Last time: DIS

\[ e^+ e^- \rightarrow q \bar{q} + X \]

\[ Q^2 = q^2 < 0 \]
\[ Q \gg \Lambda_{QCD} \text{ - part tiny} \]

NG: \( \frac{d^2 \sigma}{dx dy} \propto \left( \sum x_f q(x) Q^2 \right) \cdot (1 + (1-y)^2) \)

CC: \( \frac{d^2 \sigma}{dx dy} \propto x f_d + x f_u (1-y)^2 \) for 4\( \rightarrow e^- \)
\( x f_u (1-y)^2 + x f_d \) for \( j \rightarrow e^+ \)

\[ \Rightarrow \text{measure } f_d, f_u \]

Corrections: - non-perturbative ("power") \( \Theta \left( \frac{\Lambda_{QCD}^2}{Q^2} \right) \)

- perturbative: \( e^+ p \rightarrow e^- + X \)

ISL - QCD:

Real-gluon emission - part of "X"!

\[ dB_3 = \frac{dt \cdot dx \cdot ds}{2 \pi} \hat{P}_{q q} (x) \ dB_2 (x p q + \ldots) \]

\[ \hat{P}_{q q} (x) = \frac{4}{3} \frac{1 + x^2}{1 - x} \quad \text{(HW)} \]

\[ t = (p_q - p_{\bar{q}})^2 = \left[ p_q x + p_{\bar{q}} \right]^2 \approx \frac{1}{2} p_{\perp}^2 = -\frac{1}{2} \left( p_{\perp} \right)^2 . \]

\( t = 1 - x \)

As usual, this diverges for \( |p_{\perp}| \rightarrow 0 \) - collinear divergence.

Separates:
\[ t > \mu^2 - 3\text{-body final state} \]
\[ t < \mu^2 - 2\text{-body final state} \]

\( \mu = "\text{factorization scale}" \)
Recall in QED: \[ e^+ e^- \rightarrow \gamma \mu^+ \mu^- \text{ ISR} \]

\[ b_{2 \rightarrow 2}^{\text{NLO}} = \int_{0 \rightarrow m_\gamma^2} \frac{d t}{t} \int_{0}^{1} d x \frac{1}{2 \pi} \frac{1 + x^2}{1 - x} b_{2 \rightarrow 2}^{\text{LO}} (s, x) \]

\[ = \frac{1}{2 \pi} \int_{0}^{1} d x \frac{1 + x^2}{1 - x} \log \frac{m^2}{m_\gamma^2} b_{2 \rightarrow 2}^{\text{LO}} (s, x) \]

"e" p.d.f.: \[ f_e (x, \mu) = \frac{1}{2 \pi} \log \frac{m^2}{m_\gamma^2} \left[ \frac{1 + x^2}{1 - x} \right] + c \delta (1 - x) \]

C from probability conservation \( (c = 2/3) \)

\[ b_{2 \rightarrow 3} = \int \frac{d t}{t} \int_{0}^{1} d x \frac{1 + x^2}{1 - x} b_{2 \rightarrow 2}^{\text{LO}} (s, x) + \text{non-log-enhanced \& non-factored \ terms} \]

In QCD: \[ p_q = s_0 \cdot q \quad p' = s_0 \cdot x \cdot p \equiv s \cdot p \]

\[ f_q (s) \rightarrow f_q (s, \mu) = f_q^{\text{LO}} (s) + \int_{s_0}^{1} d s_0 \frac{d s}{2 \pi} \int_{0}^{1} d x \delta (s_0 x - s) \]

\[ \times P_{qq} (x) \log \frac{m^2}{m_{IR}^2} \cdot f_q^{\text{LO}} (s_0) \]

\[ = f_q^{\text{LO}} (s) + \frac{d s}{2 \pi} \int_{3}^{1} d \frac{s_0}{s} f_q^{\text{LO}} (s_0) P_{qq} \left( \frac{s}{s_0} \right) \log \frac{m^2}{m_{IR}^2} \]

Note that \[ f_q^{\text{LO}} (s) \equiv f_q^{\text{LO}} (s, m_{IR}) \]
This can be rewritten as

\[ \frac{\partial f_q (s, \mu^2)}{\partial \log \mu^2} = \frac{d s}{2 \pi} \int_{3}^{\infty} \frac{d s_0}{s_0} f_q (s_0) P_{q q} \left( \frac{\mu^2}{s_0} \right) \]

(leading logs to all orders in \( d s \)) \( \rightarrow \)

\[ \frac{d s (\mu)}{2 \pi} \int_{3}^{\infty} \frac{d s_0}{s_0} f_q (s_0, \mu) P_{q q} \left( \frac{\mu^2}{s_0} \right) \]

This is an exact analogue of Gribov-Lipatov equations for QED. It's called Altarelli-Parisi (AP) or GLAP (or DGLAP + Dokshitzer) equation.

Boundary condition: In QED, \( f_e (x, m_e) = \delta (1-x) \). In QCD, can set \( m^2 = m_{QCD} \), no perturbative cutoff \( \Rightarrow \) need data!

Another contribution at \( \Theta (d s) \): \( q \rightarrow q \bar{q} \) splitting

\[ P_{q q} (x) = \frac{1}{2} \left( x^2 + (1-x)^2 \right) \]

And

\[ \frac{\partial f_q}{\partial \log \mu^2} = \frac{d s (\mu)}{2 \pi} \int_{3}^{\infty} \frac{d s_0}{s_0} \left[ f_q (s_0, \mu) P_{q q} \left( \frac{\mu^2}{s_0} \right) + f_g (s_0, \mu) P_{q g} \left( \frac{\mu^2}{s_0} \right) \right] \]

Likewise

\[ \frac{\partial f_g}{\partial \log \mu^2} = \frac{d s (\mu)}{2 \pi} \int_{3}^{\infty} \frac{d s_0}{s_0} \left[ f_q (s_0, \mu) P_{g q} \left( \frac{\mu^2}{s_0} \right) + f_g (s_0, \mu) P_{g g} \left( \frac{\mu^2}{s_0} \right) \right] \]

Generalization to \( 2 M_f \) (quark + antiquark) flavors is straightforward.
As in QED, we also have

$$\hat{\sigma}_{2\to3} = \sum_{\text{terms}} \frac{1}{\mu^2} \int_0^1 \int_0^{2\pi} P(x) \hat{\sigma}^{LO}_{2\to2}(S|x) + \text{non-universal terms}$$

So that the proton-model formula becomes

$$\sigma(\ell(\nu) + \bar{\nu}(P) \to \ell'(\nu') + X) = \int_0^1 d\beta \sum_{f} f_i(3,1) \cdot \left[ \hat{\sigma}_{2\to2}^{LO}(\ell(\nu) + \bar{\nu}(P) \to \ell'(\nu') + f_i) + \hat{\sigma}_{2\to3}(\ell + f \to \ell' + f' + f'') \right]$$

\[ + \Theta(\Lambda^2) \]

Since we're computing inclusive x-section, the r.h.s. side must be independent of $\mu$. Can use pdf's evaluated at any scale, as long as contributions with $t > \mu^2$ are included in $\hat{\sigma}_{2\to3}$.

**BUT:**

$$\hat{\sigma}_{2\to3} = \sum_{\text{terms}} \frac{1}{\mu^2} \int_0^1 d\beta \frac{d\sigma}{2\pi} P(x) \hat{\sigma}_{2\to2} + \text{non-universal}$$

$$= \frac{d\sigma}{2\pi} \cdot \log \frac{Q^2}{\mu^2} \cdot C + \text{(no-log-enhancements)}$$

or $\log^2$ if also soft singularity!

\[ \Rightarrow \text{expansion in } \frac{d\sigma}{2\pi} \log \frac{Q^2}{\mu^2}, \text{potentially large. If we choose} \]

$$\mu^2 \propto Q^2, \text{then large logs are hidden inside pdf's (where they are resummed by AP equation), and the series in } \hat{\sigma} \text{ is in } \frac{d\sigma}{2\pi} \text{- fast convergence!} \]