Problems:

1. **Jet Counting and Event Shapes with Madgraph**
   
   In this problem, we will use the sample of \( e^+e^- \rightarrow u\bar{u}g \) events simulated for Problem 3 on problem set 2.

   - Use JADE algorithm to distinguish 2-jet and 3-jet events in your sample: that is, define a three-jet event as one for which
     \[
     \min_{i,j}(p_i + p_j)^2 > y_s \quad (i, j = q, \bar{q}, g),
     \]
     where \( y < 1 \) is a parameter, and label all other events as 2-jet. Plot the percentage fraction of 2-jet and 3-jet events, as a function of \( y \). Compare with the analytic prediction (LO QCD) for the three-jet fraction \( f_3 \):
     \[
     f_3(y) = \frac{2\alpha_s}{3\pi} \left[ 2\ln^2 \frac{y}{1-y} + (3-6y) \frac{y}{1-2y} + \frac{5}{2} - 6y - \frac{9}{2} y^2 + 4\text{Li}_2 \left( \frac{y}{1-y} \right) - \frac{\pi^2}{3} \right].
     \]
     (Of course \( f_2 = 1 - f_3 \).) What is the influence of generator-level cuts you introduced when generating the events on this comparison? Where and why do you expect the LO QCD prediction to fail to reproduce real data? Based on this expectation, what would be the “optimal” choice of the generator-level cuts?

   - Plot the distribution of events in your sample in the thrust variable \( T \), defined in lecture. Compare with the analytic (LO QCD) formula,
     \[
     \frac{1}{\sigma} \frac{d\sigma}{dT} = \frac{2\alpha_s}{3\pi} \left[ \frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln \frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T} \right], \quad \frac{1}{2} \leq T < 1.
     \]
     Comment on the behavior of the analytic and MG distributions near \( T = 1 \).

2. **Four-Jet Events at LEP**

   Simulate a sample of events with \( e^+e^- \rightarrow 4j \) with Madgraph. Choose \( \sqrt{s} = M_Z \) to match the LEP experiment. Impose cuts to avoid singular regions. In each event, order jets according to their energies, \( E_1 > E_2 > E_3 > E_4 \). Plot the distribution of events in the cosine of the Bengtsson-Zerwas angle, defined as
     \[
     \cos \chi_{BZ} = \frac{(P_1 \times P_2) \cdot (P_3 \times P_4)}{|P_1 \times P_2||P_3 \times P_4|}.
     \]
     Compare with the distribution measured by the L3 collaboration at LEP, see figure on the next page (taken from Phys. Lett. B248 (1990) 227). **HINT:** To speed up event generation, you can set the
maximum number of allowed QED couplings to 2. This reduces the number of diagrams while only changing the result by $\mathcal{O}(\alpha/\alpha_s) \approx 0.1$. (Make sure you understand why!)

**NOTE 1:** The figure compares the true QCD prediction with the “abelian QCD” prediction, which is what you would obtain by throwing away all diagrams containing the triple-gluon vertex. The abelian QCD prediction clearly does not fit, demonstrating the presence of the 3-gluon vertex. For extra credit, you can simulate this process in the abelian QCD, and plot the distribution in $\chi_{\text{HZ}}$ in that model. You will need to edit the SM model files in creative ways to mock up abelian QCD. Alternatively, you can add a new abelian gauge boson, a “$Z$”, using the MadGraph’s “user model” option. We will learn how to use this option in class later, but if you can’t wait, the instructions are at [http://cp3wks05.fynu.ucl.ac.be/Manual/usrmod.html](http://cp3wks05.fynu.ucl.ac.be/Manual/usrmod.html)

**NOTE 2:** If you want to understand why this works, read section 5.1 of Ellis, Stirling and Weber (ESW), pp. 158-164.

3. **Splitting Functions in QCD**

Derive the expressions for the (unpolarized) splitting functions defined in Lecture 7:

\[
\begin{align*}
\hat{P}_{gg}(z) & = \frac{1}{2} \left[ z^2 + (1-z)^2 \right] , \\
\hat{P}_{qg}(z) & = \frac{4}{3} \frac{1 + z^2}{1 - z} , \\
\hat{P}_{qg}(z) & = \hat{P}_{qg}(1-z) , \\
\hat{P}_{gg}(z) & = 6 \left[ \frac{1-z}{z} + \frac{z}{1-z} + z(1-z) \right] .
\end{align*}
\]

(5)

Note that these expressions are only valid away from $z = 1$, where the soft singularities need to be treated carefully. **HINT:** The calculations are outlined in Sec. 5.1 of ESW or Sec. 17.5 in Peskin and Schroeder.

![Figure 1: Distribution of four-jet events in $\chi_{\text{HZ}}$, measured by the L3 collaboration at LEP.](image)