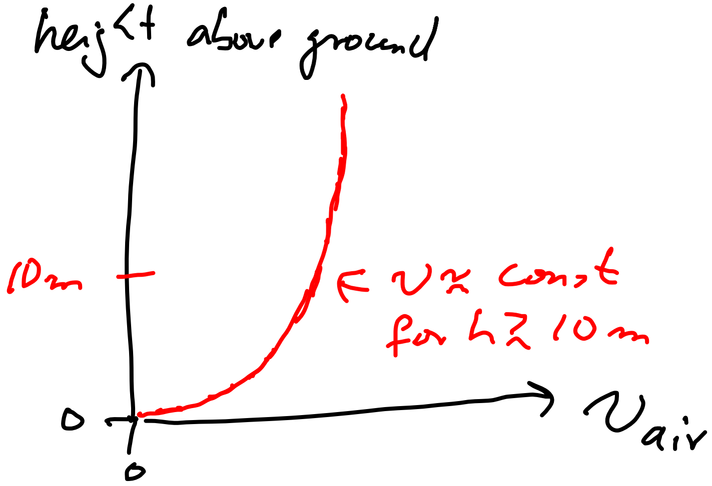
→ Fluid Friction:

- Recall: Solid on Solid friction:
  - f_s, f_k oppose relative motion of the surfaces in contact!

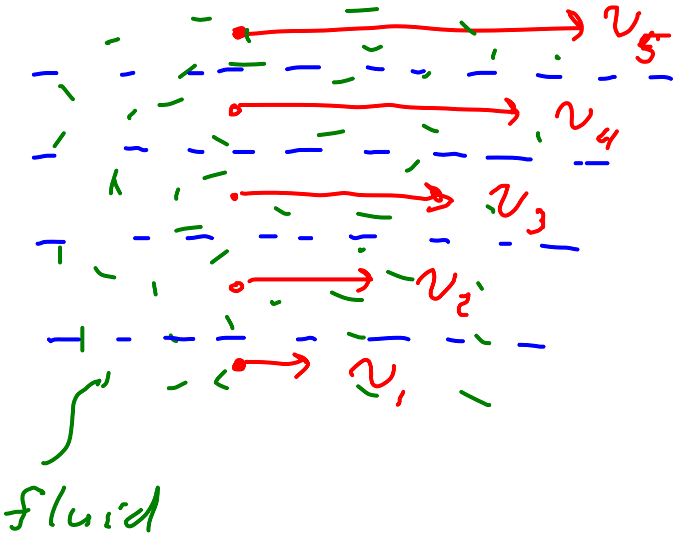
- Fluid-Solid Friction:



- $v_f = v_f(y)$ depends on height above solid surface
- $v_{fluid}(y=0)=0$ at $y=0$ (i.e. at fluid-solid interface) relative to surface (no relative motion at surface)
- Examples:
  - rain drops on car windshield
  - air near earth's surface



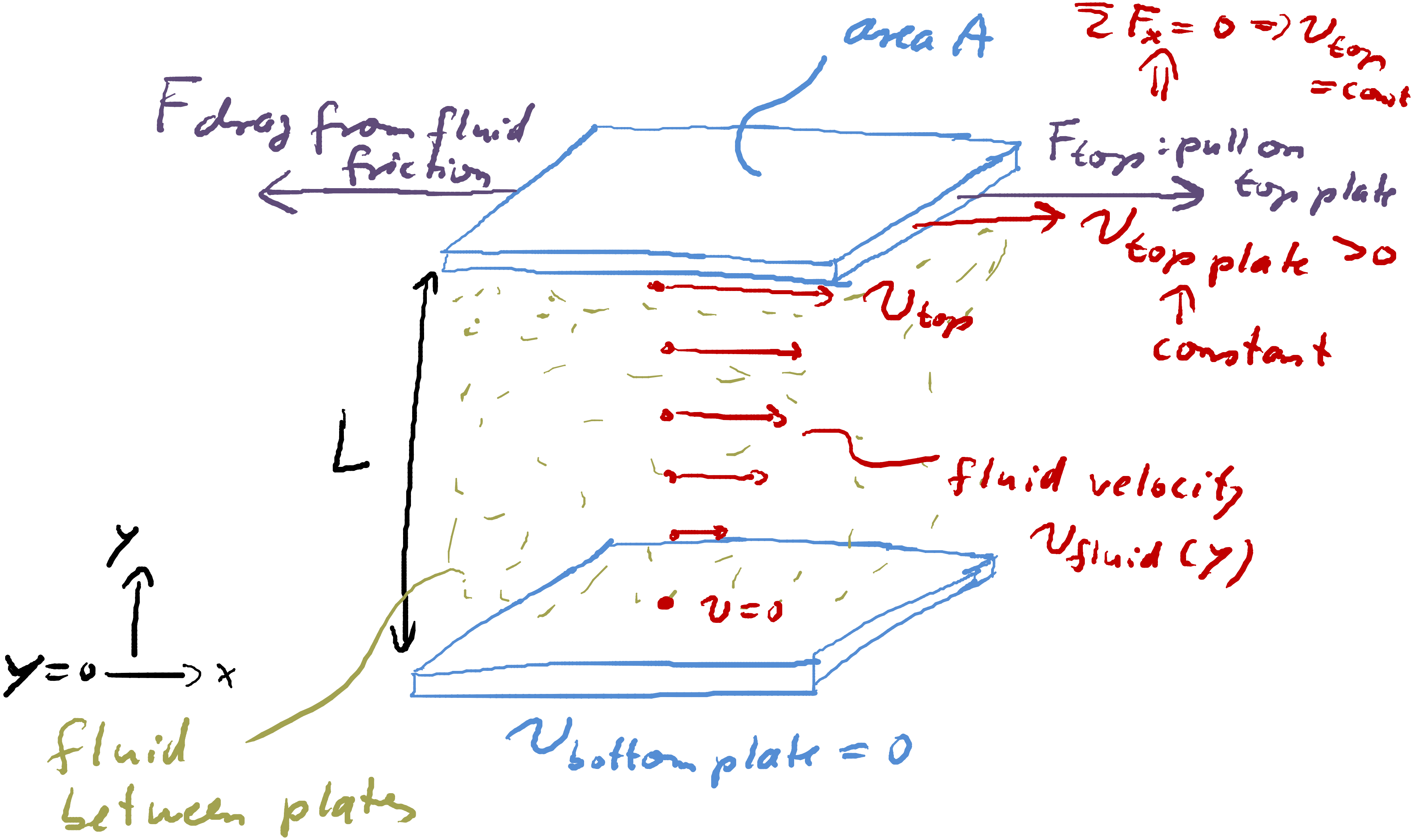
- Fluid - Fluid Friction:

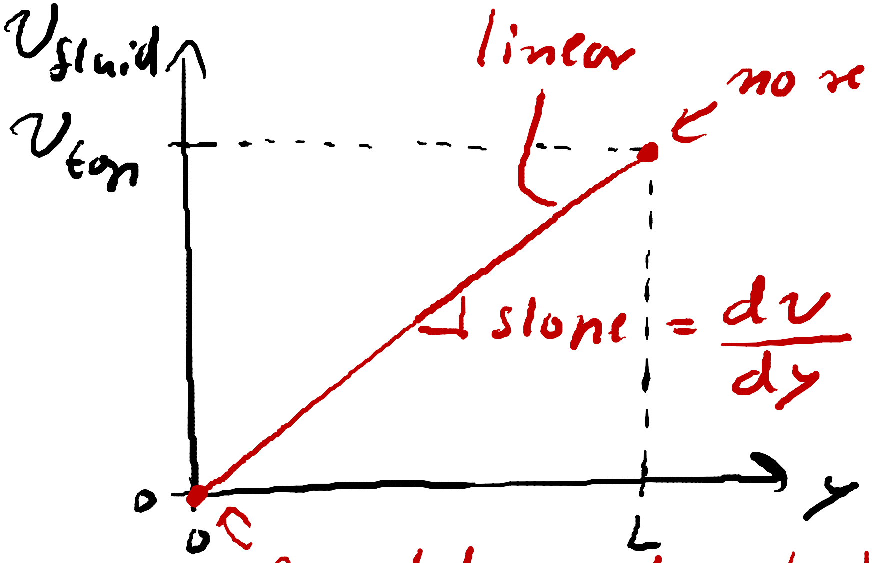


- friction forces (drag) between "layers" at different speeds
- Fluid friction opposes relative motion of adjacent fluid "layers"

$\Rightarrow$ Viscosity η = "eta"

→ specific Geomitry:





no relative motion to top plate

$$\text{slope} = \frac{dv}{dy} = \frac{v_{top}}{L}$$

no relative motion to bottom plate

$$|F_{fluid\ drag}| \propto A \frac{v_{top}}{L}$$

v_{top} ⇐ larger speed ⇒ more force

L ⇐ smaller L ⇒ more force

A ↑ larger A ⇒ more force

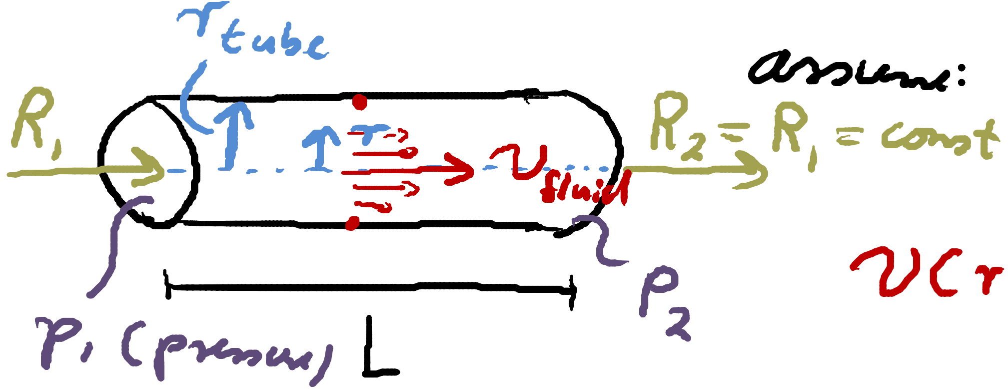
$$\frac{v_{top}}{L} = \frac{dv}{dy}$$

$$\Rightarrow \boxed{F_{\substack{fluid\\drag}} = \eta A \frac{dv}{dy}}$$

η = viscosity $[\eta] = \frac{kg}{m\,s} = Pa\cdot s$

$\eta_{water} = 10^{-3}\,Pa\cdot s$ $\eta_{air} = 1.8\cdot 10^{-5}\,Pa\cdot s$

→ Viscous, laminar flow through tube (const. height)



assume: $y = \text{const}$, $A = \text{const}$

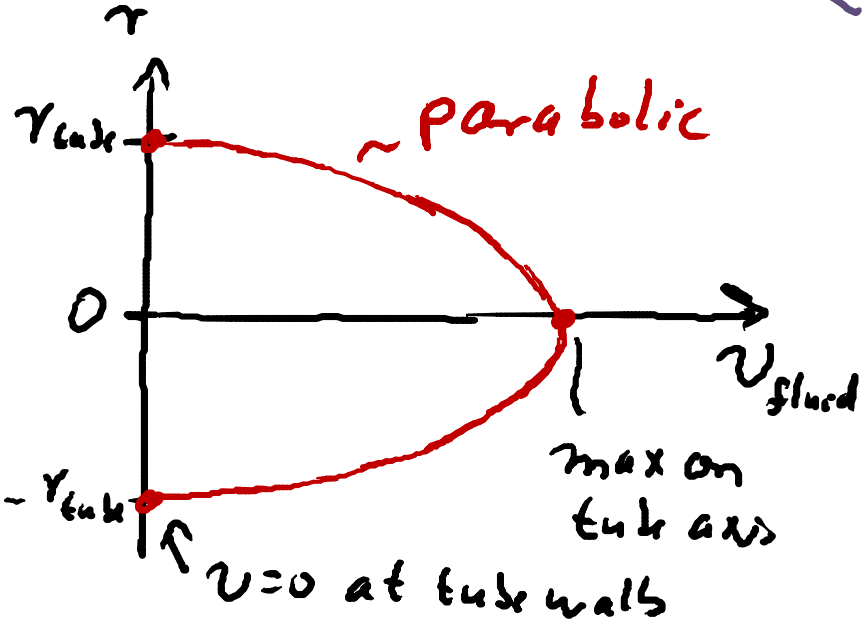
fluid speed in the tube:

$$v(r) = \frac{1}{4}\frac{1}{\eta}\frac{\Delta p}{L}\left(r_{tube}^2 - \underline{\underline{r}}^2\right)$$

↗ radial distance from tube axis

need $\Delta p = p_1 - p_2 > 0$

work by pressure difference required to provide energy lost by friction and to maintain constant flow!



Note: Bernoulli's equation only for ideal fluids, i.e. no friction ($\eta = 0$)!

$\Rightarrow$ Volume flow rate for tube:

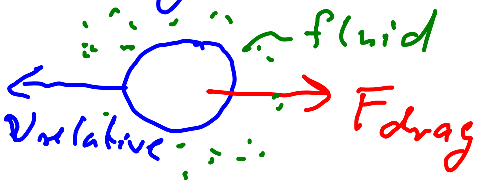
$$\underline{\underline{R}} = \text{Area} \cdot v_{avg} = \pi r_{tube}^2 \cdot v_{avg}$$ ← average of flow speed over cross-sectional area of tube

$$= \underbrace{\pi r_{tube}^2}_{\text{from area}} \cdot \frac{1}{8} \frac{1}{\eta} \frac{\Delta p}{L} \underbrace{r_{tube}^2}_{\text{from } v_{fluid}} \propto r_{tube}^4$$

$$\Rightarrow \boxed{R_{tube} = \frac{\pi}{8\eta} \frac{\Delta p}{L_{tube}} r_{tube}^4}$$ Poiseuille's equation

- for laminar, viscous flow only!
- ~ obeyed in human circulatory system

→ Fluid drag on Moving Objects:



Two regimes (extreme cases):

Turbulent drag regime

- high velocities, large objects, small η

$$\boxed{F_{drag,\,turb} = \frac{1}{2} C \rho_{fluid} A_{obj} \underline{\underline{v^2}}}$$

⇒ for sphere: $C = \frac{1}{2}$

⇒

$$F_{drag,\,turb,\,sphere} = \frac{1}{4} \rho_{fluid} \underbrace{\pi \underline{\underline{r_{sp}^2}}}_{A_{sp} = \pi r_{sp}^2} v^2$$

↗ radius of sphere

Viscous drag (laminar flow) regime

- low velocities, small objects, large η

$$F_{drag,\,visc} \propto \eta v$$

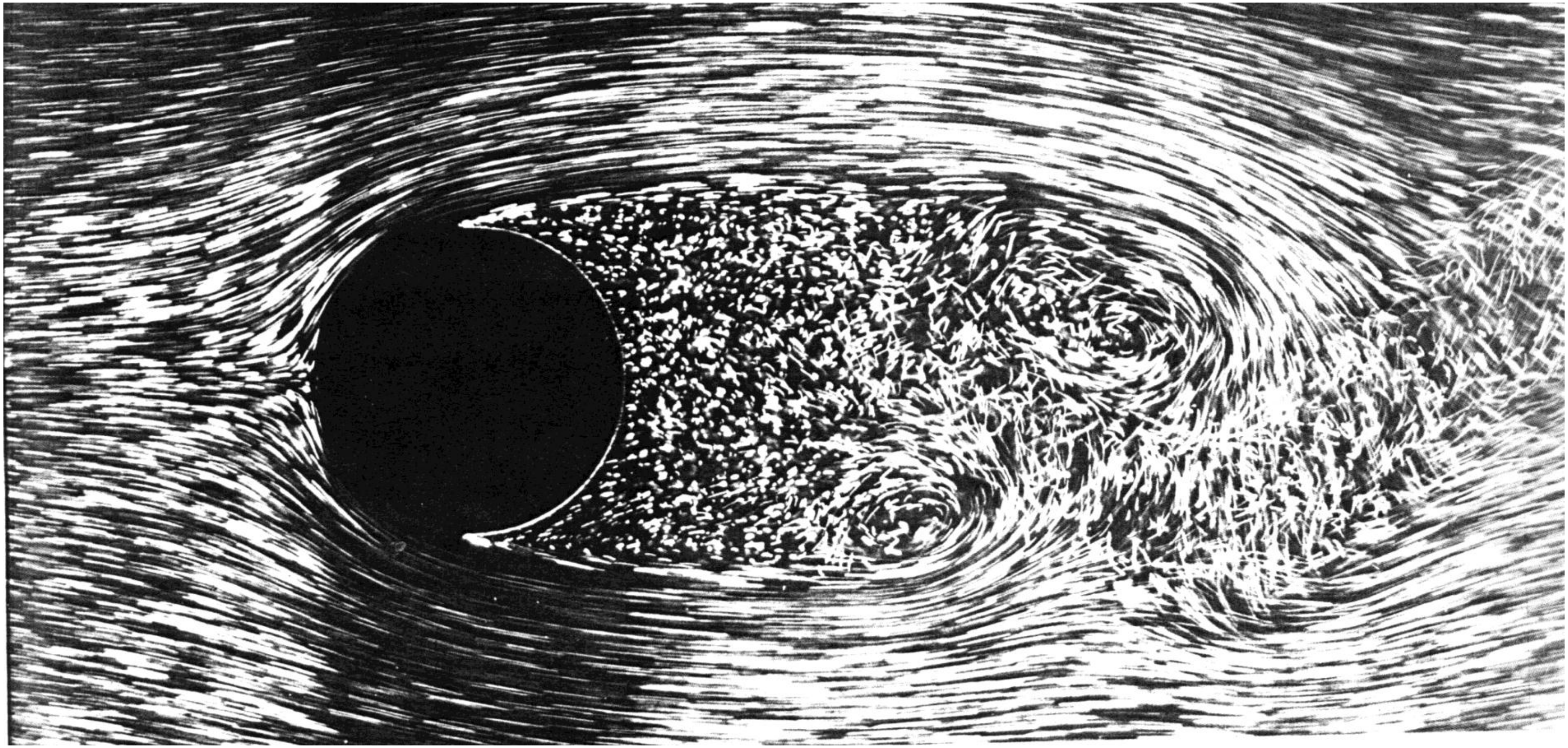
⇒ for sphere: (r_{sp} = radius of sphere)

$$\boxed{F_{drag,\,visc.,\,sphere} = 6 \pi \eta r_{sp} v}$$

Stokes' Law

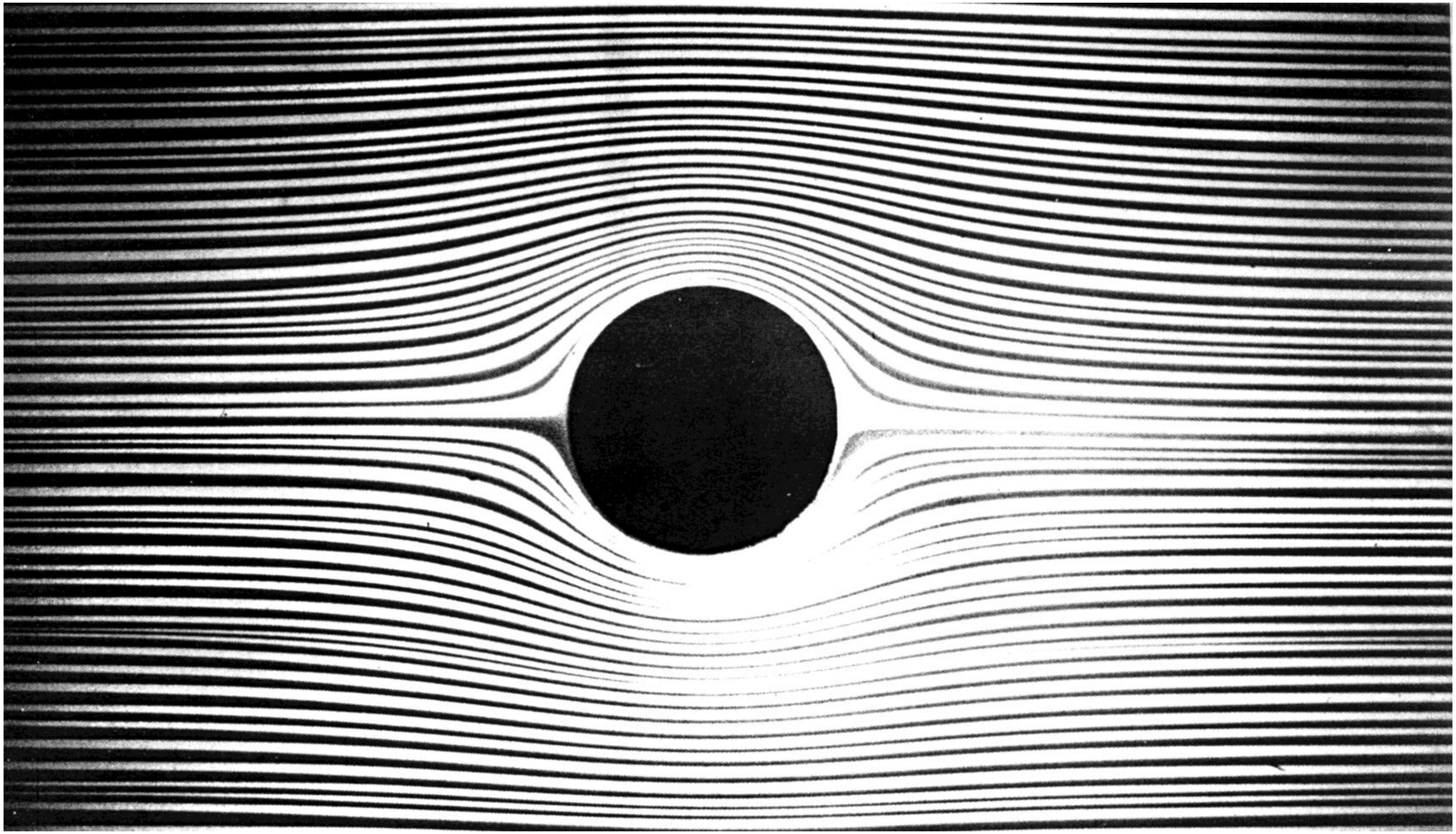
Fluid flow pattern around a moving cylinder

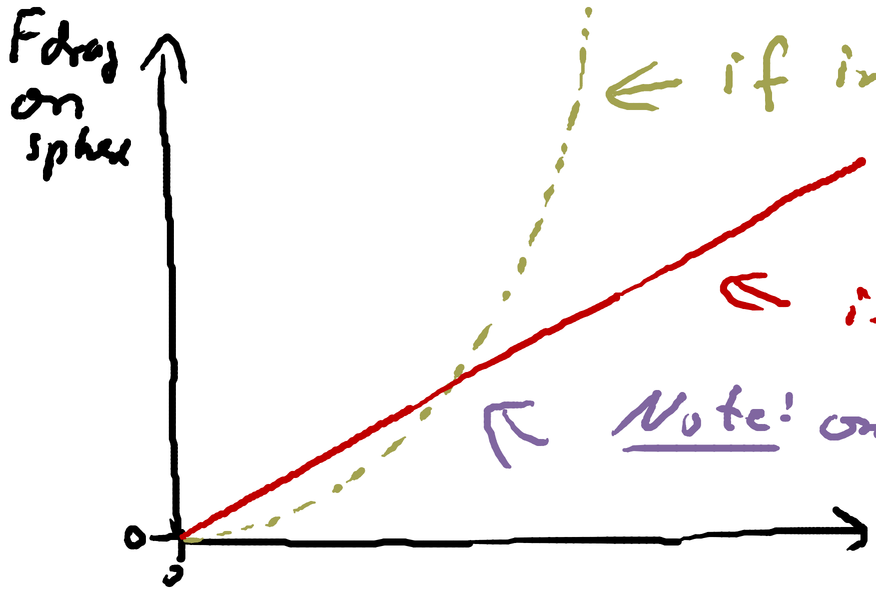
Turbulent flow: High velocities / larger diameters / "thin" fluids:



Fluid flow pattern around a moving cylinder

<u>Laminar Flow</u>: Low velocities / small diameters / "thick" fluids





r_{sp}, v = const

or v, r_{sp} = const

← if in turbulent regime: $F_{drag,turb} \propto r_{sp}^2$

$F_{drag,turb} \propto v^2$

← if in viscous regime: $F_{drag,visc} \propto r_{sp}$

$F_{drag,visc} \propto v$

<u>Note!</u> only $F_{drag,visc.}$ <u>or</u> $F_{drag,turb}$ is acting on sphere, depending on given conditions!!

what determines whether we are in the <u>viscous</u> regime <u>or</u> <u>turbulent</u> regime?

$\Rightarrow$ compare forces in regimes:

$$\frac{F_{drag,turb}}{F_{drag,visc}} \text{ on sphere} = \frac{1}{24} \underbrace{\frac{\rho_f r_{sp} v}{\eta}}_{\text{Reynolds' number (dimensionless)}} \begin{cases} \text{if } \gg 1 \Rightarrow \underline{\text{turbulent}} \\ \text{if } \ll 1 \Rightarrow \underline{\text{viscous}} \end{cases}$$

⇒ Reynold's Number:

$$\boxed{Re = \frac{\rho_{fluid}\, r\, v}{\eta}}$$

r: "typical" dimension of the object

- for sphere: ($r = r_{sphere}$) moving through fluid:

  $Re \gg 24$ → turbulent → use $F_{drag,\,turb}$

  $Re \ll 24$ → viscous/laminar → use $F_{drag,\,visc.}$

- for flow through pipes: (r = radius of pipe)

  $Re > 3000$ → turbulent

  $Re < \underbrace{2000}$ → viscous

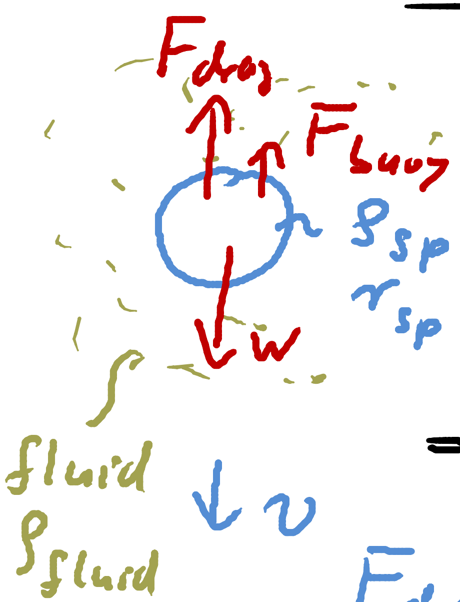
  determined experimentally

→ Terminal speed v_t of a sphere:

- at $v = v_t$ when $a \to 0$

$\Rightarrow \sum \vec{F} = 0$ at $v = v_t$

$\Rightarrow |F_{drag}| = |W| - |F_{buoy}|$



⇒ In viscous regime:

$$F_{drag,visc,sphere}(v = v_t) = 6\pi\eta r_{sp} \underline{\underline{v_t}}$$

$$W = mg = V_{sp}\rho_{sp} g = \frac{4}{3}\pi r_{sp}^3 \rho_{sp} g$$

$$F_{buoy} = W_{fluid\ disp} = \frac{4}{3}\pi r_{sp}^3 \rho_{fluid} g$$

$$v_{t,visc} = \frac{2}{9} \frac{(\rho_{sp} - \rho_{fluid})}{\eta} g r_{sp}^2 \propto \underline{\underline{r_{sp}^2}}$$