THERMAL FIELD THEORY

10/6/2014

The Plan

(1) Quantum Mechanics @ Finite Temperature
(2) Scalar Field Theory
(3) Interactions & Renormalization
(4) Fermions & Gauge Fields
(5) Finite Density
(6) Phase Transitions

References

- Laine Lecture Notes: www.laine.itp.unibe.ch/basics.pdf
- Thermal Field Theory: Le Bellac
- Finite Temp. Field Theory: Kapusta.
- Zinn-Justin: 0005272

1.1 Quantum Mechanics @ Finite Temperature

In any quantum or classical mechanical system, we are interested in the partition function

$$Z[\beta] = \mathrm{tr}\, e^{-\beta \hat{H}}$$

From this, we can calculate any observables we want, by taking derivatives with respect to a source.

$$\langle \mathcal{O} \rangle = \frac{1}{Z}\mathrm{Tr}\left[\mathcal{O}\, e^{-\beta \hat{H}}\right].$$

In some cases, if we know the energy eigenvalues, we can compute this quite quickly. For the SHO, $E_n = \omega(n+\frac{1}{2})$ so

$$Z = \sum_n e^{-\beta\omega(n+\frac{1}{2})} = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}} = \frac{1}{2\sinh(\frac{\beta\omega}{2})}.$$

However, we are usually not so lucky. Fortunately, we almost always have a basis provided by $|q\rangle$ & $|p\rangle$, & so we can write the trace as

$$\mathcal{Z} = \mathrm{tr}\left[e^{-\beta \hat{H}}\right] = \int dq \, \langle q| e^{-\beta \hat{H}} |q\rangle$$

We are actually quite familiar with this expression, as we will see. Indeed, if we take $\hat{H} = P^2/2m + V(q)$,

$$\langle q| \underbrace{e^{-\beta \hat{H} \epsilon} \cdots e^{-\beta \hat{H} \epsilon}}_{N} |q\rangle \quad \text{\& insert} \quad \mathbb{1} = \int \frac{dp_i}{2\pi} |p_i\rangle\langle p_i|$$

$$\text{or} \quad \mathbb{1} = \int \frac{dq_i}{2\pi} |q_i\rangle\langle q_i|$$

& take $\epsilon \to 0$, $N \to \infty$, we have

$$\mathcal{Z} = \lim_{N\to\infty} \int \left(\prod_{i=1}^{N} \frac{dx_i \, dp_i}{2\pi}\right) \exp\left\{-\frac{1}{\hbar}\sum_{j=1}^{N} \epsilon \left[\frac{p_j^2}{2m} - i p_j \frac{x_{j+1} - x_j}{\epsilon} + V(x_j)\right]\right\}$$

where, because we are interested in $\langle q| e^{-\beta \hat{H}} |q\rangle$, $x_{N+1} = x_1$ & $\epsilon = \beta/N$.

If we perform the Gaussian integral over p_j, then we can usually write this as

$$\int_{x(0)=x(\beta)} \mathcal{D}x(t) e^{-S_E(\beta)} = \mathcal{Z} = c \int_{x(\beta)=x(0)} \mathcal{D}x(\tau) \, e^{-\underbrace{\int_0^\beta d\tau \left(\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x(\tau))\right)}_{\text{Euclidean Action } S_E(\beta)}}$$

where $t \to -i\tau$

we see that the thermal partition function is nothing more than the quantum partition function with a periodic time coordinate, $\tau = \tau + \beta$, & the fields are periodic as well.

If we add a source

$$Z(\beta;j) = \int_{q(0)=q(\beta)} \mathcal{D}q(\tau)\, \exp\left\{-S_E(\beta) + \int_0^\beta j(\tau)q(\tau)\,d\tau\right\}$$

then we can calculate thermal averages using $\frac{\delta^2 Z}{\delta j \delta j}\frac{1}{Z}$

Lets look @ the SHO again:

$$Z(\beta;j) = C\int \mathcal{D}x(\tau)\, e^{-\int_0^\beta d\tau\left(\frac{1}{2}x(\tau)\left(-\frac{d^2}{d\tau^2}+\omega^2\right)x(\tau)\right) + \int_0^\beta d\tau j(\tau)q(\tau)}$$

We can quickly calculate the propagator, or two point function of this theory by remembering that we can integrate this Gaussian via.

$$\int \prod_{i=1}^{i=N} d\varphi_i \exp\left\{-\frac{1}{2}\varphi^T A^{-1}\varphi + j^T\varphi\right\} = Z(0)\exp\left\{\frac{1}{2}j^T A\, j\right\}$$

so the propagator $G(\tau) = \langle x(\tau)x(0)\rangle$ can be read off as the inverse of the operator

$$\left(\frac{-d^2}{d\tau^2}+\omega^2\right)G(\tau) = \delta(\tau)$$

Because time is periodic, we can expand $G(\tau)$ in terms of Matsubara frequencies.

$$G(\tau) = \sum_{n\in\mathbb{Z}} G(i\omega_n)\, e^{i\omega_n\tau}$$

where

$$\omega_n = \frac{2\pi n}{\beta} = 2\pi n T \text{ for } n\in\mathbb{Z}$$

Inserting this expansion in our equation

$$\left(-\frac{d^2}{d\tau^2}+\omega^2\right)\sum_n G(i\omega_n)e^{i\omega_n\tau} = T\sum_n e^{i\omega_n\tau}$$

$$G(i\omega_n) = \frac{1}{\omega_n^2+\omega^2}.$$

so

$$G(\tau) = \sum_{n\in\mathbb{Z}} \frac{e^{i\omega_n\tau}}{\omega_n^2+\omega^2}.$$

How can we solve this? Well, without introducing Matsubara frequencies, we know that

$$G(\tau) = Ae^{\omega\tau} + Be^{-\omega\tau}$$

& $G(\tau) = G(\beta-\tau) \longrightarrow G(\tau) = A\left(e^{\omega\tau}+e^{\omega(\beta-\tau)}\right).$

In order to fix A, we can look @ $G(0) = A(1+e^{\omega\beta})$. From our Matsubara expansion, we know that

$$G(0) = \sum_{n\in\mathbb{Z}}^{T} \frac{1}{\omega_n^2+\omega^2} = \sum_{n\in\mathbb{Z}} g(k_0 = i\omega_n+\mu).$$

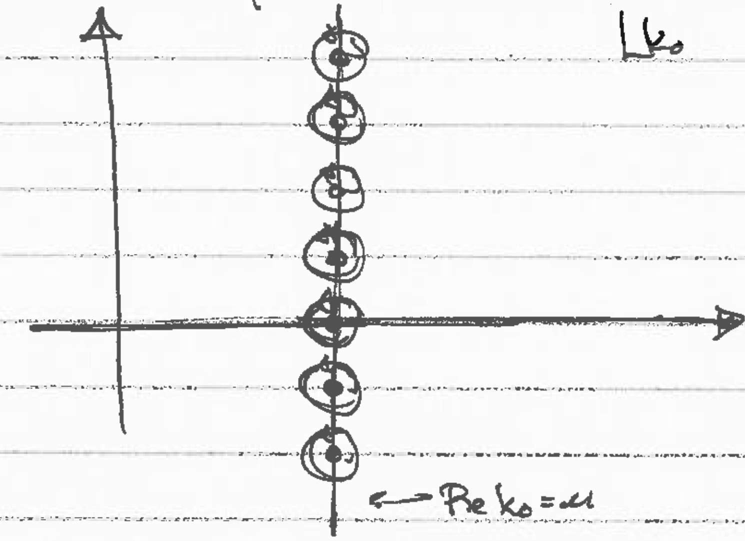
How do we deal with these sums? They show up everywhere. We have a term @ every n. coth x has a ~~pole @ ...~~ pole @ $x = i\pi n$, $n\in\mathbb{Z}$, so we can use this to write $G(0)$ as a bunch of contour integrals.

If we have a function $g(k_0)$ that is meromorphic & regular along the vertical line ~~k...~~ $\mathrm{Re}\{k_0\}=\mu$, & decreases faster than k_0^{-1} for $|k_0|\to\infty$, then we can use a lovely trick.

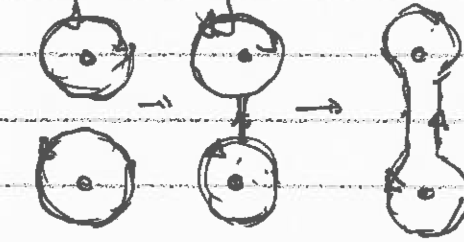
$$S = T\sum_{n\in\mathbb{Z}}{}' g(k_0 = i\omega_n + \mu) = \int_C \frac{dk_0}{2\pi i}\, g(k_0)\,\frac{1}{2}\coth\left(\frac{\beta(k_0-\mu)}{2}\right).$$

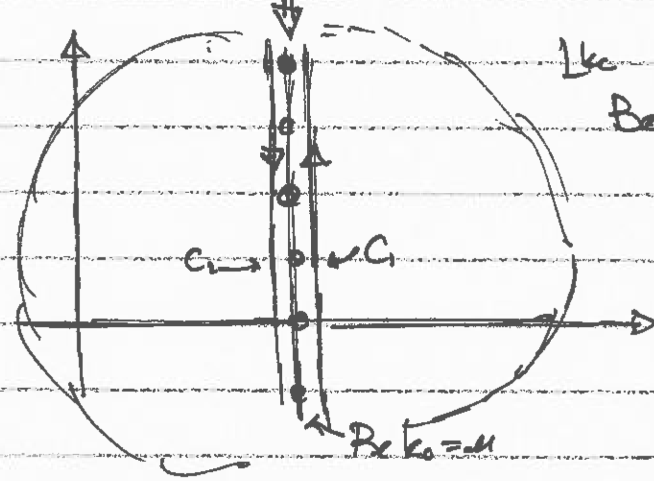
$\frac{\beta}{2}\coth\left(\frac{\beta(k_0-\mu)}{2}\right)$ has unit residue @ $k_0 = i\frac{2\pi n}{\beta} + \mu$.

where C represents the contour



We can think about deforming this contour





Because of the asymptotics of $g(k_0)$, we can close $C_1 + C_2$ with semicircles @ ∞. This will enclose the poles of $g(k_0)$ so that

$$S = +\frac{1}{2}\sum{}' \mathrm{Res}\, g(k_0)\coth\left(\frac{\beta(k_0-\mu)}{2}\right).$$

For $g(k_0 = i\omega_n)$, we have $\dfrac{1}{-k_0^2+\omega^2} = \dfrac{1}{\omega_n^2+\omega^2} = \dfrac{1}{(\omega-k_0)(\omega+k_0)}$.

$$\Rightarrow S = T\sum_{n\in\mathbb{Z}}{}' \frac{1}{\omega^2+\omega_n^2} = +\frac{1}{2}\left(\frac{1}{2\omega}\coth\left(\frac{\beta\omega}{2}\right) - \frac{1}{2\omega}\coth\left(\frac{-\beta\omega}{2}\right)\right)$$

$$= \frac{1}{2\omega}\coth\left(\frac{\beta\omega}{2}\right).$$

This fixes A such that

$$G(\tau) = \frac{1}{2\omega} \frac{\cosh\left(\left(\beta/2 - \tau\right)\omega\right)}{\sinh\left(\beta\omega/2\right)}$$

Note, since by $g(k_0) \to 0$ as $|k_0| \to \infty$ $\sum_i \text{Res}\, g(k_0) = 0$ & thus we can rewrite $S = \sum_i \frac{\text{Res}\, g(k_0)}{e^{\beta(k_0 - \omega)} - 1}$ ← BE factor

Summary:

- Interested in $Z_1 = \text{tr}\left[e^{-\beta\hat{H}}\right]$
- Found $Z_1 = \int_{x(0)=x(\beta)} \mathcal{D}q(\tau)\, e^{-S_E(\beta)}$ where $S_E(\beta)$ is the Euclidean action defined on a space with periodic imaginary time $\tau = it = \tau + \beta$.
- Periodic time naturally leads to expansions of quantities in terms of Matsubara frequencies.

§2 | The Free Scalar Field @ Finite Temperature

The previous "derivation" of the generating functional only depended on the fact that the Hamiltonian was quadratic in the conjugate momentum $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$. For a free scalar field (& the interacting one) nothing changes, except for the fact that we have more indices to keep track of.

$$\hookrightarrow Z = \text{tr}\, e^{-\beta\hat{H}} \to C \int_{\varphi(\beta)=\varphi(0)} \mathcal{D}\varphi \exp\left\{-\int_0^\beta d\tau \int d^{d}\vec{x}\, \mathcal{L}_E\right\}$$

with $\mathcal{L}_E = \frac{1}{2}\left(\frac{\partial\varphi}{\partial\tau}\right)^2 + \sum_{i=1}^{d} \frac{1}{2}\left(\frac{\partial\varphi}{\partial x_i}\right)^2 + V(\varphi)$.

Were we doing regular field theory, we'd want to fourier expand

$$\varphi(\vec{x},t) = \int \frac{d^d p}{(2\pi)^d} \tilde{\varphi}(\vec{p},t)\, e^{i\vec{p}\cdot\vec{x}-\omega_p t} = \int \frac{d^{d+1}p}{(2\pi)^d} \tilde{\varphi}(p)\, e^{ip\cdot x}$$

But periodic time allows us to expand in Matsubara frequencies

$$\varphi(\vec{x},t) = T \sum_{\omega_n} \int \frac{d^d k}{(2\pi)^d} \tilde{\varphi}(\omega_n,\vec{k})\, e^{i\omega_n \tau + i\vec{k}\cdot\vec{x}}$$

If we stick this expansion into our Boltzmann factor, this becomes

$$\exp\{-S_E(\beta)\} = \exp\left\{-\tfrac{1}{2} T \sum_{\omega_n} \tfrac{1}{V} \sum_{\vec{k}} (\omega_n^2 + \vec{k}^2 + m^2) |\tilde{\varphi}(\omega_n,\vec{k})|^2\right\}$$

$$= \prod_{\vec{k}} \exp\left\{-\tfrac{1}{2} \frac{T}{V} \sum_n (\omega_n^2 + \vec{k}^2 + m^2) |\tilde{\varphi}(\omega_n,\vec{k})|^2\right\}$$

which is Gaussian:

$$Z = \prod_{\vec{k}} \prod_n (\omega_n^2 + E_{\vec{k}}^2)^{-\frac{1}{2}} \prod_{n'} (\omega_n^2)^{\frac{1}{2}}$$

↳ No zero mode.

$$= \prod_{\vec{k}} \underbrace{T \prod_n (\omega_n^2 + E_{\vec{k}}^2)^{-\frac{1}{2}} \prod_{n'} (\omega_n^2)^{\frac{1}{2}}}_{\exp\left\{-\frac{1}{T}\left[\frac{E_{\vec{k}}}{2} + T\log(1-e^{-\beta E_{\vec{k}}})\right]\right\}}$$

$$\text{so } \log Z = \sum_{\vec{k}} \left(-\tfrac{1}{T}\left(\frac{E_{\vec{k}}}{2} + T\log(1-e^{-\beta E_{\vec{k}}})\right)\right)$$

$$\lim_{V\to\infty} \frac{\Omega}{V} = -\frac{\log Z}{\beta V} = \int \frac{d^d\vec{k}}{(2\pi)^d} \left(\frac{E_{\vec{k}}}{2} + T\log(1-e^{-\beta E_{\vec{k}}})\right).$$

In the massless limit, we find $m \ll g$

$$\frac{\Omega}{V} = \frac{T^4}{2\pi^2}\int_0^\infty dx\, x^2 \log\left(1-e^{-x}\right) = \frac{-\pi^2 T^4}{90} \leftarrow \text{Black Body Radiation!}$$

Including mass, we have

$$\Omega/V = \frac{-\pi^2 T^4}{90} + \frac{m^2T^2}{24} - \frac{m^3 T}{12\pi} - \frac{m^4}{2(4\pi)^2}\left(\log\left(\frac{m e^{\gamma_E}}{4\pi T}\right) - 3/4\right) + \dots$$

It will be useful for later to note that the $\frac{m^3 T}{12\pi}$ term is due to the Matsubara zero mode. To see this, we can convert the integral

$$\int \frac{d^dk}{(2\pi)^d}\left(\frac{E_{\vec{k}}}{2} + T\log\left(1-e^{-\beta E_{\vec{k}}}\right)\right) = T\sum_{\omega_n}\int\frac{d^dk}{(2\pi)^d}\left[\frac{1}{2}\log(\omega_n^2+E_{\vec{k}}^2) + \text{const}\right]$$

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{2E_{\vec{k}}}\left[1+2n_B(E_{\vec{k}})\right] \qquad n_B(E_{\vec{k}}) = \frac{1}{e^{\beta E_{\vec{k}}}-1}.$$

$$= T\sum_n\int\frac{d^dk}{(2\pi)^d}\frac{1}{\omega_n^2+E_{\vec{k}}^2}.$$

The zero mode contribution is divergent, but expanding $d = 3-2\epsilon$, we would find that its contribution yields

$-\frac{Tm}{4\pi} + \mathcal{O}(\epsilon)$, which upon multiplication by m & integration

yields $-\frac{Tm^3}{12\pi} + \mathcal{O}(\epsilon)$, which will be important to note later on

What about our lovely propagator? We know it should be the solution to

$$\left(-\frac{d^2}{d\tau^2} - \vec{\nabla}^2 + m^2\right) G(\tau, \vec{x}) = \delta(\tau)\,\delta(\vec{x}).$$

If we expand $G(\tau, \vec{x}) = T \sum_{\omega_n} \int \frac{d^d \vec{k}}{(2\pi)^d} \tilde{G}(i\omega_n, \vec{k}) e^{i\omega_n \tau + i\vec{k}\cdot\vec{x}}$

$$= \sum\!\!\!\!\!\!\int_{\tilde{k}} \tilde{G}(\tilde{k}) e^{i\tilde{k}\cdot\tilde{x}}.$$

then the propagator is written

$$\langle \tilde{\phi}(\tilde{P}) \tilde{\phi}(\tilde{Q}) \rangle = \frac{\delta(\tilde{P}+\tilde{Q})}{\tilde{P}^2 + m^2} = \frac{\delta_{\omega_p \omega_q} \delta(\vec{p}+\vec{q})}{\omega_n^2 + \vec{p}^2 + m^2}$$

with $\omega_n = \frac{2\pi n}{\beta}$.

$$\Rightarrow \langle \phi(x)\phi(y) \rangle = G(x-y)$$
$$= \sum\!\!\!\!\!\!\int_{\tilde{p}} e^{i\tilde{p}\cdot(x-y)} \frac{1}{\tilde{p}^2 + m^2}.$$

Using our familiar sum, we can replace the sum & integrand with

$$G(x-y) = \int \frac{d^d \vec{p}}{(2\pi)^d} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \frac{1}{2E_{\vec{p}}} \frac{\cosh\left(\left(\frac{\beta}{2} - |x_0 - y_0|\right) E_{\vec{p}}\right)}{\sinh\left(\frac{\beta E_{\vec{p}}}{2}\right)}.$$

~~When~~ For $G(|\vec{x}|)$, $|\vec{x}| \ll 1/T, 1/m$

$$G(|\vec{x}|) \simeq \frac{1}{4\pi^2 |x|^2}$$

while for $|\vec{x}| \gg 1/T$, $G(|\vec{x}|) \simeq T \frac{e^{-m|\vec{x}|}}{4\pi|\vec{x}|}$.

We see that IR behavior changes with T, while UV is unaffected. This makes sense, & is a common theme.

§3 | Interactions & Renormalization

Now lets think of adding an interaction term, so

$$\mathcal{L}_E = \frac{1}{2}\partial_\mu\varphi\,\partial_\mu\varphi + \frac{1}{2}m^2\varphi^2 + \frac{1}{4!}\lambda\varphi^4$$

We know how to deal with this from our QFT classes.

$$Z_1(\beta) = C\int \mathcal{D}\varphi\, e^{-S_0(\beta) - S_I(\beta)} = C\int \mathcal{D}\varphi\, e^{-S_0(\beta)}\, e^{-S_I}.$$

$$= C\int \mathcal{D}\varphi\, e^{-S_0(\beta)}\left(1 - S_I + \frac{S_I^2}{2} + \dots\right)$$

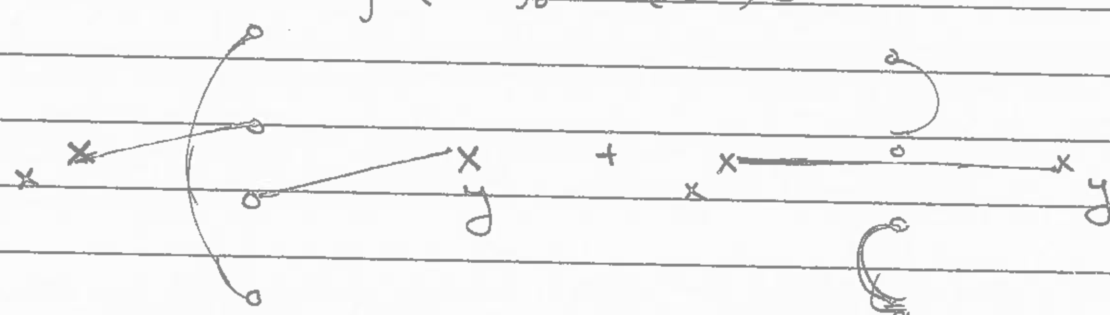
Lets look @ the propagator:

$$G(x-y) = \frac{1}{Z(\beta)}\int \mathcal{D}\varphi\, \varphi(x)\varphi(y)\, e^{-S_E(\beta)}$$

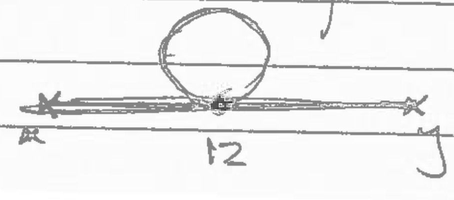
$$= \frac{1}{Z_1(\beta)}\int \mathcal{D}\varphi\, \varphi(x)\varphi(y)\left(1 - \lambda\int_0^\beta d^4z\, \varphi^4(z)\right) e^{-S_E^0(\beta)}$$

We know how to deal with this: Wick's theorem.

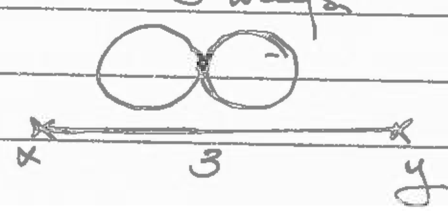
$$\int \mathcal{D}\varphi\, \varphi(x)\varphi(y)\left(1 - \lambda\int_0^\beta d^4z\, \varphi^4(z)\right) e^{-S_E^0(\beta)}.$$



4×3 Ways



3 ways



So,

$$Z(\beta)\, G(x,y) = \int \mathcal{D}\varphi \left(\varphi(x)\varphi(y)\left(1 - \lambda/4! \int d^4z\, \varphi^4(z)\right) e^{-S_E^0(\beta)} \right)$$

$$= Z_F(\beta)\Big(G_F(x-y) - \tfrac{12}{24}\lambda \int_0^\beta d^4z\, G_F(x-z)\, G_F(y-z)\, G_F(z=0)$$

$$- \tfrac{3}{24}\lambda \int_0^\beta d^4z\, G_F(x-y)\, G_F^2(z=0) \Big)$$

Fortunately, we haven't divided out $Z(\beta)$, which removes all vacuum diagrams. These leads us to

where the rules haven't changed much from what we are used to in regular QFT.

Indeed, the Feynman Rules are given by

1) Draw all topologically distinct, connected diagrams.
2) Assign a factor of $G_F(i\omega_n, \vec{k}) = \dfrac{1}{\omega_n^2 + \omega_k^2}$ to each internal line
3) Assign a factor $-\lambda\beta\, \delta\left(\sum_{i=1}^4 \omega_i\right)(2\pi)^3\, \delta^{(3)}\left(\sum_{i=1}^4 k_i\right)$ to each vertex.
4) Integrate over every internal line
$$T \sum_n \int \frac{d^3k}{(2\pi)^3}$$
5) Multiply by the symmetry factor.
6) There will be an overall $\beta(2\pi)^3\delta^{(3)}(0) = \beta V$.

So

$$G(x-y) \Rightarrow G(i\omega_n,\vec{k}) = G_F(i\omega_n,\vec{k}) + -\underbrace{\frac{\lambda}{2}}_{\text{Symmetry Factor}} G_F(i\omega_n,k)\Big($$
$$\times\left(T\sum_m \int \frac{d^3k'}{(2\pi)^3} G_F(i\omega_m,\vec{k}')\right) G_F(i\omega_n,k).$$

We can define the self-energy as

$$\triangleq\ G^{-1}(i\omega_n,\vec{k}) = G_F^{-1}(i\omega_n,\vec{k}) + \Pi(i\omega_n,\vec{k})$$

so, to first order we have $\Pi(i\omega_n,\vec{k}) = \frac{\lambda T}{2}\sum_{\omega_n}\int\frac{d^3k'}{(2\pi)^3} G_F(i\omega_n,\vec{k}')$

$$\Pi = \frac{\lambda T}{2}\int\frac{d^3k'}{(2\pi)^3}\sum_{\omega_n}\frac{1}{\omega_n^2+\omega^2} = \frac{\lambda}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\Big(\underbrace{1}_{\text{Divergent!}} + \frac{2}{e^{\beta\omega_k}-1}\Big)$$

Note that $T\to 0$, $\beta\to\infty$, what we are left with is the divergent part.

General Rule: A renor theory renormalized @ zero temperature stays renormalized.
Corollary: Temperature dependent effects are physical.

If we add a mass counterterm, then

$$\Pi = \delta m_\beta^2 = \frac{\lambda}{2}\int\frac{d^3k}{(2\pi)^3}\frac{n(\omega_k)}{\omega_k} \xrightarrow{m\to 0} \underbrace{\frac{\lambda T^2}{24}}$$

Massless Scalar Field gains a thermal mass!

~~The ...~~

What about higher order terms?

Naively we would think that the next correction is $\mathcal{O}(\lambda^2)$. This is too naive!

We can define Π by the self consistent equation

$$\Pi = \frac{\lambda}{2} T \sum_{\omega_n}{}' \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_n^2 + p^2 + m^2 + \Pi^2}$$

Lets look @ the massless case. We can use our tricks to convert this into

$$1 = \frac{\lambda}{4\pi^2} \int_1^\infty dx \frac{\sqrt{x^2-1}}{e^{\beta \Pi_\beta^{1/2} x} - 1} \quad \text{where } \Pi_\beta \equiv \Pi(T) - \Pi(0)$$

Thermal Part, or renormalized part.

Since $F(u) = \int_1^\infty dx \frac{\sqrt{x^2-1}}{e^{ux}-1} = \frac{2\pi^2}{u^2}\left(\frac{1}{12} - \frac{u}{4\pi} + \dots\right)$ $\mathcal{O}(u^2 \log u)$.

$$\Pi_\beta = \frac{\lambda T^2}{24}\left(1 - 3\left(\frac{\lambda}{24\pi^2}\right)^{1/2} + \dots\right).$$

If we resum all diagrams (1PI) then we see that the structure of perturbation theory breaks down.

But, where exactly is it breaking down?

Let's look @ the free energy order by order in λ_B & m_B.

$$\Omega^{(0)} = -\frac{\pi^2 T^4}{90} + \frac{m_B^2 T^2}{24} + -\frac{m_B^3 T}{12\pi} + \mathcal{O}(m_B^4)$$

We saw this came from the zero mode.

$$\Omega^{(1)}_1 = \tfrac{3}{4!}\lambda\left(G_F(0)\right)^2 = 3\,\text{OO} = \tfrac{3}{4}\lambda_B\left[\frac{T^4}{144} - \frac{m_B T^3}{24\pi} + \dots\right]$$

$$\Omega^{(2)} = -\tfrac{9}{4!}\lambda_B^2 \frac{T^4}{144}\,\frac{T}{8\pi m_B} + \dots$$

If we define $I(m_B, T) = T\sum_{\omega_n}\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{\omega_n^2 + E_k^2}$ to represent a vacuum bubble, $I^{n=0}$ is $\int \frac{d^dk}{(2\pi)^d}\frac{1}{E_k^2}$

we can then express the odd ~~po~~ m_B power corrections as

$$\delta_{odd}\,\Omega_1 = \frac{3}{4!}\lambda_B\, I'(0,T)\, I^{n=0} \rightarrow \text{zero mode.}$$

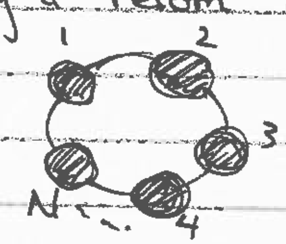
$$\delta_{odd}\,\Omega_2 = \tfrac{9}{4!}\lambda_B^2\,\underbrace{\left[I'(0,T)\right]^2}_{\text{Non-zero mode contribution.}}\,\frac{dI^{n=0}}{dm_B^2} \rightarrow \text{zero mode.}$$

we notice that $\frac{\delta_{odd}\,\Omega_1}{\delta_{odd}\,\Omega_0}$ or $\frac{\delta_{odd}\,\Omega_2}{\delta_{odd}\,\Omega_1} \sim \frac{\lambda_B T^2}{8 m_B^2} \rightarrow$ Diverges if the mass $\ll T$!

Things are going hella wrong because of IR divergences. The only way to fix this is to go deeper: resum.

From the above, we can see a pattern. Things are going wrong because of contributions that have multiple non-zeromode contributions & one zero-mode contribution. In other "words", we should try & resum "daisy" or "ring diagrams"

$$\sum_{N=2}^{\infty}$$



Noticing that

$$\Omega(T) = \langle S_I - \tfrac{1}{2} S_I^2 + \ldots + \frac{(-1)^{N+1}}{N!} S_I^N \rangle_{\text{connected}}$$
$$= \langle e^{-S_I} \rangle_{\text{connected}}$$

Odd Contribution:

$$\frac{(-1)^{N+1}}{N!} \langle S_I^N \rangle = \frac{(-1)^{N+1}}{N!} \left(\frac{\lambda_B}{4!}\right)^N \langle \overbrace{\phi\phi}^{}\underbrace{\phi\phi \;\; \phi}_{2(N-1)}\overbrace{\phi\phi}^{6}\phi \;\; \underbrace{\phi\phi\overbrace{\phi\phi}^{6}}_{2(N-2)}\ldots\overbrace{\phi\phi}^{6}\phi\phi \rangle$$

$$= \frac{(-1)^{N+1}}{N!}\left(\frac{\lambda_B}{4!}\right)^N \underbrace{6^N \left[2(N-1)\, 2(N-2) \ldots\right]}_{2^N (N-1)!} \underbrace{\left[\frac{T^2}{12}\right]^N}_{I'(0,T)} T \underbrace{\int \frac{d^d p}{(2\pi)^d}\left(\frac{1}{\vec{p}^2 + m_B^2}\right)^N}_{\text{Zero-mode term}}$$

Using induction, we can write $\int \frac{d^{d-1} p}{(2\pi)^d}\left(\frac{1}{\vec{p}^2+m_B^2}\right)^N = \frac{(-1)^N}{(N-1)!}\left(\frac{d}{dm_B^2}\right)^{N-1}\left(\frac{m_B^3}{6\pi}\right).$

Summing all contributions

$$\sum_{N=0}^{\infty} \frac{1}{N!}\left(\frac{\lambda_B T^2}{4!}\right)^N \left(\frac{d}{dm_B^2}\right)^N \left(\frac{-m_B^3 T}{12\pi}\right) = \frac{-T}{12\pi}\left(m_B^2 + \frac{\lambda_B T^2}{4!}\right)^{3/2}$$

which is finite as $m_B \to 0$!

$$P = -\frac{\partial \Omega}{\partial V} = \frac{\pi^2 T^4}{90}\left(1 - \frac{15}{8}\left(\frac{\lambda_B}{24\pi^2}\right) + \frac{15}{2}\left(\frac{\lambda_B}{4!\pi^2}\right)^{3/2} + \ldots\right).$$

Again, this continues the overall trend of thermal physics increasing IR sensitivity.

Gauge Fields & Fermions

What changes when we add fermions to the mix?

Remember the Fermionic Harmonic Oscillator.

$$\{a, a^\dagger\} = 1 \qquad a|0\rangle = 0 = a^\dagger|1\rangle$$

$$\hat{a}|1\rangle = |0\rangle \ \& \ \hat{a}^\dagger|0\rangle = |1\rangle \quad \Big] \rightarrow \text{Two States.}$$

We can introduce Grassmann valued numbers c & c^* & define

$$|c\rangle = e^{-ca^\dagger}|0\rangle = (1 - ca^\dagger)|0\rangle \Rightarrow a|c\rangle = c|0\rangle = c|c\rangle$$

$$\langle c| = \langle 0|e^{-ac^*} = \langle 0|(1 - \hat{a}c^*) \qquad \langle c|a^\dagger = \langle c|c^*.$$

$$\langle c'|c\rangle = e^{c'^*c} \quad \leftarrow \text{These are Fermionic Coherent States.}$$

Then,

$$\int dc^* dc\, e^{-c^*c} \langle -c|\hat{A}|c\rangle = \int dc^* dc \{1 - c^*c\} \langle 0|(1 + ac^*)\hat{A}(1 - ca^\dagger)|0\rangle$$

$$= \langle 0|\hat{A}|0\rangle + \langle 1|\hat{A}|1\rangle = \mathrm{Tr}\, \hat{A}.$$

If $\hat{A} = e^{-\beta \mathcal{H}}$, then we can split this operator up a bunch & perform the same song & dance we did for the scalar field. However, now we see that our field has to be anti-periodic!

$\Rightarrow$ Applying this to the Dirac Field:

$$Z = \int_{\substack{\bar{\Psi}(0,\vec{x}) = -\bar{\Psi}(\beta,\vec{x}) \\ \Psi(0,\vec{x}) = -\Psi(\beta,\vec{x})}} \mathcal{D}\Psi\, \mathcal{D}\bar{\Psi}\, \exp\left\{-\int_0^\beta d\tau \int d^d\vec{x}\, \mathcal{L}_E\right\}$$

$$\mathcal{L}_E = \bar{\Psi}(\hat{\gamma}_\mu \partial_\mu + m)\Psi$$

$\hookrightarrow$ Euclidean Dirac Matrices.

Fortunately, because of the antiperiodicity, Fermionic Matsubara Frequencies include no zero modes $\omega_n = \frac{2\pi}{\beta}(n+\frac{1}{2})$, $n \in \mathbb{Z}$.

Otherwise, nothing much changes. $\Omega = -4 \sum\!\!\!\!\int_{p_F} \frac{1}{2} \log(\tilde{P}_F^2 + m^2) + \text{const}$

4 Spinor DoF.

We don't have to do much work with our sums, either.

Let $S_f = T \sum_{\omega_n^f} f(\omega_n)$.

$$
\begin{aligned}
S_f &= T[\ldots + f(-3\pi T) + f(-\pi T) + f(\pi T) + f(3\pi T) + \ldots] \\
&= T[\ldots f(-3\pi T) + f(-2\pi T) + f(-\pi T) + f(0) + \ldots] \\
&\quad - T[f(-2\pi T) + f(0) + f(2\pi T) + \ldots] \\
&= 2\tfrac{T}{2}[\ldots f(-6\pi \tfrac{T}{2}) + f(-4\pi \tfrac{T}{2}) + f(-2\pi \tfrac{T}{2}) + f(0) + \ldots] \\
&\quad - T[f(-2\pi T) + f(0) + f(2\pi T) + \ldots] \\
&= 2 S_b(\tfrac{T}{2}) - S_b(T)
\end{aligned}
$$

$\Rightarrow$ Very often the $S_b(T) \propto \frac{1}{e^{\beta x} - 1}$.

$$
\Rightarrow 2\left(\frac{1}{e^{2\beta x} - 1}\right) - \frac{1}{e^{\beta x} - 1} = \frac{1}{e^{\beta x} - 1}\left[\frac{2}{e^{\beta x} + 1} - 1\right]
$$

$$
= -\frac{1}{e^{\beta x} + 1} \leftarrow \text{Fermi-Dirac!}
$$

What about Gauge Fields? A little more complicated, but the story goes through. Gauge Fix $\to$ Get Hamiltonian, $\to$ Impose Constraint on states, recover gauge

$$
\hookrightarrow \delta_{\hat{G}.c.} = \int_{-\pi}^{\pi} \frac{d\theta_i}{2\pi} e^{i\theta_i \hat{G}}
$$

$$Z_{phys} = C\oint^{+} \mathcal{D}A_\mu^a \int^{+} \mathcal{D}\bar{c}^a \mathcal{D}c^a \int^{-} \mathcal{D}\psi \mathcal{D}\bar{\psi} \times$$

$$\exp\left\{-\int_0^\beta d\tau \int_V d^{d}\vec{x} \left[\tfrac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \tfrac{1}{2\xi} G^a G^a + \bar{c}^a \frac{\delta G^a}{\delta\theta^b} c^b + \bar{\psi}(\hat{\gamma}_\mu \bar{D}_\mu + m)\psi\right]\right\}$$

As we saw for the scalar, do the gluons pick up a thermal mass? Yes!

Thermal Gluon Mass: $m_E^2 = g^2 T^2 \left(\frac{N_c}{3} + \frac{N_f}{6}\right)$ for massless quarks.

$$\Rightarrow \langle A_\mu^a A_\nu^b \rangle = \frac{\delta^{ab}\delta_{\mu\nu}\,\delta(\tilde{P}+\tilde{Q})}{\tilde{P}^2 + \delta_{\mu 0}\delta_{\nu 0} m_E^2}$$

← Color Electric Force Gets Screened by Thermal Plasma.

§5 Finite Density

Now, we know that when we are dealing with a non-relativistic theory, particle number is conserved. Not so in relativistic theories!

↳ How do conserved quantities play out in QFT?

If we have a conserved charge, then we will have to use a Grand Canonical Ensemble.

$$Z = \mathrm{tr}\left[e^{-\beta(\hat{H} - \mu \hat{Q})}\right]$$

~~Eq~~ a Where does μ come from?

We want to restrict to zero-charge states!

$$Z = \sum_{\psi_0}{}' \langle \psi_0 | e^{-\beta \hat{H}} | \psi_0 \rangle = \sum_{\psi}{}' \langle \psi | \delta_{\hat{Q}0} e^{-\beta \hat{H}} | \psi \rangle$$

$$= \int \sum_{\psi}{}' \langle \psi | e^{-ic\hat{Q}} e^{-\beta \hat{H}} | \psi \rangle$$

$$= \sum_{\psi}{}' \langle \psi | e^{-\beta(\hat{H} - \mu \hat{Q})} | \psi \rangle \rightarrow$$ μ plays the role of a constant, imaginary gauge field

Lets see how this works with the complex scalar:

$$\mathcal{L}_M = \partial_\mu \varphi \partial^\mu \varphi^* - m \varphi \varphi^* - \lambda (\varphi \varphi^*)^2$$

Perform the split:

$$\mathcal{H} = \tfrac{1}{2}\left(\Pi_1^2 + \Pi_2^2 + (\partial_i \varphi_1)^2 + (\partial_i \varphi_2)^2 + m^2 \varphi_1^2 + m^2 \varphi_2^2\right) + \tfrac{1}{4}\lambda(\varphi_1^2 + \varphi_2^2).$$

The Langrangian has the conserved current

$$J^\mu_\mu = i(\varphi \partial_\mu \varphi^* - \varphi^* \partial_\mu \varphi)$$

$$\Rightarrow \quad Q = \int d^3\vec{x}\, J^0 \rightarrow -\mu Q = \int d^3x\, \mu (\Pi_1 \varphi_2 - \Pi_2 \varphi_1).$$

$\Rightarrow$ Combining $\int d\Pi_1 \exp\left\{-\left(\tfrac{1}{2}\Pi_1^2 + \Pi_1(-i\mathring{\varphi}_1 - \mu\varphi_2)\right)\right\}$

$$\propto \exp\left\{-\tfrac{1}{2}\left(\frac{\partial \varphi_1}{\partial \tau} + i\mu\varphi_2\right)^2\right\}.$$

$$Z(T,\mu) \propto \int_{\text{Lo}}^{+} \mathcal{D}\varphi \exp\left\{-\int_0^\beta d\tau \int d^3x \left((\partial_\tau - \mu)\varphi^*(\partial_\tau + \mu)\varphi + \partial_i\varphi^* \partial_i \varphi + m^2 \varphi^* \varphi + \lambda(\varphi^* \varphi)^2\right)\right\}.$$

$$S_E = \sum\!\!\!\!\!\!\int_{\tilde{P}_0} \tilde{\varphi}^*(\tilde{P}_0)\left[(\omega_n - i\mu)^2 + \vec{p}^2 + m^2\right]\tilde{\varphi}(P_0).$$

$$\downarrow$$

$$\Omega(T,\mu) = \int \frac{d^d p}{(2\pi)^d}\left\{E_{\vec{p}} + T\log\left(1 - e^{-\frac{E+\mu}{T}}\right) + T\log\left(1 - e^{-\frac{E-\mu}{T}}\right)\right\}$$