Relativistic Quantum Field Theory, Spring 2014

Homework Assignment # 5

(Due Tuesday, March 18, in lecture.)

Lectures and Reading Assignments:

Readings are from “An Introduction to QFT” by Peskin and Schroeder (PS); “Quantum Field Theory” by Srednicki (Sr); “Quantum Field Theory in a Nutshell” by Zee (Z).

- Lec 10, 3/06 (Thu): Photon Spin Sum and Compton Scattering. PS Sec. 5.5. UV Divergences in QED. PS Sec. 10.1. Electron Self-Energy. PS Sec. 7.1, Sr Sec. 62
- Lec 11, 3/11 (Tue): Vacuum Polarization. PS Sec. 7.5, Sr Sec. 62.
- Lec 12, 3/13 (Thu): Vertex Function. Anomalous Magnetic Moment. PS Sec. 6.2, 6.3; Sr Sec. 63, 64

Problems:

1. **Compton Scattering**: Consider the process

   \[ e^- (p_1) + \gamma (p_2) \rightarrow e^- (p_3) + \gamma (p_4). \]  

   Work in the center-of-mass frame of the colliding particles. Assume that the colliding particles are unpolarized, and the spins of the final-state particles are undetected. Denote the incoming electron’s energy as \( E \) and the incoming photon’s energy as \( \omega \). **Note:** You can use PS Sec. 5.5 when you work on this problem, but make sure you perform and understand all intermediate steps of the calculation.

   (a) Write down the invariant matrix element for this process. Show explicitly that the Ward identity is satisfied: replacing a photon polarization vector with its momentum vector in the matrix element yields zero. Show that this works for both the incoming and the outgoing photons.

   (b) Compute the differential cross section of this reaction, \( d\sigma/d\cos\theta \), where \( \theta \) is the angle between the three-vectors \( p_1 \) and \( p_3 \). Express the answer in terms of \( E, \omega, \theta, \) the electron mass \( m \), and the fine-structure constant \( \alpha \).

   (c) Sketch the angular distribution of the outgoing photons, in two regimes: \( E \sim m \) and \( E \gg m \).

   (d) Compute the leading behavior of the total cross section in the limit \( E \gg m \).

2. **UV Divergences in QED**: In this problem, you will prove some statements that were made in Lecture 10 in the discussion of the structure of the UV divergences of QED.

   (a) Show that for any QED Feynman diagram, \( 4L - P_f - 2P_{\gamma} = 4 - \frac{2}{\alpha}N_f - N_{\gamma} \), where: \( L \) is the number of loops, \( P_f \) is the number of fermion propagators, \( P_{\gamma} \) is the number of photon propagators, \( N_f \) is the number of external fermions and antifermions, and \( N_{\gamma} \) is the number of external photons.
Problem 10.1

3. Technical Naturalness: In the lecture, we saw that the electron mass renormalization is proportional to the mass itself, and diverges logarithmically. In this problem, you will acquire a deeper understanding of these facts.

(a) Show that the QED Lagrangian can be equivalently rewritten as

\[ L = \bar{\psi}^\dagger \sigma^\mu D_\mu \psi + \bar{\psi}^\dagger \sigma^\mu D_\mu \psi - m \left( \bar{\psi}^\dagger L \psi_R + \bar{\psi}^\dagger R \psi_L \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2} \]

where \( \psi_L \) and \( \psi_R \) are Weyl spinor fields in the left- and right-handed spin-1/2 representations of the Lorentz group, respectively; \( D_\mu = \partial_\mu + ieA_\mu \) is the covariant derivative; and \( \sigma_\mu = (1, \sigma) \) and \( \bar{\sigma}_\mu = (1, -\sigma) \), with \( \sigma \) denoting the 3-vector of Pauli matrices.

(b) Show that this Lagrangian is symmetric under the “vector U(1)” global symmetry: \( \psi_L \to e^{i\alpha} \psi_L \), \( \psi_R \to e^{i\alpha} \psi_R \), where \( \alpha \) is an \( x \)-independent constant. Note: This is equivalent to \( \Psi \to e^{i\alpha} \Psi \) in the Dirac notation.

(c) Show that this Lagrangian is symmetric under the “axial U(1)” global symmetry: \( \psi_L \to e^{i\alpha} \psi_L \), \( \psi_R \to e^{-i\alpha} \psi_R \), but only if \( m = 0 \). Note: This is equivalent to \( \Psi \to e^{i\alpha\gamma_5} \Psi \) in the Dirac notation.

(d) Assume that all symmetries of the original Lagrangian must be preserved by the radiative corrections derived from it. Construct a symmetry argument why \( \delta m \propto m \). Then, use dimensional analysis to argue that \( \delta m \) can be at most logarithmically divergent.

Note: This is an example of a very general phenomenon in QFTs, first discussed by ’t Hooft in the 1970s. If setting a Lagrangian parameter to 0 enhances the symmetry of the theory, then radiative corrections to this parameter must be proportional to the parameter itself. So, if the bare value of the parameter is small (compared to the cutoff, in the case of dimensionful parameters), the radiative corrections will not make it large – the small value is “natural”.