Recap

Lecture 21

**Magnetic Induction:**

- $E_{\text{mf}}$ induced by changing flux through a conducting loop:
  - Faraday's law: $E = -\frac{d\Phi}{dt}$ through loop
  - Resulting induced current: $i_{\text{ind}} = \frac{E}{R} \ll \text{Resistance of the loop}$

- Lenz's Law: An induced current has a direction such that the magnetic field due to the induced current opposes the change in the magnetic flux that induces the current.

\[ \Phi = \mathbf{B} \cdot \mathbf{A} < 0 \text{ (here)} \]

\[ \frac{d\Phi}{dt} < 0 \text{ (here) } \Rightarrow E > 0 \]

\[ \Rightarrow \text{current flows in } + \text{ direction} \]
Today:

• Inductors and their inductance
• RL circuits
• Energy density of a magnetic field
Inductors and Inductance $L$:

Recall: Capacitors and Capacitance

- Capacitor: produces electric field between two plates
  - Symbol: $\cap \cap$
  - Described by capacitance: $C = \frac{Q}{\Delta V_c}$ (depends on geometry of capacitor)
  - Energy of electric field in capacitor: $U = \frac{1}{2} C \Delta V_c^2$

Now:

- Inductor: produces a magnetic field around current-carrying wire
  - Symbol: $\mathcal{L}$
  - Described by inductance $L$
Consider a solenoid with \( N \) windings/turns:

- Magnetic flux through central region of area \( A \):
  \[ \Phi_B \]

- Same flux goes through each winding of solenoid = are "linked" by shared flux.

- Define **magnetic flux linkage** = \( N \Phi_B \)

- Define **Inductance** \( L \) of solenoid/inductor:
  \[ L = \frac{N \Phi_B}{i} \]

Units:
\[ \frac{L}{\frac{\Phi_B}{i}} = \frac{Tm^2}{A} = 1 \text{ henry} = 1 \text{ H} \]
for a long solenoid of length $l$ with $N$ turns

$B_{\text{inside}} = \mu_0 \frac{N}{l} i \Rightarrow \Phi = \mu_0 \frac{N}{l} i \frac{A}{\pi}$

\[ \Rightarrow \text{Inductance of a solenoid:} \]

\[ L = \frac{N \Phi_B}{i} = \frac{N \mu_0 \frac{N}{l} i A}{i} = \mu_0 \frac{N^2 A}{l} \]

\[ \text{# of turns} \]

\[ \text{Cross-sectional area} \]

\[ \text{depends on geometry of solenoid only} \]

\[ \text{length} \]
Self-induction in an Inductor:

Suppose that we let current $i$ in the solenoid change with time $\psi$

magnetic flux $\Phi_B$ change with time $\psi$

According to Faraday’s law: an emf will be ‘self-induced’ in the solenoid, that opposes the change in current flux (note: only one solenoid involved here!)

For ideal inductor without resistance $R$ in wire

$\epsilon_L = 10V_L = 132 V$
\[ = \text{from Faraday's Law:} \]

\[ E_L = -N \frac{d\Phi_B}{dt} \quad \text{for solenoid with} \ N \ \text{turns} \]

\[ = -\frac{d(N\Phi_B)}{dt} = -\frac{d(Li)}{dt} \quad \text{since} \ L = \frac{N\Phi_B}{i} \]

\[ = -L \frac{di}{dt} \]

\[ = \text{if current is changing in an inductor} \ (\frac{di}{dt} \neq 0) \]

\[ \text{then there is a self-induced emf in the inductor:} \]

\[ E_{\text{self-induced}} = -L \frac{di}{dt} \quad \text{rate of change of current} \]
At time $t = 0$ move the switch to position a.

Current $i$ begins to flow but the self-induced emf $\mathcal{E}_L$ in the inductor $L$ opposes the rise in current.

$\rightarrow$ Current starts out at 0 at $t=0$ and then increases until it approaches a steady state value asymptotically.
At time $t = 0$ move the switch to position $a$.

After a very long time what will be the magnitude of the steady state current in the circuit?

A. 0  B. $|\mathcal{E}_L|/R$  C. $\mathcal{E}/R$  D. $(\mathcal{E} - |\mathcal{E}_L|)/R$  E. Both answers C & D above.
At time $t = 0$ move the switch to position a.

**RL circuit: Rise of current**

Use loop rule: $\mathcal{E} + \Delta V_R + \Delta V_L = 0$

$$\Rightarrow \mathcal{E} - iR - L\frac{di}{dt} = 0 \quad \text{differential equation for current rise}$$

At $t = 0$ : $i(t=0) = 0 \quad \text{initial condition}$

**Solution:** $i(t) = \frac{\mathcal{E}}{R} \left[ 1 - e^{-\frac{tR}{L}} \right]$  

Check: $i(t=0) = 0 \vee \frac{di}{dt} = -\frac{\mathcal{E}}{L} e^{-\frac{tR}{L}}$
**RL circuit: Rise of current**

Current in inductor:

\[ i(t) = \frac{\varepsilon}{R} [1 - e^{-t/\tau}] \]

With time constant \( \tau = \frac{L}{R} \)

Change in electric potential across the inductor:

\[ 10V_e = \int \frac{di}{d\tau} = \frac{\varepsilon}{R} e^{-t/\tau} \]

Note:
- At \( t = 0 \), \( i = 0 \) and \( 10V_e = \varepsilon \)
- At \( t \to \infty \), \( i = \frac{\varepsilon}{R} \) and \( 10V_e = 0 \)

i.e. inductor eventually behaves like an ordinary connecting wire.
The switch has been in position a for a very long time.

At time $t = 0$ move the switch to position b.

Current $i$ begins to decrease, but the self-induced emf $\mathcal{E}_L$ in the inductor L slows down the decrease in current.

$\rightarrow$ Current starts out at the equilibrium value, and then decays to zero over time.
**RL circuit: Decay of current**

At time \( t = 0 \), switch to position b using a make-before-break switch.

Use loop rule: \( \Delta V_R + \Delta V_L = 0 \)

\[
\Rightarrow -iR - L \frac{di}{dt} = 0
\]

\( \frac{di}{dt} \) differential equation for current decay

At \( t=0 \) : \( i(t=0) = i_0 \) \( \Rightarrow \) initial current

\( \Rightarrow \) solution: \( i(t) = i_0 e^{-\frac{tR}{L}} \)

Check: \( i(t=0) = i_0 \) \( \Rightarrow \frac{di}{dt} = -i_0 \frac{R}{L} e^{-\frac{tR}{L}} \)
**RL circuit: Decay of current**

Current in inductor:
\[ i(t) = i_0 e^{-\frac{t}{\tau}} \]

with time constant \( \tau = \frac{L}{R} \)

for current decay

Change in electric potential across the inductor:
\[ 10V_L(t) = 10 \frac{di}{dt} \]
\[ = i_0 R e^{-\frac{t}{\tau}} \]

Note:
If current has reached steady state value \( i = \frac{E}{R} \) before moving switch to position B:
\[ i_0 = \frac{E}{R} \Rightarrow 10V_L(t=0) = E \]