Dynkin Diagrams
or
Everything You Ever Wanted to Know About Lie Algebras (But Were Too Busy To Ask)

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1 Introduction

This is a compilation of the definitions that I'll present, and of the important "algorithms" that are used to obtain cool stuff from Dynkin diagrams. I've taken pretty much everything from *Lie Algebras In Particle Physics*, by Howard Georgi, with the exception of the mistakes, which I came up with all by myself.

2 Definition

2.1 Cartan Subalgebra

The Cartan Subalgebra of a Lie algebra is the largest subset of elements of the algebra that can diagonalized simultaneously; said differently, it is the largest subset of elements of the algebra that all commute with one another. Following Georgi, the m elements of the Cartan Subalgebra will be written H^i, $i = 1, ..., m$.

2.2 Rank

The Rank of a Lie algebra is the number of elements contained in its Cartan Subalgebra.

2.3 Weights

The Weights, or Weight Vectors, of a representation, are vectors which contain the eigenvalues of the elements of the Cartan Subalgebra. As an example, if $H^1 = Diag(a, b, c)$ and $H^2 = Diag(d, e, f)$, the weight vectors are $w^1 = (a, d)$, $w^2 = (b, e)$, and $w^3 = (c, f)$.

2.3.1 Interesting Facts About Weights

-Weights live in an n-dimensional space, where n is the rank of the Lie Algebra
-The number of Weights associated with a representation is equal to the dimension of this representation, although some Weights may appear more than once in certain representations.

2.4 Roots

The Roots, or Roots Vectors, of a Lie algebra, are the Weight Vectors of its adjoint representation. We will use the notation α^i to denote the roots, with the index i indicating which particular root we are talking about, and NOT the ith element of the root vector.

Roots are very important, because they can be used both to *define* Lie algebras and to build their representations. We will see that Dynkin Diagrams are, in fact, really only a way to encode information about roots.

The number of Roots is equal to the dimension of the Lie algebra, which is also equal to the dimension of the adjoint representation; therefore, we can associate a Root to every element of the algebra. The operators associated with the root α is written E_α. All elements of the Cartan subalgebra are associated with the zero root.

Maybe the most important things about Roots is that they allow us to move from one Weight to another. More precisely, if $|w\rangle$ is a vector with weight w, then $E_\alpha|w\rangle$ is a vector with weight $w + \alpha$ (or vanishes). We can use this to find the roots of an algebra if we know one of its representations, by simply looking at the difference between the weights of that representation.

Also very important: the operators E_α, $E_{-\alpha}$, and the linear combination of Cartan generators $\alpha \cdot H$ (where here the $\cdot$ symbol indicate a dot product between the elements of the α vector and the vector whose elements are Cartan generators) form a SU(2) sub algebra (within some normalization factors). E_α acts like a raising operator for this sub algebra, and $E_{-\alpha}$ like a lowering operator.

There are too many other interesting and important facts about roots to list here; consult Georgi ch. 6 and ch. 8 for more.

2.5 Positive Roots

Positive Roots are defined to be Roots whose first non-vanishing element is positive.

2.6 Simple Roots

Simple Roots are Positive Roots which cannot be expressed as the sum of other positive roots. Simple roots will always form a linearly independent, complete set of vectors, but will in general not be orthonormal.

Simple Roots are magic. Their diverse properties are listed in Georgi ch. 8. If you know the simple roots of a Lie algebra, you can express all other roots as sums of the simple roots, and you can then use the roots to obtain the commutation relations of the algebra. This means that Lie algebras are uniquely defined by their simple roots. Simple roots can also be used to construct all the representation of an algebra, and to learn how these representations transform under subalgebras.

Dynkin Diagrams are used to record the structure of the system of simple roots of an algebra, telling us all we need to do to unleash their awesomeness.

3 Construction

3.1 Fundamental Weights

The Fundamental Weights of a Lie algebra are the set of n vectors μ^k, $k = 1, ..., n$, which live in a n-dimensional space, with n the rank of the Lie algebra, and which satisfy $\frac{2\mu^k \cdot \alpha^j}{(\alpha^j)^2} = \delta_{jk}$. They will be linearly independent, but in general will *not* be orthonormal, just like the roots.

The Fundamental Weights are extremely useful for building representations of a Lie algebra, for the following reason: the highest weight of any representation is always of the form $\sum_k l_k \mu^k$, where the l^k are non-negative integers called the *Dynkin Coefficients*. All representations are then uniquely identified by the set of Dynkin coefficients to which they corresponds. From the Dynkin coefficient we get the highest weight, and from the highest weight we can build the entire representation by applying the lowering operators corresponding to simple roots, as we shall see.

3.2 Cartan Matrix

The Cartan Matrix A is defined by $A_{ji} \equiv \frac{2\alpha^j \cdot \alpha^i}{(\alpha^i)^2}$. Here I'm following Georgi in using the index j to talk about rows and i to talk about columns, because even though that feels weird, I'm too scared to deviate from him. It is very easy to obtain the Cartan Matrix of a Lie algebra from its Dynkin diagram, using the following facts:

-The diagonal elements are always 2.
-For two distinct roots α^i and α^j, we have $A_{ji} * A_{ij} = 4\cos^2\theta_{ij}$ and $\frac{A_{ji}}{A_{ij}} = \frac{(\alpha^j)^2}{(\alpha^i)^2}$.

Since the Dynkin diagram tells us the angle and relative length between any two roots, we can use these two conditions to extract the Cartan matrix without much pain.

A very useful property of the Cartan matrix is that the jth row corresponds to the Dynkin coefficients of the jth simple root; that is, $\alpha^j = \sum_i A_{ji}\mu^i$.

3.3 Building Any Representation Using the Cartan Matrix

There is a straightforward algorithm that allows us to build any representation of a Lie algebra if we know its Cartan matrix. Unfortunately, this is one of those things which is simple to do, but hard to explain. I'll give it a shot here, but it's probably necessary to work through an example to see how this work. See Georgi ch. 8 and ch. 9, or the examples given at the blackboard.

The algorithm is:

i) Write down an ordered row of non-negative integers representing the Dynkin coefficients for the highest weight of the representation you want to build. This, of course, represents the highest weight of this representation.

ii) This highest weight state is also the highest weight of SU(2) subalgebras, corresponding to each simple roots of the Lie algebra. We can find the m quantum number of this state for the root α^i by dividing the ith Dynkin coefficient by 2. As an example, if the Dynkin coefficients are $[2, 1]$, then it is a $|j = 1, m = 1\rangle$ state under the SU(2) sub algebra associated with the simple root α^1, and a $|j = 1/2, m = 1/2\rangle$ state under the SU(2) subalgebra associated with the simple root α^2.

iii) Complete the SU(2) multiplets found above by drawing lines that descend from the highest weight state and adding rows of Dynkin coefficients for each member of the multiplet. We can find the Dynkin coefficients using the following rule: the Dynkin coefficients of a state which is obtained by "descending" from a state which is directly above it in the SU(2) multiplet corresponding to the root α^j is obtained by adding the jth line of the Cartan matrix to the Dynkin coefficients of the state right above it. "Descending" in a multiplet associated with the SU(2) subalgebra of the simple root α^i is equivalent to acting on states with the operator $E_{-\alpha^j}$, since this is the operator (to within a normalization factor) that corresponds to the lowering operator of this SU(2) sub algebra. So if μ is the highest weight, "descending" corresponds to obtaining to obtaining the weights of the representation, which will be equal to $\mu - \sum_i \alpha^i$, with the simple root α appearing in the sum being given by the "path" taken to get to the weight.
(At the risk of repeating myself, I'm afraid that this written explanation may be pretty difficult to understand, but it's really very simple: the examples worked out in Georgi should convince you of this).

iv) Descend this way, completing SU(2) multiliplets and adding rows of Dynkin coefficients, until you reach a weight from which it is not possible to descend anymore (i.e. a state completing the representation.

In this way, one obtains all the weights of a representation. We are not quite done however, because more than one state could be associated with any one weight. This will only be a possibility if there is more than one "path" that leads from the highest weight state to the weight in question, and even then it may be that this weight is uniquely attributed to a state. Verifying that a weight is attributed to a unique state is, once again, straightforward, but cumbersome. There seems to be a shortcut, however: if a weight has $m \neq 0$ for at least one of the SU(2) subalgebras, then it is associated with a unique state. I'm not 100% sure about this though, because Georgi never actually explicitly states it, but in practice he uses this rule in a couple of examples, so it's probably reliable. When a weight has $m = 0$ for all SU(2) sub algebra, then there's nothing to be done but to actually look at the different "paths" that leads to this state from the highest weight, and check to see if they give distinct states. The details are in Georgi 9.2.

I'll call this kind of construction a "Weight Diagram". I think this name is also used for graphs that display explicitly the weight vectors in n-dimensional space, but I'm pretty sure the context will eliminate risks of confusion.

3.4 Obtaining the Commutation Relations With the Cartan Matrix

We can use the procedure given above to obtain the adjoint representation, which can be identified as that representation that has all the simple roots as weights. Then, all non-vanishing weights in this representation can be uniquely associated with a generator, and this generator can be expressed as a multiple commutator of simple roots, since we know, by construction, how to obtain any root in term of the simple roots. Then the jacobi identity can be used to evaluate the commutators of all generators, giving us the commutation relation of this Lie algebra. Again: straightforward, but cumbersome. Look at Georgi 8.10 for an example.

4 Deconstruction

Another very useful property of Dynkin diagrams is that they tell us about the subalgebras, simple and semi-simple, of any Lie algebra.

4.1 Non-Maximal Subalgebras

If we take any Dynkin diagram and remove a circle from it, the result is either another Dynkin diagram, or a set of disconnected Dynkin diagrams. These corresponds to subalgebras of the of the Lie algebra that we started with. If the Dynkin diagrams are disconnected, then this corresponds to a *direct product* of the two subgroups associated with each disconnected diagram. In addition, each circle removed corresponds to a U(1) factor. The subalgebras obtained this way are called *Non-Maximal*, because they are of lower rank than the original algebra (when one ignores the U(1) factors).

4.2 Maximal Subalgebras and Extended Dynkin Diagrams

An algebra may also have *Maximal Subalgebras*, that is, subalgebras which are of the same rank as itself. To obtain these, we need to write down the *Extended Dynkin Diagram* that corresponds to each individual Dynkin diagram; these are obtained by adding a circle to the original Dynkin diagram, as shown in Georgi p. 252. You can then remove a circle, any circle, and the diagram that you obtain corresponds to a maximal subalgebra.

4.3 Representation Of Subalgebra

Finally, you can use the weight diagrams that we discussed above to see how a representation of a Lie algebra transform under its subalgebras. This is easy when the weight diagram has already been written. For a non-maximal sub algebra, you just have to remove the lines that correspond to the action of the simple root that you have removed from the Dynkin diagram. You will get a set of disconnected box diagram, each corresponding a representation of the sub algebra. The U(1) factors are a little trickier: they are proportional to $\mu^k \cdot \beta$, where μ^k is the fundamental weight associated with the simple root α^k that has been removed, and β is any weight which is part of a given representation of the subalgebra.

The only thing missing is the equivalent of this procedure, but for a maximal sub algebra. There are a couple of extra steps here. First, you build the extended Dynkin diagram, and you associate the extra circle with the *lowest* root of the algebra. Then, you can express the lowest root as a sum of simple roots and use this to define a Dynkin coefficient associated with the lowest root, which you can then attribute to all the weights of a representation. Finally, you now remove from the weight diagram the Dynkin coefficient associated with the circle that you remove from the extended Dynkin diagram to obtain the maximal sub algebra, and the box diagram will then break into disconnected box diagrams, each associated with a representation of the sub algebra. An example of this procedure is given in Georgi, p.305-306.

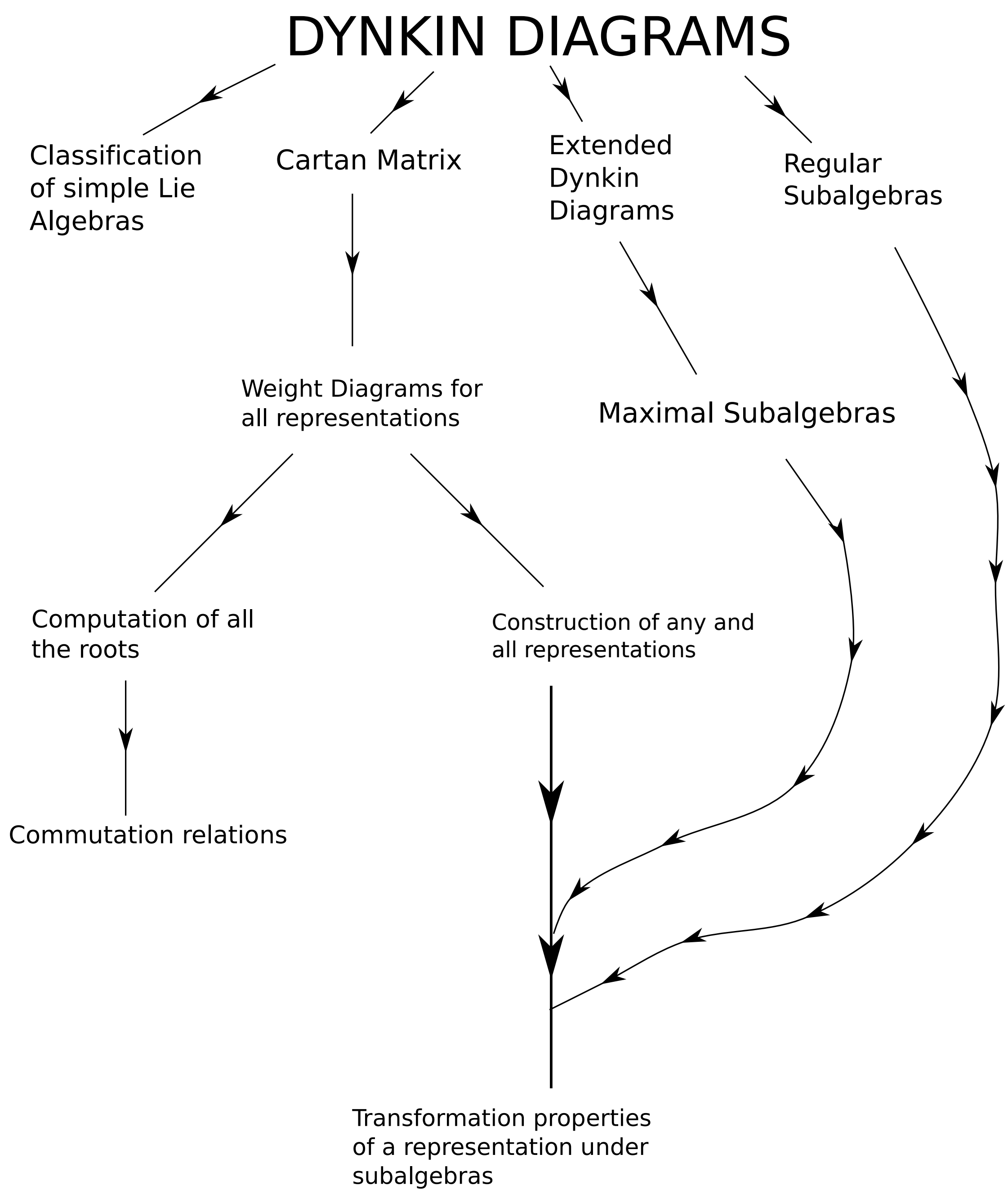

Figure 1: Helpful (?) flowchart indicating the information that can be derived from Dynkin diagrams.