Warped Penguins: Derivations
a.k.a. a pedagogical guide to bulk fermions in RS

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Abstract

This is an extra note detailing the derivations for our paper, *Warped Penguins* (`arXiv:1001.late`). It is an 'official' set of calculations starting from a pedagogical background. Please note that the main paper has the primary results and that this is meant to be supplementary material that is *not* meant to replace our arXiv paper.

Contents

1 What you already know

The reader is assumed to already be familiar with the Randall-Sundrum model and have some facility with working with bulk fields. A good summary can be found in the TASI lecture notes by Csáki et al. [1, 2]

2 Vielbeins, Spin Connections, and Antisymmetrization

How does one write down the fermionic action for a curved spacetime, such as the Randall-Sundrum scenario? In the semiclassical limit where gravity is treated as a classical background for quantum fields, the effects of the curvature of space on the action are:

1. **Covariant element.** The measure is promoted to a covariant volume element,

$$d^dx \to d^dx\sqrt{|g|},$$

   where g is the determinant of the metric.

2. **Metric.** The contraction of Lorentz indices must be accounted for with explicit factors of the nontrivial metric.

3. **Covariant derivatives.** The derivative operators must be promoted to the appropriate covariant derivatives. In particular, the covariant derivative acting on fermions is the spin covariant derivative.

4. **Vielbeins.** The γ matrices which act on spinor indices are define on the tangent space. Vielbeins must be inserted to convert spacetime indices to tangent space indices.

The first two are rather well known and should be familiar for those who have worked with scalar fields on curved backgrounds. Those requiring further background are encouraged to review their favorite general relativity book. The remaining two items are not necessarily covered in a typical general relativity or field theory course and are worth discussing a bit further.

2.1 Vielbeins

The γ matrices which obey the Clifford algebra are only defined for flat spaces. Another way to say this is that they live on the tangent space of our spacetime. In order to define curved-space generalizations of objects like the Dirac operator $i\gamma^\mu\partial_\mu$, we need a way to go from spacetime indices M to tangent space indices a. **Vielbeins** are geometric objects which do precisely this: $e^a_\mu(x)$. The completeness relations associated with vielbeins allow them to be interpreted as a sort of "square root" of the metric in the sense that

$$g_{MN}(x) = e^a_M(x)e^b_M(x)\eta_{ab}, \tag{2.1}$$

where η_{ab} is the flat (e.g. Minkowski) metric on the tangent space. For our particular purposes we will need the inverse vielbein, $E_a^M(x)$, defined such that

$$E_a^M(x)e_N^a(x) = \delta^M{}_N \tag{2.2}$$
$$E_a^M(x)e_N^b(x) = \delta_a{}^b. \tag{2.3}$$

The capital 'E' for the inverse vielbein is just a silly notation picked up from somewhere that helps visually distinguish e_μ^a from its inverse. Spacetime indices are raised and lowered using the spacetime metric $g_{MN}(x)$ while tangent space indices are raised and lowered using the flat metric $\eta_{ab}(x)$.

A more physically motivated way of thinking about the vielbein is in terms of Einstein's equivalence principle, which states that at any point one can always set up a coordinate system such that the metric is flat (Minkowski) at that point. Thus for each point x in space there exists a family of coordinate systems that are flat at x. For each point we may choose one such coordinate system, which we call a frame. By general covariance one may define a map that transforms to this flat coordinate system at each point. This is the vielbein. One can see that it is a kind of local gauge transformation, and indeed this is the basis for treating gravity as a gauge theory built upon diffeomorphism invariance. In slightly more mathematical, the vielbein represents the frame bundle on the spacetime.

2.2 Spin covariant derivative

Moving on we come to the **spin covariant derivative**. By now we are familiar that the covariant derivative is composed of a partial derivative term plus connection terms which depend on the particular object being differentiated. For example, the covariant derivative on a spacetime vector V^μ is

$$D_M V^N = \partial_M V^N + \Gamma^N_{ML} V^L. \tag{2.4}$$

Now that we are armed with a vielbein, however, the object V didn't necessarily need to have a spacetime vector index, μ. We could convert it into an object with a tangent space index, a. (And using gamma/Pauli matrices we can convert this into spinor indices.) We would then like to define a covariant derivative acting on the tangent space vector V^a,

$$D_M V^a = \partial_M V^a + \omega^a_{Mb} V^b, \tag{2.5}$$

where the quantity ω^a_{Mb} is called the **spin covariant derivative**. Consistency of the two equations implies

$$D_M V^a = e^a_N D_M V^N. \tag{2.6}$$

This is sufficient to determine the spin connection. We won't prove the result here, but one can look up the appropriate references (e.g. your favorite general relativity of differential geometry books). The result is [3]

$$\omega_M^{ab} = \frac{1}{2} g^{RP} e_R^{[a} \partial_{[M} e_{P]}^{b]} + \frac{1}{4} g^{RP} g^{TS} e_R^{[a} e_T^{b]} \partial_{[S} e^c_{P]} e^d_M \eta_{cd} \tag{2.7}$$
$$= \frac{1}{2} e^{Na} \left(\partial_M e_N^b - \partial_N e_M^b\right) - \frac{1}{2} e^{Nb} \left(\partial_M e_N^a - \partial_N e_M^a\right) - \frac{1}{2} e^{Pa} e^{Rb} \left(\partial_P e_{Rc} - \partial_R e_{Rc}\right) e^c_M. \tag{2.8}$$

When acting on spinors one needs the appropriate structure to convert the a, b tangent space indices into spinor indices. This is provided by

$$\sigma_{ab} = \frac{1}{4}\left[\gamma_a, \gamma_b\right] \tag{2.9}$$

so that the appropriate spin covariant derivative is

$$D_M = \partial_M + \frac{1}{2}\omega_M^{ab}\sigma_{ab}. \tag{2.10}$$

2.3 Antisymmetrization and Hermiticity

Finally, the fermionic action takes the form

$$S = \int d^d x\, \sqrt{|g|}\; \overline{\Psi}\left(iE_a^M \gamma^a \overleftrightarrow{D_M} - m\right)\Psi, \tag{2.11}$$

where the antisymmetrized covariant derivative is defined by

$$\overleftrightarrow{D_M} = \frac{1}{2}D_M - \frac{1}{2}\overleftarrow{D_M}. \tag{2.12}$$

This last point is somewhat subtle. The canonical form of the fermionic action must be antisymmetric in this derivative in order for the operator to be Hermitian (and thus for the action to be real). In flat space we are free to integrate by parts in order to get an only right-acting Dirac operator. This is a very nice thing to say and ‘makes sense,’ but the actual meaning is a little bit subtle.

Hermiticity is defined with respect to an inner product. The inner product in this case is given by

$$\langle \Psi_1 | \mathcal{O}\Psi_2\rangle = \int d^5 x\, \sqrt{|g|}\; \overline{\Psi_1}\mathcal{O}\Psi_2. \tag{2.13}$$

A manifestly Hermitian operator is given by $\mathcal{O}_H = \frac{1}{2}\left(\mathcal{O} + \mathcal{O}^\dagger\right)$, where we recall that

$$\langle \Psi_1 | \mathcal{O}^\dagger \Psi_2\rangle = \langle \mathcal{O}\Psi_1 | \Psi_2\rangle \tag{2.14}$$

$$= \int d^5 x\, \sqrt{|g|}\; \overline{\mathcal{O}\Psi_1}\Psi_2. \tag{2.15}$$

The definition of an inner product on the phase space of a quantum field theory is a nontrivial matter in the study of QFT on curved spacetimes. Since our spacetime is not warped in the time direction there is no ambiguity in picking a canonical Cauchy surface to quantize our fields and we may follow the usual procedure of Minkowski space quantization with the usual Minkowski spinor inner product.

As a sanity-check, consider the case of the partial derivative operator ∂_μ on flat space time. The Hermitian conjugate of the operator is the left-acting derivative, $\overleftarrow{\partial_\mu}$, by which we really mean

$$\int d^d x\, \overline{\Psi_1}\partial^\dagger \Psi_2 = \langle \Psi_1 | \partial_\mu^\dagger \Psi_2\rangle = \langle \partial_\mu \Psi_1 | \Psi_2\rangle = \int d^d x\, \overline{\partial_\mu \Psi_1}\Psi_2 = \int d^d x\, \overline{\Psi_1}\overleftarrow{\partial_\mu}\Psi_2 = \int d^d x\, \overline{\Psi_1}\left(-\partial_\mu\right)\Psi_2.$$

In the last step we've integrated by parts and dropped the boundary term. We see that the Hermitian conjugate of the partial derivative is negative itself. Thus the partial derivative is not a Hermitian operator. This is why the momentum operator is given by $\hat{P}_\mu = i\partial_\mu$, since the above analysis then yields $\hat{P}_\mu^\dagger = \hat{P}_\mu$ (again dropping the boundary term and noting that the i flips sign under the bar).

Now we can explicitly write out what we mean by the left-acting derivative in eq. (2.11). The operator $iE_a^M\gamma^a D_M$ is not Hermitian and needs to be made Hermitian by writing it in the form $\mathcal{O}_H = \frac{1}{2}\left(\mathcal{O} + \mathcal{O}^\dagger\right)$. Thus we may write a manifestly Hermitian Dirac operator as,

$$\overline{\Psi}\,(\text{Dirac})\,\Psi = \overline{\Psi}\left[\frac{1}{2}\left(iE_a^M\gamma^a D_M\right) + \frac{1}{2}\left(iE_a^M\gamma^a D_M\right)^\dagger\right]\Psi \tag{2.16}$$

$$= \overline{\Psi}\frac{i}{2}E_a^M\gamma^a D_M\Psi + \overline{\frac{i}{2}E_a^M\gamma^a D_M\Psi}\Psi \tag{2.17}$$

$$= \overline{\Psi}\frac{i}{2}E_a^M\gamma^a D_M\Psi - \frac{i}{2}E_a^M\overline{\gamma^a D_M\Psi}\Psi, \tag{2.18}$$

where we've used the fact that E_a^M is a real function with no spinor indices. The second term on the right-hand side can be massaged further,

$$\overline{\gamma^a D_M\Psi}\Psi = \Psi^\dagger\overleftarrow{D_M}^\dagger\gamma^{a\dagger}\gamma^0\Psi \tag{2.19}$$

$$= \Psi^\dagger\overleftarrow{D_M}^\dagger\left(\gamma^0\gamma^a\gamma^0\right)\gamma^0\Psi \tag{2.20}$$

$$= \Psi^\dagger(\overleftarrow{\partial_M} + \omega_M^{bc}\sigma^{bc\dagger})\gamma^0\gamma^a\Psi \tag{2.21}$$

$$= \overline{\Psi}\overleftarrow{D_M}\gamma^a\Psi \tag{2.22}$$

$$= \overline{\Psi}\gamma^a\overleftarrow{D_M}\Psi. \tag{2.23}$$

Note that we have used that $\gamma^{M\dagger} = \gamma^0\gamma^M\gamma^0$ and, in the last line, that $[\sigma^{bc}, \gamma^a] = 0$. Finally, putting this all together, we can write down our manifestly real fermion action (i.e. manifestly Hermitian Dirac operator) as in eq. (2.11),

$$S = \int d^dx\,\sqrt{|g|}\;\overline{\Psi}\left(iE_a^M\gamma^a\overleftrightarrow{D_M} + m\right)\Psi \tag{2.24}$$

$$= \int d^dx\,\sqrt{|g|}\left(\frac{i}{2}\overline{\Psi}E_a^M\gamma^a D_M\Psi - \frac{i}{2}\overline{D_M\Psi}E_a^M\gamma^a\Psi - m\overline{\Psi}\Psi\right). \tag{2.25}$$

All of this may seem a little pedantic since integration by parts allows one to go back and forth between the 'canonical' form and the 'right-acting only' form of the fermion kinetic operator (Dirac operator). Our interest, however, is to apply this to the Randall-Sundrum background where integration by parts will generally introduce boundary terms and so it is crucial to take the canonical form of the Dirac operator as the starting point.

2.4 Further Reading

We've been necessarily terse here, listing only the immediately relevant results. The material touched upon, however, goes much deeper and play key roles in the elegant differential geometric structure of physics. The curious reader is encouraged to pursue these topics further by looking at the appropriate differential geometry and general relativity references.

Vielbeins (lit. "many legs") are often referred to as 'vierbeins' ("four legs") or 'tetrads' in general relativity books (pick up your favorite and see[1]). They are often mentioned in associated with the Cartan formalism of gravity or as frame fields (or frame bundles for the mathematically inclined). They also appear in quantum field theory as key ingredients to promoting local supersymmetry to supergravity [4, 5, 6]. The spin connection requires a little more mathematical investment to appreciate, but an excellent and accessible introduction can be found in chapter 12 of Green, Schwarz, Witten, volume 2 [7]. Finally, the fermionic action on a curved spacetime is presented in Bertlmann's textbook [8].

An important caveat should be made here: it is the opinion of the author that Bertlmann makes some misguided statements in this section of his book. While the statements in his book are all correct for manifolds without boundaries, they are presented somewhat backwards and are misleading for manifolds with boundaries (e.g. the RS orbifold). He starts with the right-acting action and shows that integration by parts allows one to convert the right-acting action into the canonical antisymmetrized action. As noted above this is not true for spacetimes with boundaries. It's somewhat ironic that the original Grossman-Neubert paper on bulk RS fermions [9] cites Bertlmann when writing the action. The analysis is still correct, however, since the chiral boundary conditions imposed on the fermions end up cancelling the incorrect boundary terms that appear when Bertlmann's method is applied to the RS spacetime[2].

An excellent book for a broader picture of the application of differential geometry to physics, Göckeler and Schucker provides fantastic breadth and depth in a compact volume written in a way that will appeal to particle theorists.

Finally, a nice reference touching on many of these topics as applied to 5D spaces in quantum field theory is Sundrum's TASI lectures [10] and pedagogical write-up [3].

3 The Randall-Sundrum Bulk Fermion Action

We now specialize to the case of the Randall-Sundrum background,

$$ds^2 = (R/z)^2 \left(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2\right). \tag{3.1}$$

We use the 'modern' notation where the metric is conformally flat. This is the natural choice for invoking the AdS/CFT correspondence [11], though the generation of hierarchies is not as clear as the original choice of coordinates with an exponential warp factor[3].

One can convert back and forth between these notations using

$$z = Re^{ky} \qquad\qquad k = 1/R. \tag{3.2}$$

[1]If you don't find it, then you're reading an undergraduate textbook.

[2]It may be useful to recall that the metric is covariantly constant.

[3]In these conformal coordinates the generation of exponential hierarchies is based on a radius stabilization mechanism, e.g. Goldberger-Wise.

3.1 The spin connection

In these coordinates we may write the vielbein and inverse vielbein as

$$e^a_M(z) = \frac{R}{z}\delta^a_M \qquad\qquad E^M_a(z) = \frac{z}{R}\delta^M_a. \tag{3.3}$$

We may write out the spin connection term of the covariant derivative as

$$\omega^{ab}_M = \underbrace{\frac{1}{2}g^{RP}e^{[a}_R\partial_{[M}e^{b]}_{P]}}_{\omega^{ab}_M(1)} + \underbrace{\frac{1}{4}g^{RP}g^{TS}e^{[a}_R e^{b]}_T\partial_{[S}e^a_{P]}e^d_M\eta_{cd}}_{\omega^{ab}_M(2)}. \tag{3.4}$$

This can be simplified using the fact that the vielbein only depends on z. The first part is

$$\begin{aligned}
\omega^{ab}_M(1) = & \ \frac{1}{2}g^{RP}e^a_R\partial_{[M}e^b_{P]} - \frac{1}{2}g^{RP}e^b_R\partial_{[M}e^a_{P]} & (3.5)\\
= & \ \frac{1}{2}g^{RP}e^a_R\partial_M e^b_P - \frac{1}{2}g^{RP}e^a_R\partial_P e^b_M - \frac{1}{2}g^{RP}e^b_R\partial_M e^a_P + \frac{1}{2}g^{RP}e^b_R\partial_P e^a_M & (3.6)\\
= & -\frac{1}{2z}g^{RP}e^a_R e^b_P\delta^5_M + \frac{1}{2z}g^{RP}e^a_R e^b_M\delta^5_P + \frac{1}{2z}g^{RP}e^b_R e^a_P\delta^5_M - \frac{1}{2z}g^{RP}e^b_R e^a_M\delta^5_P & (3.7)\\
= & -\frac{1}{2z}\eta^{ab}\delta^5_M + \frac{1}{2z}g^{R5}e^a_R e^b_M + \frac{1}{2z}\eta^{ba}\delta^5_M - \frac{1}{2z}g^{R5}e^b_R e^a_M & (3.8)\\
= & -\frac{1}{2z}\eta^{ab}(\delta^5_M - \delta^5_M) + \frac{1}{2z}g^{R5}\left(e^a_R e^b_M - e^b_R e^a_M\right) & (3.9)\\
= & -\frac{1}{2z}\delta^R_5\left(\delta^a_R\delta^b_M - \delta^b_R\delta^a_M\right) & (3.10)\\
= & \ \frac{1}{2z}\delta^{[a}_M\delta^{b]}_5, & (3.11)
\end{aligned}$$

where we've used $\partial_M e^b_P = -\frac{1}{z}e^b_P\delta^5_M$ and the completeness relation $g^{MN}e^a_M e^b_M = \eta^{ab}$. The second part is given by

$$\begin{aligned}
\omega^{ab}_M(2) = & \ \frac{1}{4}g^{RP}g^{TS}e^a_R e^b_T\partial_{[S}e^c_{P]}e^d_M\eta_{cd} - \frac{1}{4}g^{RP}g^{TS}e^b_R e^a_T\partial_{[S}e^c_{P]}e^d_M\eta_{cd} & (3.12)\\
= & \ \frac{1}{4}g^{RP}g^{TS}e^a_R e^b_T\partial_S e^c_P e^d_M\eta_{cd} - \frac{1}{4}g^{RP}g^{TS}e^b_R e^a_T\partial_S e^c_P e^d_M\eta_{cd} & \\
& -\frac{1}{4}g^{RP}g^{TS}e^a_R e^b_T\partial_P e^c_S e^d_M\eta_{cd} + \frac{1}{4}g^{RP}g^{TS}e^b_R e^a_T\partial_P e^c_S e^d_M\eta_{cd} & (3.13)\\
= & -\frac{1}{4z}g^{RP}g^{TS}e^a_R e^b_T\delta^5_S e^c_P e^d_M\eta_{cd} + \frac{1}{4z}g^{RP}g^{TS}e^b_R e^a_T\delta^5_S e^c_P e^d_M\eta_{cd} & \\
& +\frac{1}{4z}g^{RP}g^{TS}e^a_R e^b_T\delta^5_P e^c_S e^d_M\eta_{cd} - \frac{1}{4z}g^{RP}g^{TS}e^b_R e^a_T\delta^5_P e^c_S e^d_M\eta_{cd} & (3.14)\\
= & \ \frac{1}{4z}\left(-\eta^{ac}g^{T5}e^b_T e^d_M\eta_{cd} + \eta^{bc}g^{T5}e^a_T e^d_M\eta_{cd} + g^{R5}\eta^{bc}e^a_R e^d_M\eta_{cd} - g^{R5}\eta^{ac}e^b_R e^d_M\eta_{cd}\right) & (3.15)\\
= & \ \frac{1}{4z}\left(\delta^T_5\delta^b_T\delta^d_M\delta^a_d - \delta^T_5\delta^a_T e^d_M\delta^b_d - \delta^R_5\delta^a_R\delta^d_M\delta^b_d + \delta^R_5\delta^b_R\delta^d_M\delta^a_d\right) & (3.16)\\
= & \ \frac{1}{2z}\left(\delta^b_5\delta^a_M - \delta^a_5\delta^b_M - \delta^a_5\delta^b_M + \delta^b_5\delta^a_M\right) & (3.17)\\
= & \ \frac{1}{2z}\delta^{[a}_M\delta^{b]}_5. & (3.18)
\end{aligned}$$

Note that these vanish identically for $M = 5$. We can now write out the spin-connection part of the covariant derivative,

$$\frac{1}{2}\omega_M^{ab}\sigma_{ab} = \frac{1}{2}\left(\frac{1}{z}\delta_M^{[a}\delta_5^{b]}\right)_{M\neq 5}\frac{1}{4}\left[\gamma_a,\gamma_b\right] \tag{3.19}$$

$$= \frac{1}{4z}\left(\gamma_M\gamma_5 + \delta_M^5\right), \tag{3.20}$$

where we've inserted a factor of δ_M^5 to cancel the $(\gamma_5)^2$ when $M = 5$. (Note that the natural convention is that $(\gamma^5)^2 = -\mathbb{1}$ since this is what satisfies the 5D Clifford algebra.)

Finally, the spin connection part of the covariant derivative is

$$\frac{1}{2}\omega_M^{ab}\sigma_{ab} = \frac{1}{4z}\left(\gamma_M\gamma_5 + \delta_M^5\right) \tag{3.21}$$

so that the spin covariant derivative is

$$D_M = \begin{cases} \partial_\mu + \frac{1}{4z}\gamma_\mu\gamma_5 & \text{if } M = \mu \\ \partial_5 & \text{if } M = 5. \end{cases} \tag{3.22}$$

Now after all that pencil-pushing, we can say something rather anticlimactic: the spin connection drops out of the action.

$$S = \int d^5x \left(\frac{R}{z}\right)^4 \frac{i}{2}\overline{\Psi}\delta_a^M\gamma^a D_M\Psi - \frac{i}{2}\delta_a^M\overline{D_M\gamma^a\Psi}\Psi \tag{3.23}$$

$$= \int d^5x\, \frac{i}{2}\left(\frac{R}{z}\right)^4 \left(\overline{\Psi}\gamma^M\overleftrightarrow{\partial_M}\Psi + \frac{1}{4z}\overline{\Psi}\gamma_\mu\gamma_5\gamma^\mu\Psi - \frac{1}{4z}\overline{\gamma_\mu\gamma_5\gamma^\mu\Psi}\Psi\right), \tag{3.24}$$

The two spin connection terms cancel since $\overline{\gamma_\mu\gamma^5\gamma^\mu\Psi}\Psi = \overline{\Psi}\gamma_\mu\gamma_5\gamma^\mu\Psi$, so that upon including a bulk mass term,

$$S = \int d^5x\, \frac{i}{2}\left(\frac{R}{z}\right)^4 \overline{\Psi}\gamma^M\overleftrightarrow{\partial_M}\Psi - \int d^5x\, \frac{i}{2}\left(\frac{R}{z}\right)^5 m\overline{\Psi}\Psi \tag{3.25}$$

$$= \int d^5x\, \frac{i}{2}\left(\frac{R}{z}\right)^4 \overline{\Psi}\left(\gamma^M\overleftrightarrow{\partial_M} - \frac{c}{z}\right)\Psi, \tag{3.26}$$

where $c = mR = m/k$ is a dimensionless parameter that is the ratio of the bulk mass to the curvature. As we will see, the bulk mass does not contribute directly to the 4D Kaluza-Klein mass spectrum of the model. Instead, c determines the localization of the 5D wavefunction. This, in turn, determines the overlap with the Higgs field and the contribution to masses from electroweak symmetry breaking. More comprehensive discussions can be found in the original paper by Grossman and Neubert [9] or the review by Gherghetta [11].

3.2 Right-acting RS Fermionic Action

When deriving the Dirac equation from the variational principle we set all of our operators to be right-acting, i.e. acting on Ψ, so that we can vary with respect to $\overline{\Psi}$ to get an operator equation for Ψ. Obtaining this is from eq. (3.26) is now a straightforward matter of integration by parts of the left-acting derivative term. Note that it is now crucially important that we pick up a derivative acting on the metric/vielbein factor $(R/z)^4$. We would have missed this term if he had mistakenly written our original 'canonical action,' eq. (2.11), as being right-acting only.

The integration by parts for the $M = \mu = 0, \cdots, 4$ terms proceeds trivially since these directions have no boundary and the metric/vielbein factor is independent of them. Doing the $M = 5$ integration by parts we find

$$S = \int d^4x \int_{R'}^{R} dz \left(\frac{R}{z}\right)^4 \overline{\Psi} \left(i\partial\!\!\!/ + \frac{i}{2}\gamma^5 \overleftrightarrow{\partial_5} - \frac{c}{z}\right) \Psi \tag{3.27}$$

$$= \int d^4x \int_{R'}^{R} dz \left(\frac{R}{z}\right)^4 \overline{\Psi} \left(i\partial\!\!\!/ + i\gamma^5\partial_5 - i\frac{2}{z}\gamma^5 - \frac{c}{z}\right) \Psi + (\text{boundary term})\big|_{R'}^{R}. \tag{3.28}$$

The term in the parenthesis can be identified with the Dirac operator for the Randall-Sundrum model with bulk fermions. (This 'definition' is up to conventions regarding the inclusion of the mass term and factors of i.) The boundary term takes the form

$$(\text{boundary}) = (R/z)^4 \left(\psi\chi - \overline{\chi}\overline{\psi}\right)\big|_{R'}^{R}, \tag{3.29}$$

where we've written out the Dirac spinor Ψ in terms of two-component Weyl spinors χ and ψ. This term vanishes when we impose chiral boundary conditions, which we shall review in the next section. The final form of the RS fermion action is

$$S = \int d^4x \int_{R'}^{R} dz \left(\frac{R}{z}\right)^4 \overline{\Psi} \left(i\partial\!\!\!/ + i\gamma^5\partial_5 - i\frac{2}{z}\gamma^5 - \frac{c}{z}\right) \Psi. \tag{3.30}$$

In terms of Weyl spinors this has the form

$$S = \int d^4x \int_{R'}^{R} dz \left(\frac{R}{z}\right)^4 \begin{pmatrix}\psi & \overline{\chi}\end{pmatrix} \begin{pmatrix} -\partial_5 + \frac{2-c}{z} & i\partial\!\!\!/ \\ i\overline{\partial\!\!\!/} & \partial_5 - \frac{2+c}{z} \end{pmatrix} \begin{pmatrix}\chi \\ \overline{\psi}\end{pmatrix}, \tag{3.31}$$

where $\overline{v\!\!\!/} = v_\mu \overline{\sigma}^\mu$, $v\!\!\!/ = v_\mu \sigma^\mu$.

3.3 Chiral boundary conditions

Recall that 5D theories are vectorlike, meaning that the fundamental spinor representation is a Dirac spinor (containing both left- and right-handed components) rather than a chiral Weyl spinor. This can be understood by consider γ^5. In four dimensions $\gamma^5 \sim \gamma^0\gamma^1\gamma^2\gamma^3$ is a special operator that can be used to project chiralities via $P_{L,R} = \frac{1}{2}(1 \pm i\gamma^5)$, noting the different normalization from usual QFT texts (see the Appendix on conventions). In 5D, however, γ^5 is just the gamma matrix corresponding to the z direction and there is no analogous 'special' gamma matrix. The $\gamma^0, \cdots, \gamma^5$ form a basis for the four component spinor representation of the Clifford algebra. One can find a good discussion of the Clifford algebra and spinor representation in various dimensions in the appendices of volume 2 of Polchinksi [12].

The vector nature of 5D spinors is an immediate problem for model-building since the Standard Model is manifestly chiral and there appears to be no way to write down a chiral fermion without immediately introducing a partner fermion of opposite chirality and the same couplings[4]. To get around this problem we can relax our requirement that the bulk 5D fermion be chiral. Phenomenologically, all that is strictly required is that the zero modes of these fermions are chiral.

We can project out the zero modes of the bad-chirality (Weyl spinor) components of a bulk Dirac 5D fermion by imposing chiral boundary conditions: namely that these bad-chirality components vanish on the branes. This prohibits this component from having a zero mode since zero modes have a trivial 5D profile and the only such profile compatible with the chiral boundary conditions is identically zero everywhere. For left-chiral boundary conditions, $\psi = 0$ on the branes, while for right-chiral boundary conditions $\chi = 0$ on the branes. Thus we are guaranteed that both terms in eq. (3.29) vanish at $z = R, R'$ for either chirality.

Imposing these chiral boundary conditions are completely equivalent to the statement that the Randall-Sundrum compactified extra dimension is an orbifold. From a phenomenological point of view the language of boundary conditions is preferred since it avoids potential ambiguities with the sign of the fermion mass term. Further the language of boundary conditions best connects to the actual process of solving partial differential equations that we will follow.

These chiral boundary conditions cancel the boundary term that appears when converting a right-acting Dirac operator into a canonical Dirac operator that we discussed in section 2.4. It is precisely because of these chiral boundary conditions (i.e. orbifolding) that the Grossman Neubert paper on bulk neutrinos in RS had the correct fermion action despite using the erroneous right-acting action from Bertlmann's text.

Boundary conditions for compact spaces were first discussed in this light in the paper by the 'Three Musketeers[5],' and Grojean [13].

4 Determining the propagator

We now have a Dirac operator which we shall write as

$$\mathcal{D} = i\partial\!\!\!/ + i\gamma^5\partial_5 - i\frac{2}{z}\gamma^5 - \frac{c}{z}, \tag{4.1}$$

so that we may write down the RS Dirac equation as

$$(R/z)^4\,\mathcal{D}\Psi = 0, \tag{4.2}$$

where we've included the warp factors from the metric and vielbein. The propagator from point x' to x, $\Delta(x - x')$, is then defined to be the Green's function of the Dirac operator,

$$(R/z)^4\,\mathcal{D}\Delta(x - x') = i\delta^{(5)}(x - x'), \tag{4.3}$$

[4]The same problem is found in $\mathcal{N} > 1$ supersymmetric models.

[5]Jay Hubisz is apparently d'Artagnan. I believe Christophe should be George Villiers for sheer irony.

that satisfies the appropriate boundary conditions. This is, at least, what we would write down from our intuition based on flat space quantum field theory. However, one must be careful since the $\delta^{(5)}(x-x')$ is not written in a manifestly covariant way. This equation turns out to be correct, but we should note that one should be more careful with the derivation. We must be sure that the right-hand side of this equation is correct in powers of the warp factor.

4.1 Proper derivation of the Green's function equation

Recalling the path integral formalism, let us write down the action with the appropriate fermionic sources,

$$S = \int d^5x \sqrt{|g|} \left\{ \frac{i}{2} E_a^M \overline{\Psi} \gamma^a \overleftrightarrow{D_M} \Psi + \overline{J}\Psi + \overline{\Psi} J \right\}. \tag{4.4}$$

Varying with respect to $\overline{\Psi}$, we get the equation of motion

$$(R/z)^4 \, \mathcal{D}\Delta(x-x') = (R/z)^5 J, \tag{4.5}$$

where the right-hand side now has a factor of $\sqrt{|g|}$. Usually in quantum field theory we set $J(x) = i\delta^{(d)}(x-x')$. The factor of i is the usual factor obtained in 4D Minkowski QFT, e.g. from variation of the generating functional. This source requires some modification, however, since the δ function is not covariant with respect to diffeomorphism invariance. The correct covariant δ function is

$$\delta^{(d)}(x-x') \to \delta^{(d)}\left(\sqrt{|g|}(x-x')\right) = \frac{\delta^{(d)}(x-x')}{\sqrt{|g|}}. \tag{4.6}$$

The inverse power of $\sqrt{|g|}$ is precisely what is necessary to cancel the factor of $\sqrt{|g|}$ from the integration measure in the action, and we are left with precisely eq. (4.3).

In the rest of this section we shall derive the 'mixed position-momentum space' propagator on the RS background, $\Delta(p, z, z')$, where a fermion of 4-momentum p propagates from a position z' to z in the fifth dimension. This was discussed for scalars and spin-2 particles in Giddings, Katz, Randall [14] and Randall and Schwartz [15]. A pedagogical introduction the derivation and calculation of these propagators in flat compactifications is presented in Puchwein and Kunzst [16], though we shall follow what we feel is a more intuitive derivation.

4.2 The position space propagator

Let us define our RS Dirac operator as

$$\mathcal{D} = i\left(\gamma^M \partial_M - \frac{2}{z}\gamma^5\right) - \frac{c}{z}, \tag{4.7}$$

so that the action takes the form

$$S = \int d^4x \int dz \; (R/z)^4 \; \overline{\Psi}\mathcal{D}\Psi. \tag{4.8}$$

The equation of motion that we must solve is

$$(R/z)^4 \mathcal{D}\Delta(x) = i\delta^{(5)}(x - x'), \tag{4.9}$$

where $\Delta(x)$ is the Dirac Green's function that, upon imposing appropriate boundary conditions, we will identify with the bulk fermion propagator. Because of the Dirac structure in $\mathcal{D}$ this equation is difficult to disentangle to solve. We shall make use of a handy trick analogous to the usual 4D case where we 'square' the Dirac equation into a scalar equation, where the phrase 'square' is used colloquially and with no mathematical rigor[6]. Define the 'conjugate' (also a colloquial term) Dirac operator as

$$\mathcal{D}^* = -i\left(\gamma^M \partial_M - \frac{2}{z}\gamma^5\right) - \frac{c}{z}. \tag{4.10}$$

One can then note that the product of these operators takes the form

$$\mathcal{D}\mathcal{D}^* = \left(\gamma^M \partial_M - \frac{2}{z}\gamma^5\right)^2 + \frac{c^2}{z^2} - i\underbrace{\gamma^M\left(\partial_M \frac{c}{z}\right)}_{-\gamma^5 \frac{c}{z^2}}. \tag{4.11}$$

The first term on the right-hand side is

$$\left(\gamma^M \partial_M - \frac{2}{z}\gamma^5\right)^2 = \partial_M\partial^M - \gamma^M\partial_M\left(\gamma^5\frac{2}{z}\right) \underbrace{-\frac{2}{z}\gamma^M\gamma^5\partial_M - \frac{2}{z}\gamma^5\gamma^M\partial_M}_{\text{cancels for } M\neq 5} + \frac{4}{z^2}\left(\gamma^5\right)^2 \tag{4.12}$$

$$= \partial_M\partial^M + \frac{2}{z^2}\left(\gamma^5\right)^2 - \frac{4}{z}\left(\gamma^5\right)^2\partial_5 + \frac{4}{z^2}\left(\gamma^5\right)^2 \tag{4.13}$$

$$= \partial_M\partial^M - \frac{6}{z^2} + \frac{4}{z}\partial_5. \tag{4.14}$$

Thus, finally, we are left with a 'squared' Dirac operator

$$\mathcal{D}\mathcal{D}^* = \partial_M\partial^M + \frac{1}{z^2}\left(c^2 + i\gamma^5 c - 6 + 4z\partial_5\right), \tag{4.15}$$

where one should recall that in our conventions $i\gamma^5 = \mathrm{diag}(-\mathbb{1}, \mathbb{1})$. The operator is not *scalar*, but it is at least manifestly diagonal and can be decomposed into scalar equations on Weyl spinors. For simplicity we shall write

$$\mathcal{D}\mathcal{D}^* = \begin{pmatrix} \mathcal{D}\mathcal{D}^*_- & 0 \\ 0 & \mathcal{D}\mathcal{D}^*_+ \end{pmatrix}, \qquad \mathcal{D}\mathcal{D}^*_\pm = \partial_\mu\partial^\mu - \partial_5^2 + \frac{4}{z}\partial_5 + \frac{c^2 \pm c - 6}{z^2}. \tag{4.16}$$

[6]Even though this is not an honest conjugate or 'squaring' of the Dirac operator, there is an interesting mathematical digression to be made regarding the actual conjugate Dirac operator. In flat space, the Dirac operator squares to the Klein-Gordon operator exactly. In general curved spaces the square of the (massless) Dirac operator differs from the square of the covariant derivative (box) by a polynomial in the curvature. This is the so-called Weitzenböck decomposition: $\mathcal{D}^2 = \nabla^*\nabla + \frac{1}{4}R$. See, for example, N. Hitchin, "The Dirac Operator," in *Invitations to Geometry and Topology*.

Let us now write down a diagonal 2×2 matrix of scalar functions in Weyl spinor space, $F(x) = \mathrm{diag}(F_-(x), F_+(x))$ such that

$$(R/z)^4\, \mathcal{D}\mathcal{D}^* F(x) = \left(\frac{R}{z}\right)^4 \begin{pmatrix} \mathcal{D}\mathcal{D}^*_- & 0 \\ 0 & \mathcal{D}\mathcal{D}^*_+ \end{pmatrix} \begin{pmatrix} F_-(x) & 0 \\ 0 & F_+(x) \end{pmatrix} = i\delta^{(5)}(x - x'). \tag{4.17}$$

Solving for $F_\pm(x)$ reduces to the usual program of solving partial differential equations with boundary conditions for scalar functions. The trick is that upon finding the F functions, we automatically (i.e. by construction) get solutions for the Dirac Green's function equation, eq. (4.3),

$$\Delta(x) = \mathcal{D}^* F(x) = \begin{pmatrix} \partial_5 F_-(x) - \frac{2+c}{z} F_-(x) & -i\sigma^\mu \partial_\mu F_+(x) \\ -i\overline{\sigma}^\mu \partial_\mu F_-(x) & -\partial_5 F_+(x) + \frac{2-c}{z} F_+(x) \end{pmatrix}. \tag{4.18}$$

4.3 Mixed Position-Momentum Space Propagator

Thus far all of our calculations have been in 5D position space. Since each z-slice of the Randall Sundrum AdS space is flat with respect to the 4D Minkowski directions we may make use of the usual method of Fourier transforming to momentum space for these directions. Note that on the compactified (z) direction we do *not* want to go into momentum space. It is important to work in position space for the z direction because we want to be able to describe brane-localized operators and the overlap of bulk fields.

Let's start with a review of how this Fourier transform works for a general Green's function equation for an operator $\mathcal{O}(\partial_\mu, \partial_5)$. The Green's function equation is

$$\mathcal{O}(\partial_\mu, \partial_5) f(x - x', z, z') = i\delta^{(4)}(x - x')\delta^{(5)}(z - z'). \tag{4.19}$$

We first perform a Fourier transform on the 4D coordinates with respect to a dummy variable p',

$$\int đ^4 p'\, e^{ip'x}\, \mathcal{O}(ip', \partial_5) f(p', z, z') = i\delta(z - z') \int đ^4 p\, e^{ip'x}, \tag{4.20}$$

where we've used 4D Lorentz invariance to shift x' to zero. We can now project upon a particular momentum mode p by multiplying both sides by e^{-ipx} and integrating over d^4x.

$$\int d^4x \int đ^4 p' e^{i(p-p')x}\, \mathcal{O}(ip', \partial_5) f(p', z, z') = i\delta(z - z') \int d^4x\, đ^4 p' e^{i(p'-p)x} \tag{4.21}$$

$$\int đ^4 p' \delta^{(4)}(p' - p) \mathcal{O}(ip', \partial_5) f(p', z, z') = i\delta^{(4)}(z - z') \int đ^4 p\, \delta(p' - p). \tag{4.22}$$

From this we finally derive the mixed position-momentum space Green's function equations,

$$\mathcal{O}(ip, \partial_5) f(p, z, z') = i\delta(z - z'). \tag{4.23}$$

Thus we are now left with a one-dimensional Green's function equation to solve in position spaece. For our 'Dirac-squared' operator above, eq. (4.16), this yields

$$\mathcal{D}\mathcal{D}_\pm F_\pm(p, z, z') = \left(-p^2 - \partial_5^2 + \frac{4}{z}\partial_5 + \frac{c^2 \pm c - 6}{z^2}\right) F_\pm(p, z, z'). \tag{4.24}$$

4.4 Solving the differential equation

We now follow the usual procedure for solving the Green's function equation that one is familiar with from electromagnetism. The calculations can now become rather messy, so we've provided a model calculation for a bulk fermion on a flat extra dimension in Appendix B. We split the solution into two parts corresponding to the solution $F^{>}_{\pm}(p,z,z')$ for $z > z'$ and the solution $F^{<}_{\pm}(p,z,z')$ for $z < z'$. In each of these regimes one avoids the δ function on the right-hand side of the Green's function equation so that one only needs to find the solution to the homogenous differential equation:

$$\left(-p^2 - \partial_5^2 + \frac{4}{z}\partial_5 + \frac{c^2 \pm c - 6}{z^2}\right) F^{<,>}_{\pm}(p,z,z') = 0. \tag{4.25}$$

This has a general solution

$$F^{<,>}_{\pm}(p,z,z') = A^{<,>}_{\pm} z^{5/2} J_{c\pm 1/2}(\chi_p z) + B^{<,>}_{\pm} z^{5/2} Y_{c\pm 1/2}(\chi_p z). \tag{4.26}$$

The $J_{c\pm1/2}$ and $Y_{c\pm1/2}$ are Bessel functions of the first and second kinds (Y is occasionally called N for Neumann function). χ_p is the shorthand used in the literature for $\sqrt{p^2}$, where $p^2 = p_\mu p^\mu$. The eight coefficients $A^{<,>}_{\pm}$ and $B^{<,>}_{\pm}$ are determined by boundary conditions on the brane (chiral/orbifold boundary conditions) and matching conditions at the point $z = z'$.

Csáki, Grojean, Hubiz, Shirman, and Terning ('the Three Musketeers, Christophe, and Jay') [13] noted that since the Dirac equation is first order one doesn't need separate boundary conditions for the Weyl spinor fields χ and ψ that live in a Dirac fermion. In fact, specifying separate boundary conditions overconstrains the system. The point is that the (bulk) Dirac equation evaluated at the boundaries convert ψ boundary conditions into χ boundary conditions and vice versa.

Our approach is rather different but makes their observation more manifest. By 'squaring' the Dirac equation we now work with a second order scalar equation. There are no ambiguities about the Dirac equation coupling different components of the Green's function. In the previous point of view, imposing the Dirac equation on the branes to obtain boundary conditions for the opposite-chirality Weyl spinor is automatically satisfied by construction in eq. (4.18). As a second-order equation there is now an additional constant to be solved for, but this is taken care of by an additional matching condition. Thus the two approaches are completely equivalent, though the present procedure has the benefit of additional clarity.

Our first boundary condition we'll impose is the **jump conditon** at $z = z'$ obtained from integrating the Green's function equation over a sliver about the source at z'.

$$\int_{z'-\epsilon}^{z'+\epsilon} dz \left(\frac{R}{z}\right)^4 \left[-p^2 - \partial_5^2 + \frac{4}{z}\partial_5 + \frac{c^2 \pm c - 6}{z^2}\right] F_{\pm}(p,z,z') = i \int_{z'-\epsilon}^{z'+\epsilon} dz\, \delta(z - z'), \tag{4.27}$$

where we've explicitly included the $(R/z)^4$ term since we are now working with the inhomogeneous Green's function. Only the terms with ∂_5s survive the limit when $\epsilon \to 0$.

$$-\partial_5\, F_{\pm}(p,z,z')\big|_{z'-\epsilon}^{z'+\epsilon} - \frac{4}{z}\, F_{\pm}(p,z,z')\big|_{z'-\epsilon}^{z'+\epsilon} + \int_{z'-\epsilon}^{z'+\epsilon} dz \left(\partial_5 \frac{4}{z}\right) F_{\pm}(p,z,z') = i \left(\frac{R}{z'}\right)^{-4} \tag{4.28}$$

The last term drops out as $\epsilon \to 0$, so that the jump boundary condition takes the form

$$\left(\partial_5 + \frac{4}{z'}\right) F_\pm^{>}(p,z',z') - \left(\partial_5 + \frac{4}{z'}\right) F_\pm^{<}(p,z',z') = -i\left(\frac{z'}{R}\right)^4 . \tag{4.29}$$

Here used $F_\pm(p,z'+\epsilon,z') = F_\pm^{>}(p,z',z')$ and $F_\pm(p,z'-\epsilon,z') = F_\pm^{<}(p,z',z')$.

The next boundary condition that we would like to impose is continuity of $F_\pm(p,z,z')$ at $z=z'$. This comes about because even though we've seen above that the first derivative is discontinuous at z' (as one expects for a second order differential equation), but non-singular. Thus we impose

$$F_\pm^{>}(p,z',z') = F_\pm^{<}(p,z',z'). \tag{4.30}$$

Note that plugging this back into the jump condition causes the $(4/z')$ terms to cancel.

The remaining boundary conditions come from the chiral (orbifold) bonditions on the branes, $z=R',R$. To impose these, let us write the Dirac Green's function eq. (4.18) as

$$\Delta = \mathcal{D}^* F = \begin{pmatrix} \Delta^{11} & \Delta^{12} \\ \Delta^{21} & \Delta^{22} \end{pmatrix} = \begin{pmatrix} \Delta_{\chi\psi} & \Delta_{\chi\chi} \\ \Delta_{\psi\psi} & \Delta_{\psi\chi} \end{pmatrix}, \tag{4.31}$$

where we've broken down the Dirac propagator into a matrix of Weyl propagators. For example, $\Delta_{\chi\psi}(p,z,z')$ is the propagator of a ψ (chiral right-handed spinor) at z' propagating into a χ (chiral left-handed spinor) at z with four-momentum p. One may see this heuristically by recalling that the propagator can be written heuristically as

$$\Delta \sim \Psi\overline{\Psi} \sim \begin{pmatrix} \chi\psi & \chi\overline{\chi} \\ \overline{\psi}\psi & \overline{\psi}\overline{\chi} \end{pmatrix}. \tag{4.32}$$

From this point on our boundary conditions will depend on whether we are consider left-handed fermion propagators Δ_L or right-handed fermion propagators Δ_R, where it is understood that the handedness we refer to is the chirality of the zero mode. For a bulk propagator of zero-mode chirality $X \in \{L,R\}$, our boundary conditions impose that the 'wrong chirality' component of the field cannot propagate *to* either brane. For example, a left-handed bulk fermion $\Psi = \chi \oplus \psi$ must have its right-handed component vanish at the boundary $\psi(R') = \psi(R) = 0$. Explicitly, we impose

$$0 = \Delta^L_{\psi\chi}(p,R,z') = \Delta^L_{\psi\psi}(p,R,z') \tag{4.33}$$
$$0 = \Delta^L_{\psi\chi}(p,R',z') = \Delta^L_{\psi\psi}(p,R',z') \tag{4.34}$$

$$0 = \Delta^R_{\chi\chi}(p,R,z') = \Delta^R_{\chi\psi}(p,R,z') \tag{4.35}$$
$$0 = \Delta^R_{\chi\chi}(p,R',z') = \Delta^R_{\chi\psi}(p,R',z'). \tag{4.36}$$

More succinctly,

$$\Delta^L(p,z,z')\big|_{z=R,R'} = \begin{pmatrix} \Delta_{\chi\psi} & \Delta_{\chi\chi} \\ 0 & 0 \end{pmatrix} \qquad \Delta^R(p,z,z')\big|_{z=R,R'} = \begin{pmatrix} 0 & 0 \\ \Delta_{\psi\psi} & \Delta_{\psi\chi} \end{pmatrix}. \tag{4.37}$$

One can now count that for a given chirality this gives eight equations for eight unknowns. One might ask whether we've missed something, since in principle our boundary conditions should also prohibit wrong-chirality modes of fermion propagators coming *from* either brane, i.e. $z' = R, R'$. The short answer is that these boundary conditions are automatically satisfied by when imposing the *to*-the-brane boundary conditions.

$$\Delta^L(p,z,z')\big|_{z'=R,R'} = \begin{pmatrix} 0 & \Delta_{\chi\chi} \\ 0 & \Delta_{\psi\chi} \end{pmatrix} \qquad \Delta^R(p,z,z')\big|_{z'=R,R'} = \begin{pmatrix} \Delta_{\chi\psi} & 0 \\ \Delta_{\psi\psi} & 0 \end{pmatrix}. \tag{4.38}$$

Just to show off that we can write this yet another way, let us say

$$P_R\, \Delta^L(p,z,z')\big|_{z=R,R'} = P_R \mathcal{D}^*\, F^L(p,z,z')\big|_{z=R,R'} = 0 \tag{4.39}$$

$$P_L\, \Delta^R(p,z,z')\big|_{z=R,R'} = P_L \mathcal{D}^*\, F^R(p,z,z')\big|_{z=R,R'} = 0. \tag{4.40}$$

From eq. (4.18), we may write these in terms of the F^L and F^R functions,

$$0 = -i\bar{\sigma}^\mu \partial_\mu\, F^L_-(p,z,z')\big|_{z=R,R'} \qquad 0 = \left(-\partial_5 + \frac{2}{z} - \frac{c}{z}\right) F^L_+(p,z,z')\big|_{z=R,R'} \tag{4.41}$$

$$0 = -i\sigma^\mu \partial_\mu\, F^R_+(p,z,z')\big|_{z=R,R'} \qquad 0 = \left(\ \partial_5 - \frac{2}{z} - \frac{c}{z}\right) F^R_-(p,z,z')\big|_{z=R,R'}. \tag{4.42}$$

We note that p can take any value so that these equations can be written

$$0 = F^L_-(p,z,z')\big|_{z=R,R'} \qquad 0 = \left(\partial_5 - \frac{2}{z} + \frac{c}{z}\right) F^L_+(p,z,z')\big|_{z=R,R'} \tag{4.43}$$

$$0 = F^R_+(p,z,z')\big|_{z=R,R'} \qquad 0 = \left(\partial_5 - \frac{2}{z} - \frac{c}{z}\right) F^R_-(p,z,z')\big|_{z=R,R'}. \tag{4.44}$$

Note that the left- and right-handed boundary conditions are related by $F_+ \leftrightarrow F_-$ (which translates to $\chi \leftrightarrow \psi$) and $c \leftrightarrow -c$.

Now we've written all of our boundary conditions in terms of constraints on the F functions.

4.5 Addendum: Jump Condition

This is a quick *post-facto* aside to mention that the derivation above for the jump condition of eq. (4.27),

$$\int_{z'-\epsilon}^{z'+\epsilon} dz\, \left(\frac{R}{z}\right)^4 \left[-p^2 - \partial_5^2 + \frac{4}{z}\partial_5 + \frac{c^2 \pm c - 6}{z^2}\right] F_\pm(p,z,z') = i \int_{z'-\epsilon}^{z'+\epsilon} dz\, \delta(z-z'). \tag{4.45}$$

The above derivation of the jump condition cheated a little in shifting the $(R/z)^4$ back and forth, even though the final result is correct. Here we re-derive the jump condition so that we may proceed with a good conscience. Performing the integration by parts for the left-hand side of the above equation and dropping terms that vanish in the $\epsilon \to 0$ limit, we find

$$\left[-\left(\frac{R}{z}\right)^4 \partial_5 F + 4\frac{R^4}{z^5} F\right]_{z'-\epsilon}^{z'+\epsilon} - \int_{z'-\epsilon}^{z'+\epsilon} dz\, \left(-\partial_5 \left(\frac{R}{z}\right)^4 \partial_5 F + \partial_5\left(4\frac{R^4}{z^5}\right) F\right). \tag{4.46}$$

Note that expressions of the form z^{-5} and z^{-4} are continuous at $z' \in [R, R']$ and we have shown above that $F(z = z' + \epsilon) = F(z = z' - \epsilon)$. The only discontinuity that can appear, then, is in the expression $\partial_5 F$. Thus two the terms multiplied by 4 both vanish in the $\epsilon \to 0$ limit, so that we're left with a jump condition

$$\left[-\left(\frac{R}{z}\right)^4 \partial_5 F\right]_{z'-\epsilon}^{z'+\epsilon} + \int_{z'-\epsilon}^{z'+\epsilon} dz\, \partial_5 \left(\frac{R}{z}\right)^4 \partial_5 F = i. \tag{4.47}$$

The remaining integral should be performed by integrating by parts once more,

$$\int_{z'-\epsilon}^{z'+\epsilon} dz\, \partial_5 \left(\frac{R}{z}\right)^4 \partial_5 F = \left[\partial_5 \left(\frac{R}{z}\right)^2 F\right]_{z'-\epsilon}^{z'+\epsilon} - \int_{z'-\epsilon}^{z'+\epsilon} dz\, \partial_5^2 \left(\frac{R}{z}\right)^4 F. \tag{4.48}$$

Now both sides on the right-hand side are independent of ∂_5 and thus vanish in the $\epsilon \to 0$ limit. Thus our jump condition reduces to

$$-\left(\frac{R}{z'}\right)^4 \left[\partial_5 F\right]_{z'-\epsilon}^{z'+\epsilon} = i. \tag{4.49}$$

Summary of boundary conditions:

Before proceeding, let us summarize the boundary conditions here:

$$\partial_5 F_\pm^>(p, z', z') - \partial_5 F_\pm^<(p, z', z') = -i \left(\frac{z'}{R}\right)^4. \tag{4.50}$$

$$F_\pm^>(p, z', z') = F_\pm^<(p, z', z'). \tag{4.51}$$

$$0 = F_-^L(p, z, z')\big|_{z=R,R'} \qquad 0 = \left(\partial_5 - \frac{2}{z} + \frac{c}{z}\right) F_+^L(p, z, z')\big|_{z=R,R'} \tag{4.52}$$

$$0 = F_+^R(p, z, z')\big|_{z=R,R'} \qquad 0 = \left(\partial_5 - \frac{2}{z} - \frac{c}{z}\right) F_-^R(p, z, z')\big|_{z=R,R'}. \tag{4.53}$$

Now all that remains is to determine the coefficients $A_\pm^{<,>}$ and $B_\pm^{<,>}$ from eq. (4.26). This can be solved straightforwardly using *Mathematica*.

4.6 Solution of Coefficients

Define the following 'antisymmetric' auxiliary functions,

$$S_c^\pm(x, y) = J_{c\pm\frac{1}{2}}(x) Y_{c\pm\frac{1}{2}}(y) - J_{c\pm\frac{1}{2}}(y) Y_{c\pm\frac{1}{2}}(x) \tag{4.54}$$

$$S_c^\times(x, y) = J_{c+\frac{1}{2}}(x) Y_{c-\frac{1}{2}}(y) - J_{c-\frac{1}{2}}(y) Y_{c+\frac{1}{2}}(x) \tag{4.55}$$

$$S_c^\div(x, y) = J_{c-\frac{1}{2}}(x) Y_{c+\frac{1}{2}}(y) - J_{c+\frac{1}{2}}(y) Y_{c-\frac{1}{2}}(x). \tag{4.56}$$

Even though $S_c^{\div}(x,y) = -S_c^{\times}(y,x)$, we define it separately for future simplicity.

The results of the *Mathematica* output is not pretty, but we will beautify it along the way[7]. To provide robustness against typos, we'll start by explicitly providing the complete result. Writing the common factor $C = \frac{i\pi z'^{5/2}}{2R^4}$, the left-handed results are

$$\begin{aligned}
A_+^{L<} &= -CY_{c-\frac{1}{2}}(\chi_p R)\frac{S_c^{\times}(\chi_p z', \chi_p R')}{S_c^{-}(\chi_p R, \chi_p R')} & \qquad A_+^{L>} &= -CY_{c-\frac{1}{2}}(\chi_p R')\frac{S_c^{\times}(\chi_p z', \chi_p R)}{S_c^{-}(\chi_p R, \chi_p R')} \\
B_+^{L<} &= \quad CJ_{c-\frac{1}{2}}(\chi_p R)\frac{S_c^{\times}(\chi_p z', \chi_p R')}{S_c^{-}(\chi_p R, \chi_p R')} & \qquad B_+^{L>} &= \quad CJ_{c-\frac{1}{2}}(\chi_p R')\frac{S_c^{\times}(\chi_p z', \chi_p R)}{S_c^{-}(\chi_p R, \chi_p R')}
\end{aligned}$$

$$\begin{aligned}
A_-^{L<} &= -CY_{c-\frac{1}{2}}(\chi_p R)\frac{S_c^{-}(\chi_p z', \chi_p R')}{S_c^{-}(\chi_p R, \chi_p R')} & \qquad A_-^{L>} &= -CY_{c-\frac{1}{2}}(\chi_p R')\frac{S_c^{-}(\chi_p z', \chi_p R)}{S_c^{-}(\chi_p R, \chi_p R')} \\
B_-^{L<} &= \quad CJ_{c-\frac{1}{2}}(\chi_p R)\frac{S_c^{-}(\chi_p z', \chi_p R')}{S_c^{-}(\chi_p R, \chi_p R')} & \qquad B_-^{L>} &= \quad CJ_{c-\frac{1}{2}}(\chi_p R')\frac{S_c^{-}(\chi_p z', \chi_p R)}{S_c^{-}(\chi_p R, \chi_p R')}.
\end{aligned}$$

Note that the As and Bs differ by $Y \to J$ and swapping $R \leftrightarrow R'$. (This swap also gives the relative sign due to the antisymmetry of the S_c^- auxiliary function in the denominator.) The $+$ and $-$ values differ by swapping $S_c^{\times} \to S_c^-$ in the numerator only. The right-handed equations are very similar,

$$\begin{aligned}
A_+^{R<} &= -CY_{c+\frac{1}{2}}(\chi_p R)\frac{S_c^{+}(\chi_p z', \chi_p R')}{S_c^{+}(\chi_p R, \chi_p R')} & \qquad A_+^{R>} &= -CY_{c+\frac{1}{2}}(\chi_p R')\frac{S_c^{+}(\chi_p z', \chi_p R)}{S_c^{+}(\chi_p R, \chi_p R')} \\
B_+^{R<} &= \quad CJ_{c+\frac{1}{2}}(\chi_p R)\frac{S_c^{+}(\chi_p z', \chi_p R')}{S_c^{+}(\chi_p R, \chi_p R')} & \qquad B_+^{R>} &= \quad CJ_{c+\frac{1}{2}}(\chi_p R')\frac{S_c^{+}(\chi_p z', \chi_p R)}{S_c^{+}(\chi_p R, \chi_p R')}
\end{aligned}$$

$$\begin{aligned}
A_-^{R<} &= -CY_{c+\frac{1}{2}}(\chi_p R)\frac{S_c^{\div}(\chi_p z', \chi_p R')}{S_c^{+}(\chi_p R, \chi_p R')} & \qquad A_-^{R>} &= -CY_{c+\frac{1}{2}}(\chi_p R')\frac{S_c^{\div}(\chi_p z', \chi_p R)}{S_c^{+}(\chi_p R, \chi_p R')} \\
B_-^{R<} &= \quad CJ_{c+\frac{1}{2}}(\chi_p R)\frac{S_c^{\div}(\chi_p z', \chi_p R')}{S_c^{+}(\chi_p R, \chi_p R')} & \qquad B_-^{R>} &= \quad CJ_{c+\frac{1}{2}}(\chi_p R')\frac{S_c^{\div}(\chi_p z', \chi_p R)}{S_c^{+}(\chi_p R, \chi_p R')}.
\end{aligned}$$

These right-handed results are obtained from the left-handed results by swapping the sign of the $1/2$ in the index of the explicit Bessel function. Doing this in the denominator also changes the S_c^- to S_c^+. Finally, we make the replacements in the numerator: $S_c^{\times} \to S_c^+$ and $S_c^- \to S_c^{\div}$. If all of this still looks rather complicated, note that one can simplify by writing out all the arguments in terms of dimensionless variables e.g. write all length scales in units of $1/\chi_p$, essentially setting $\chi_p = 1$ in all the arguments.

Our task now is to write this out in a succinct, human-readable form. We see a rather nice symmetry where all of the indices are transformed according to

$$L \leftrightarrow R \tag{4.57}$$

$$\pm \leftrightarrow \mp. \tag{4.58}$$

[7] "You can put lipstick on a pig, but it's still a pig," Barack Obama, 9 September 2008.

Note that this only holds for *indices*, so that, e.g.

$$A_+ \to A_- \tag{4.59}$$

$$S_c^+ \to S_c^- \tag{4.60}$$

$$c + \frac{1}{2} \to c - \frac{1}{2}, \tag{4.61}$$

but the overall sign on the C coefficient does not change. This symmetries is not so surprising given the symmetry between the left- and right-handed boundary conditions.

Recall the general solution of our F functions, eq. (4.26),

$$F_\pm^{<,>}(p, z, z') = A_\pm^{<,>} z^{5/2} J_{c\pm 1/2}(\chi_p z) + B_\pm^{<,>} z^{5/2} Y_{c\pm 1/2}(\chi_p z). \tag{4.26}$$

We recall that all of the B coefficients carry an overall minus sign, so the z-dependent Bessel functions can also be written in terms of our auxiliary functions, eqs. (4.54-4.56).

$$F_+^{L<} = \alpha^L \left(zz'\right)^{5/2} S_c^\times\left(\chi_p z', \chi_p R'\right) S_c^\div(\chi_p R, \chi_p z) \tag{4.62}$$

$$F_+^{L>} = \alpha^L \left(zz'\right)^{5/2} S_c^\times\left(\chi_p z', \chi_p R\right) S_c^\div(\chi_p R', \chi_p z) \tag{4.63}$$

$$F_-^{L<} = \alpha^L \left(zz'\right)^{5/2} S_c^-\left(\chi_p z', \chi_p R'\right) S_c^-(\chi_p R, \chi_p z) \tag{4.64}$$

$$F_-^{L>} = \alpha^L \left(zz'\right)^{5/2} S_c^-\left(\chi_p z', \chi_p R\right) S_c^-(\chi_p R', \chi_p z) \tag{4.65}$$

$$F_+^{R<} = \alpha^L \left(zz'\right)^{5/2} S_c^+\left(\chi_p z', \chi_p R'\right) S_c^+(\chi_p R, \chi_p z) \tag{4.66}$$

$$F_+^{R>} = \alpha^L \left(zz'\right)^{5/2} S_c^+\left(\chi_p z', \chi_p R\right) S_c^+(\chi_p R', \chi_p z) \tag{4.67}$$

$$F_-^{R<} = \alpha^L \left(zz'\right)^{5/2} S_c^\div\left(\chi_p z', \chi_p R'\right) S_c^\times(\chi_p R, \chi_p z) \tag{4.68}$$

$$F_-^{R>} = \alpha^L \left(zz'\right)^{5/2} S_c^\div\left(\chi_p z', \chi_p R\right) S_c^\times(\chi_p R', \chi_p z). \tag{4.69}$$

We have defined the overall coefficients as

$$\alpha^L = \frac{i\pi}{2R^4} \frac{1}{S_c^-(\chi_p R, \chi_p R')} \qquad \alpha^R = \frac{i\pi}{2R^4} \frac{1}{S_c^+(\chi_p R, \chi_p R')}. \tag{4.70}$$

It is actually sufficient to just write down the $F^<$ functions and note that the $F^>$ functions can be obtained by $\{z \leftrightarrow z'\}$. (One should remember that $S^\times$ and $S^\div$ are related by a minus sign.) The final form of the propagator is

$$\Delta(p, z, z') \equiv \mathcal{D}^* F(p, z, z') = \begin{pmatrix} \partial_5 F_-(x) - \frac{2+c}{z} F_-(x) & -i\sigma^\mu \partial_\mu F_+(x) \\ -i\bar{\sigma}^\mu \partial_\mu F_-(x) & -\partial_5 F_+(x) + \frac{2-c}{z} F_+(x) \end{pmatrix}. \tag{4.71}$$

Note the manifest $z \leftrightarrow z'$ symmetry that one expects from the Green's function.

5 KK mode profiles

It is now useful to include a quick discussion of the zero mode (more generally n^{th}-mode) profile since the external states for processes are interested in are typically zero modes. The profiles are derived pedagogically in Csáki's TASI 04 lectures [2], but we repeat the derivation here because of ideological differences[8].

The 4D equations of motion for the KK modes can be obtained from varying the bulk RS fermionic action, eq. (3.31). Even though the fundamental fermion representation in 5D is the Dirac spinor, we shall write the equations of motion in terms of Weyl spinors in anticipation of KK-reduction to 4D modes. The result is

$$0 = i\overline{\sigma} \cdot \partial\chi + \partial_5\overline{\psi} - \frac{c+2}{z}\overline{\psi} \tag{5.1}$$

$$0 = i\sigma \cdot \partial\overline{\psi} - \partial_5\chi - \frac{c-2}{z}\chi. \tag{5.2}$$

We now expand our Weyl spinors in terms of an orthonormal[9] basis of left-handed ($f_n(z)$) and right-handed ($g_n(z)$) 5D profiles[10], i.e. we make the usual Kaluza-Klein decomposition ansatz

$$\chi(x,z) = \sum_n f_n(z)\chi_n(x) \tag{5.3}$$

$$\overline{\psi}(x,z) = \sum_n g_n(z)\overline{\psi}_n(x). \tag{5.4}$$

The functions f_n and g_n are real so one never has to worry about picking up signs during complex conjugation. By assumption the KK modes each satisfy 4D Dirac equations for different KK masses m_n so that

$$0 = i\overline{\partial\!\!\!/}\chi_n(x) - m_n\overline{\psi}_n(x) \tag{5.5}$$

$$0 = i\partial\!\!\!/\overline{\psi}_n(x) - m_n\chi_n(x), \tag{5.6}$$

where there is no implied sum over n. Plugging this into the 5D equation of motion we get the 5D profiles must satisfy

$$0 = g_n'(z) + m_n f_n(z) - \frac{c+2}{z}g_n(z) \tag{5.7}$$

$$0 = f_n'(z) - m_n g_n(z) + \frac{c-2}{z}f_n(z). \tag{5.8}$$

Note that in this last step we have implicitly used the orthogonality of the 4D fields so that these equations also do not have an implied sum over n. We can now set $m_n = 0$ and solve for the

[8]Namely we disagree with their normalization of σ^0. This shows up as an overall sign in the action so that it makes no difference... but really, guys, who uses $\sigma^0 = -1$? (The authors of those lectures claim that this was to match up with the original literature [13].)

[9]Orthonormality is defined with respect to the fermion inner product of Section 2.3.

[10]Here again we differ in notation from Csáki et al's TASI lecture [2], this time via the choice of denoting left-handed 5D profiles by f_n and right-handed by g_n rather than vice versa. The present notation is more standard in current literature. Note that we will also alternately use the notation where all profiles are denoted by f with the particle species and chirality denoted explicitly, e.g. f_E^n for a right-chiral n-mode electron and f_L^n for its left-chiral $SU(2)$ doublet sister.

zero-mode profile, obtaining (up to normalization)

$$f_0 = A_0 \left(\frac{z}{R}\right)^{2-c} \tag{5.9}$$

$$g_0 = B_0 \left(\frac{z}{R}\right)^{c+2}, \tag{5.10}$$

where we've defined A_0 and B_0 to be constants of dimension 1/2. We can go on and solve for these constants by demanding that the zero mode fermions are canonically normalized. For example, plugging into the action and integrating over dz,

$$\begin{aligned}
S &= \int d^4x \int_R^{R'} dz \left(\frac{R}{z}\right)^4 i\overline{\chi}\overline{\sigma}^\mu \partial_\mu \chi + \cdots && (5.11)\\
&= \int d^4x \int_R^{R'} dz \left(\frac{R}{z}\right)^4 iA_0 \left(\frac{z}{R}\right)^{2-c} \overline{\chi}_0\overline{\sigma}^\mu \partial_\mu A_0 \left(\frac{z}{R}\right)^{2-c} \chi_0 + \cdots && (5.12)\\
&= A_0^2 \int d^4x \int_R^{R'} dz \left(\frac{z}{R}\right)^{-2c} i\overline{\chi}_0\overline{\sigma}^\mu \partial_\mu \chi_0 + \cdots && (5.13)\\
&= A_0^2\, R^{2c} \int d^4x \left[\frac{z^{1-2c}}{1-2c}\right]_R^{R'} i\overline{\chi}_0\overline{\sigma}^\mu \partial_\mu \chi_0 + \cdots && (5.14)\\
&= A_0^2\, R^{2c} \frac{1}{1-2c}\left[(R')^{1-2c} - R^{1-2c}\right] \int d^4x\, i\overline{\chi}_0\overline{\sigma}^\mu \partial_\mu \chi_0 + \cdots && (5.15)
\end{aligned}$$

In order to get the right normalization, the overall prefactor must be unity, thus

$$\begin{aligned}
A_0^2 &= \frac{1-2c}{R^{2c}\left[(R')^{1-2c} - R^{1-2c}\right]} && (5.16)\\
&= \frac{1-2c}{R^{2c}(R')^{1-2c}\left[1-(R/R')^{1-2c}\right]} && (5.17)\\
&= \frac{1}{R'}\left(\frac{R'}{R}\right)^{2c} \frac{1-2c}{1-(R/R')^{1-2c}}. && (5.18)
\end{aligned}$$

Finally, we may write the zero mode left-handed fermion (i.e. zero mode left-chiral fermion associated with a vector-like 5D fermion with left-handed boundary conditions).

$$\begin{aligned}
\chi_0(x,z) &= \chi_0(x) f_0(z) && (5.19)\\
&= \frac{1}{\sqrt{R'}}\left(\frac{R'}{R}\right)^{c} \sqrt{\frac{1-2c}{1-(R/R')^{1-2c}}} \left(\frac{z}{R}\right)^2 \left(\frac{z}{R}\right)^{-c} \chi_0(x) && (5.20)\\
&= \frac{1}{\sqrt{R'}} \sqrt{\frac{1-2c}{1-(R/R')^{1-2c}}} \left(\frac{z}{R}\right)^2 \left(\frac{z}{R'}\right)^{-c} \chi_0(x) && (5.21)\\
&\equiv \frac{1}{\sqrt{R'}} \left(\frac{z}{R}\right)^2 \left(\frac{z}{R'}\right)^{-c} f_c \chi_0(x), && (5.22)
\end{aligned}$$

where we've defined the prefactor f_c. Note that we haven't considered the special case $c = 1/2$ where the zero-mode fermion profile is 'flat' relative to the warping. In this case Eq. (5.14) is no longer correct and the term in the brackets is replaced by a term with logs[11].

Similarly, for right-handed fermions

$$S = \int d^4x \int_R^{R'} dz \left(\frac{R}{z}\right)^4 i\psi\overline{\sigma}^\mu\partial_\mu\overline{\psi} + \cdots \tag{5.23}$$

$$= B_0^2 \int d^4x \int_R^{R'} dz \left(\frac{z}{R}\right)^{2c} i\psi_0\overline{\sigma}^\mu\partial_\mu\overline{\psi}_0 + \cdots \tag{5.24}$$

$$= B_0^2 \frac{1}{R^{2c}}\frac{1}{1+2c}\left[(R')^{1+2c} - R^{1+2c}\right] \int d^4x\, i\psi_0\overline{\sigma}^\mu\partial_\mu\psi_0 + \cdots . \tag{5.25}$$

Fixing the normalization,

$$B_0^2 = \frac{R^{2c}\,(1+2c)}{\left[(R')^{1+2c} - R^{1+2c}\right]} \tag{5.26}$$

$$= \frac{1}{R'}\left(\frac{R'}{R}\right)^{-2c} \frac{1+2c}{1-(R/R')^{1+2c}}, \tag{5.27}$$

as a sanity-check we note that this reproduces the left-handed coefficient when $c \to -c$. The zero mode right-handed fermion is

$$\overline{\psi_0}(x,z) = \overline{\psi_0}(x) f_0(z) \tag{5.28}$$

$$= \frac{1}{\sqrt{R'}}\left(\frac{R'}{R}\right)^{-c}\sqrt{\frac{1+2c}{1-(R/R')^{1+2c}}}\left(\frac{z}{R}\right)^2\left(\frac{z}{R}\right)^c\overline{\psi_0} \tag{5.29}$$

$$= \frac{1}{\sqrt{R'}}\sqrt{\frac{1+2c}{1-(R/R')^{1+2c}}}\left(\frac{z}{R}\right)^2\left(\frac{z}{R'}\right)^c\overline{\psi_0} \tag{5.30}$$

$$\equiv \frac{1}{\sqrt{R'}}\left(\frac{z}{R}\right)^2\left(\frac{z}{R'}\right)^c g_c\overline{\psi_0}, \tag{5.31}$$

where we've defined the coefficient g_c similar to f_c above. Note, however, that these are simply related by

$$g_c = f_{-c}, \tag{5.32}$$

and so it is common to say that right-handed fermions take negative values for c.

The next important point is the localization of these zero modes. Even though we've written down the 5D profiles for the zero modes, the localization is nontrivial since one must take into account the effect of the warp factor. Recall that we used the following normalization conditions,

$$A_0^2 \int_R^{R'} dz \left(\frac{z}{R}\right)^{-2c} = B_0^2 \int_R^{R'} dz \left(\frac{z}{R}\right)^{2c} = 1. \tag{5.33}$$

[11] The authors thank Felix Yu for bringing this up. This can lead to some confusion in the normalization when dealing with fields in the 'conformal limit.' See the discussion in Gherghetta and Pomarol [17] below equation (50).

A tricky way to understand the localization of the zero modes is to play with the limits of integration of the integrals, remembering that they already include the effect of the warp factor. Let's first focus on the A_0 integral for fermions with left-handed boundary conditions.

- Sending the IR brane to infinity, $R' \to \infty$, we see that the A_0 integral remains convergent for $c > 1/2$. This means that for $c > 1/2$ the fermion is localized on the UV brane.
- On the other hand, sending the UV brane to zero, $R \to 0$, we see that the integral remains convergent for $c < 1/2$ and we say that the fermion is localized on the IR brane.

For fermions with right-chiral boundary conditions, the conditions get an overall sign.

- For $R' \to \infty$, the B_0 integral is finite for $c < -1/2$.
- For $R \to 0$ it is finite for $c > -1/2$.

So we see the generic feature of extra dimensional theories on an interval where the bulk mass of a field determines that field's localization. For fermions the transition between IR and UV localization occurs at $|c| = 1/2$.

For completeness, let us mention the higher KK modes, for which one must solve eqns. (5.7-5.8) for nonzero m_n. This is solved by combining the two coupled first-order differential equations to obtain second-order equations,

$$f_n'' + \frac{4}{z}f_n' + \left(m_n^2 - \frac{c^2+c-6}{z}\right) f_n = 0 \tag{5.34}$$

$$g_n'' + \frac{4}{z}g_n' + \left(m_n^2 - \frac{c^2-c-6}{z}\right) g_n = 0. \tag{5.35}$$

We already know the general solution to these equations, since they are precisely what we solved in eq. (4.25). Indeed, this procedure is exactly our mixed position-momentum space trick of squaring the Dirac operator into a scalar operator acting on each Weyl subspace, i.e. decoupling the χ and ψ. Explicitly, the solutions (before imposing boundary conditions) are

$$f_n(z) = z^{5/2}\left(A_n J_{c+1/2}(m_n z) + B_n Y_{c+1/2}(m_n z)\right) \tag{5.36}$$

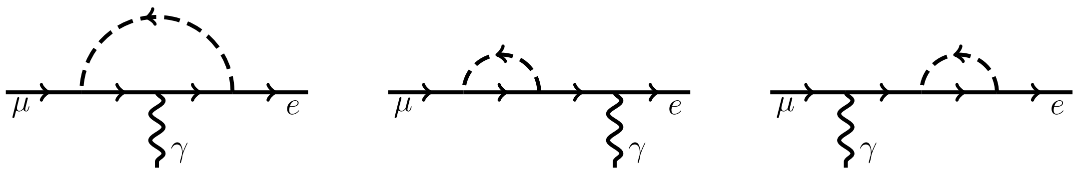
$$g_n(z) = z^{5/2}\left(A_n J_{c-1/2}(m_n z) + B_n Y_{c-1/2}(m_n z)\right), \tag{5.37}$$

where we used the bulk equations of motion to set the coefficients of each term to be the same betwee the f_n and g_n functions. The A_n and B_n are ultimately determined by the boundary conditions and imposing canonical normalization for each mode, as we did for the zero-mode.

6 Calculation of Penguins

We now discuss the practical calculation of penguin-diagrams in this warped extra dimension. We shall focus here on the technical calculation. A discussion of the finiteness of the penguin is left for our main paper. For concreteness, we will consider the leptonic penguin[12] $\mu \to e\gamma$.

[12] Purists might argue that these diagrams are not 'real' penguins because they don't have any 'feet.' The *essence* of the penguin diagram, however, is that it is a loop-mediated flavor changing neutral current that can occur on-



(a) Vertex correction (b) Muon correction (c) Electron correction

Figure 1: Higgs-mediated loops contributing to $\mu \to e\gamma$. Explicit mass insertions are suppressed.

6.1 Gauge-Invariant Amplitude

We now pause for a morality play regarding the kinds of terms that contribute to the physical amplitude. Consider the scalar contribution to the Standard Model process, illustrated in Fig. 1. The external leg corrections are linearly divergent while the 1PI diagram is log divergent. Naively one has to worry about regularizing these diagrams and this becomes an ordeal. However, we know that this process is finite. (See, for example, the heuristic arguments in our paper.) The divergences conspire to cancel each other exactly. This can be seen from the Ward identity which tells us that replacing the external photon polarization by its momentum causes the amplitude to vanish, $q_\mu \mathcal{M}^\mu = 0$. This means that the amplitude can be written in the form

$$\mathcal{M}^\mu = a\,\overline{u}\left[p^\mu + (p+q)^\mu - (m_\mu + m_e)\gamma^\mu\right]u + b\,q^\mu\,\overline{u}\gamma^\mu u + c\,q^2\overline{u}u, \tag{6.1}$$

where a, b, c are coefficients with the appropriate dimensions. This satisfies the Ward identity since the a term vanishes upon contraction with q: $\overline{u}\,m_\mu \not{q} = \overline{u}\,\not{p}\not{q} = \overline{u}\,p\cdot q$, and similarly with $m_e\,u$. The b and c terms also vanish upon contraction with q^μ since $q^2 = 0$ on shell; these terms don't contribute in the physical amplitude since $q\cdot\epsilon = 0$. Thus we are left with a physical amplitude of the form

$$\mathcal{M}^\mu = a\,\overline{u}\left[p^\mu + (p+q)^\mu - (m_\mu + m_e)\gamma^\mu\right]u. \tag{6.2}$$

We can now massage this into a gauge invariant tensor operator,

$$\begin{aligned}
\mathcal{M}^\mu &= a\,\overline{u}\left[p^\mu + (p+q)^\mu - \not{p}\gamma^\mu - \gamma^\mu\left(\not{p}+\not{q}\right)\right]u & (6.3)\\
&= a\,\overline{u}\left[p^\mu + (p+q)^\mu - \frac{1}{2}\not{p}\gamma^\mu - \frac{1}{2}\not{p}\gamma^\mu - \frac{1}{2}\gamma^\mu\left(\not{p}+\not{q}\right) - \frac{1}{2}\gamma^\mu\left(\not{p}+\not{q}\right)\right]u & (6.4)\\
&= a\,\overline{u}\left[p^\mu + (p+q)^\mu - p^\mu + \frac{1}{2}\gamma^\mu\not{p} - \frac{1}{2}\not{p}\gamma^\mu - (p+q)^\mu + \frac{1}{2}\left(\not{p}+\not{q}\right)\gamma^\mu - \frac{1}{2}\gamma^\mu\left(\not{p}+\not{q}\right)\right]u & (6.5)\\
&= a\,\overline{u}\frac{1}{2}\left[\not{q}, \gamma^\mu\right]u. & (6.6)
\end{aligned}$$

This is now manifestly in the form of a gauge-invariant term

$$\mathcal{M} = \epsilon_\mu \mathcal{M}^\mu = a\,\overline{u}\sigma^{\mu\nu}u\,F_{\mu\nu}. \tag{6.7}$$

shell due to the radiation of a boson. Whether or not that boson pair produces (hence forming the 'feet' of the penguin) is irrelevant to the point that the penguin is the leading order contribution to neutral-current flavor violation. For those who are still perturbed, note that there is no 'official' definition of the penguin diagram. The original reference is Ellis et al, [18]. For more background on the etymology of the term, see, for example, the memorable quote by John Ellis in Shifman's ITEP lectures [19].

Let us now note that the vector terms that contributed to this amplitude are finite and proportional to the external fermion masses. In this simple example we see that the only contribution to the physical amplitude comes from the term proportional to p^μ. In particular, the vector terms that one obtains from evaluating all three diagrams in Fig. 1 can either be converted into terms proportional to the mass of the external fermions (which reduces the overall degree of divergence) or must otherwise cancel (these include the terms containing divergences). Our general strategy, then, will be to identify the a coefficient as the p^μ term after using the Clifford algebra and the external spinor equations of motion. This allows us to directly write the finite physical amplitude without worrying about regularization of potentially divergent terms that ultimately cancel.

Let us be clear on this: the *amplitude* is composed of several terms. As written they all start out proportional to γ^μ from the photon coupling. These terms each contain additional Dirac structure such as $\not{p}$ and all of our favorite friends. We can massage these terms using the Clifford algebra to (anti-)commute $\not{p}$ past a γ^μ. The anticommutator term gives us a p^μ, which will contribute to the physical amplitude. The game we play is thus:

1. Simplify the amplitude's Dirac structure as much as possible.
2. Use the equations of motion to get rid of $\not{p}$ and $(\not{p}+\not{q})$ terms when they're next to the appropriate external spinor.
3. One will be left with terms like $\overline{u}(p+q)\not{p}\gamma^\mu u(p)$ which one cannot use the equations of motion to simplify. Now use the Clifford algebra to anticommute the slashed-momentum across the γ^μ. This allows one to use the equations of motion, but one also gets a leftover term of the form $p^\mu \overline{u}(p+q)u(p)$. This is precisely the term that we're interested in.

6.2 Operators

Now let us pause and note that it is particularly useful to think about the contributions to this process in terms of **effective operators**. This means that we write down an **effective Lagrangian** which encodes 1PI loops and allows us to write down *tree-level* 'skeleton diagrams' for $\mu \to e\gamma$. This still requires us to calculate the same loop diagrams as we would otherwise, but we will see that combined with the analysis of the form of the amplitude above, this will allow us to identify the bare minimum that we have to calculate[13]. The effective operators are constrained by chirality and gauge invariance. For example, the leading order operator takes the form

$$a_{k\ell}\frac{e_5}{16\pi^2}H\cdot L_i\sigma^{\mu\nu}(Y_{5ik}y^\dagger_{5k\ell}y_{5\ell j})\overline{E}_j F_{\mu\nu}. \tag{6.8}$$

The indices on the coefficient $a_{k\ell}$ refer to the flavor structure of internal propagators. This is just the 5D operator that gives a tree level $\mu \to e\gamma$ coupling with no additional insertions. If we set the fields to their zero modes, then the coefficient a here is exactly a contribution to the coefficient a in the amplitude in Eq. (6.7). In fact, if we only consider this operator, then both as are exactly the same. It's no coincidence that this **effective operator** and the **form of the amplitude** take

[13]The moral is that one shouldn't confuse the effective operators from the amplitude. Indeed, there can be an operator that takes the exact form of the amplitude and if this is the only such operator, then it is equal to the amplitude. However, in general there can be other operators that can be combined to contribute to the amplitude.

the same form, the effective operator is just the most trivial contribution to the amplitude and they are both constrained by gauge invariance.

After plugging in the wave functions for the fermion and photon zero modes, carefully including all warp factors, and matching the gauge coupling we find the relevant 4D operator for the zero modes contributing to $\mu \to e\gamma$ to be given by

$$a_{k\ell} R'^2 \frac{e}{16\pi^2} \frac{v}{\sqrt{2}} \left(f_{L_i} Y_{ik} Y^\dagger_{k\ell} Y_{\ell j} f_{-E_j} \right) \overline{L}_i^{(0)} \sigma^{\mu\nu} E_j^{(0)} F_{\mu\nu}^{(0)} + \text{h.c.} \tag{6.9}$$

in the gauge eigenbasis, i.e. before diagonalizing the Yukawa matrix. Since Y is anarchic we simplify the indices by writing

$$a_{k\ell} Y_{ik} Y^\dagger_{k\ell} Y_{\ell j} = a Y_*^3 \tag{6.10}$$

where we've used the anarchic limit[14] and the observation that the $a_{k\ell}$ do not vary strongly for different values of k and ℓ. This observation can be checked explicitly, as can be found in the handy plot in our paper. We can go to the standard 4D mass eigenbesis using the bi-unitary transformation

$$y = U_L y^{(\text{diag})} U_R^\dagger, \tag{6.11}$$

where the magnitudes of the elements of the unitary matrices $U_{L,R}$ are set by the hierarchies in the flavor constants

$$(U_L)_{ij} \sim \frac{f_{L_i}}{f_{L_j}} \quad \text{for} \quad f_{L_i} < f_{L_j}. \tag{6.12}$$

The traditional parameterization for the $\mu \to e\gamma$ amplitude is written as

$$\frac{-iC_{L,R}}{2m_\mu} \overline{u}_{L,R}(p)\, \sigma^{\mu\nu}\, u_{R,L}(p+q) F_{\mu\nu} \tag{6.13}$$

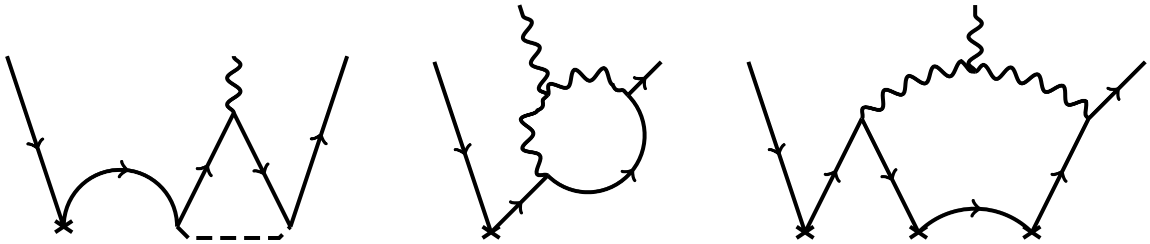
where $u_{L,R}$ are the left and right handed Dirac spinors for the μ. Comparing (6.9) with (6.13) and using the magnitudes for the off-diagonal terms in the U_L rotation matrix for the anarchic case in (6.12), we find that in the mass eigeinbasis

$$\begin{aligned} C_L &= a R'^2 \frac{e}{16\pi^2} Y_*^3 \frac{v}{\sqrt{2}} 2m_\mu f_{L_2} f_{-E_1}, \\ C_R &= a R'^2 \frac{e}{16\pi^2} Y_*^3 \frac{v}{\sqrt{2}} 2m_\mu f_{L_1} f_{-E_2}. \end{aligned} \tag{6.14}$$

[14] In particular, if $W = Y/Y^*$ is a matrix containing only elements ± 1 with randomly assigned signs, then

$$(WW^\dagger W)_{ij} = \sum_{i=1}^{9} \text{Rand}(+,-).$$

This must sum to an odd integer. It is a simple exercise in elementary probability to show that the value is ± 1 for just under half of the possible random distributions.



(a) External mass insertion (b) Bulk W loop (c) "Muffin" diagram

Figure 2: Higher order diagrams that incorporate contributions from other operators.

The actual $\mu \to e\gamma$ branching fraction is given by

$$\mathrm{Br}(\mu \to e\gamma) = \frac{12\pi^2}{(G_F m_\mu^2)^2}(C_L^2 + C_R^2) < 1.2 \cdot 10^{-11}. \tag{6.15}$$

While the generic expression of $\mathrm{Br}(\mu \to e\gamma)$ depends on the individual wave functions $f_{L,-E}$, since the product $C_L C_R$ is fixed by the physical lepton masses and $C_L^2 + C_R^2 \geq 2C_L C_R$ one can put a lower bound on the branching ratio

$$\mathrm{Br}(\mu \to e\gamma) \geq 3a^2 \frac{\alpha}{4\pi}\left(\frac{R'^2}{G_F}\right)^2 \frac{m_e}{m_\mu} Y_*^4. \tag{6.16}$$

If we want the lightest gauge KK modes to be accessible at the LHC, since $m_{\mathrm{KK}} \sim 2.4/R'$, we need to choose $1/R' \sim 1 - 1.51/\mathrm{TeV}$. Then for a given value of a we will get a generic upper bound on Y_*

$$Y_* \leq 0.18\, a^{-\frac{1}{2}}. \tag{6.17}$$

6.3 Other operators

One might consider more operators that could contribute. For example, one might consider the brane-localized kinetic terms

$$\frac{b^L_{ij}}{16\pi^2}\bar{L}_l D(y^\dagger y)_{ij} L_j + \frac{b^E_{ij}}{16\pi^2}\bar{E}_i D(yy^\dagger)_{ij} E_j, \tag{6.18}$$

or the the higher-order W-loop induced operators

$$\frac{c^L_{ij}}{16\pi^2}\bar{L}_i \not{D} D^2 L_j + \frac{c^E_{ij}}{16\pi^2}\bar{E}_i \not{D} D^2 E_j. \tag{6.19}$$

These come from diagrams such as those in Fig. 2. More generally, one can build additional gauge-invariant 5D bulk operators from powers of the covariant derivative,

$$\frac{b}{16\pi^2}\overline{L}_i \not{D}^{2n} E_j \qquad\qquad \frac{c}{16\pi^2}\overline{L}_i \not{D}^{2n+1} L_j,$$

but these do not contribute at leading-loop order since one can apply the equations of motion to convert these operators into contributions to kinetic term, i.e. wavefunction renormalization[15]. In other words, $\not{D}^n \to \not{D}$. In particular, this means that one may ignore diagrams with a bulk loop and an external chirality-flipping mass insertion on the IR brane, e.g. W loop diagrams.

6.4 R' dependence

Let's discuss how to write out an effective operator more concretely. We'll focus on the a operator. We would like to pull out all dimensionful factors and write everything in terms of a dimensionless coefficient. We would also like to translate this from a 5D operator into an operator for zero mode fermions. We start with the 'full' form of the 5D operator (note the covariant δ function to force the operator to be IR brane-localized):

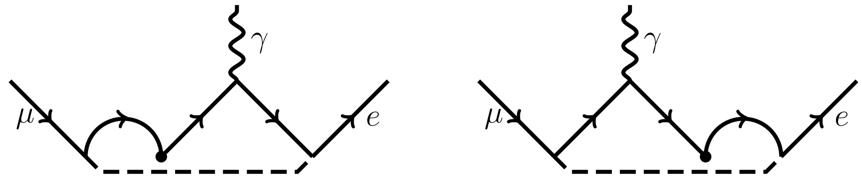
$$\mathcal{O}_a = a\sqrt{G} g_5 H L \sigma^{\mu\nu} \left(y y^\dagger y \right) \overline{E} \delta(\sqrt{G_{55}}(z - R')) F_{\mu\nu}. \tag{6.20}$$

For simplicity we're dropping flavor indices. Let's check dimensions:

$$[g_5] = -\frac{1}{2} \tag{6.21}$$
$$[H] = 1 \tag{6.22}$$
$$[\Psi] = 2 \tag{6.23}$$
$$[F_{\mu\nu}] = \frac{5}{2} \tag{6.24}$$
$$[\delta(z - R)] = 1 \tag{6.25}$$
$$[y] = -1. \tag{6.26}$$

All other terms are dimensionless so that upon including a d^5x this term is dimensionless in the effective action and thus the coefficient a is indeed dimensionless. The 4D operator is dimension 6, so when we write out an operator for (4D) zero modes, we expect a coefficient of dimension -2. Since this operator is IR brane localized, the intuitive choice of scale is R' (or, if the operator were to have turned out to be UV sensitive, it would have been $1/\Lambda^2$.) We can now insert our 4D

[15] It is a nontrivial fact often used in effective field theory that one may use the equations of motion to relate quantum operators. This has been explained pedagogically by Politzer in Section 12 of [20] and Weinberg in Section 7.7 of [21]. We thank Witek Skiba for explaining this to us.



(a) Mass insertion before γ (b) Mass insertion after γ

Figure 3: 5D one-loop diagrams contributing to the operator (6.34). The horizontal dashed line represents a brane-localized Higgs and the dot represents a Higgs vev insertion.

fields and check explicit powers of R:

$$\sqrt{G}\delta(\sqrt{G_{55}}(z-R')) = \left(\frac{R}{R'}\right)^4 \delta(z-R') \tag{6.27}$$

$$g_5 F_{\mu\nu} = eF^{(0)}_{\mu\nu} \tag{6.28}$$

$$\langle H\rangle = \frac{v}{\sqrt{2}}\left(\frac{R'}{R}\right) \tag{6.29}$$

$$L(x,z)(\cdots)\overline{E}(x,z) = \frac{1}{R'}\left(\frac{R'}{R}\right)^4 f_L f_E \chi^{(0)}(x)(\cdots)\psi^{(0)}(x) \tag{6.30}$$

$$yy^\dagger y = R^3 YY^\dagger Y \tag{6.31}$$

$$\sigma^{\mu\nu} = \left(\frac{R'}{R}\right)^2 \sigma^{\mu\nu}_{\text{flat}}. \tag{6.32}$$

The result is

$$aR'^2 \frac{e}{16\pi^2}\frac{v}{\sqrt{2}}\bar{L}_i f_{L_i}(YY^\dagger Y)_{ij} f_{-E_j}\sigma^{\mu\nu}E_j F_{\mu\nu} + \text{h.c.} \tag{6.33}$$

Of course we could have just guessed this form from dimensional analysis and noting that the operator is localized on the IR brane so must depend on the warped down scale $(1/R')$.

6.5 Weyl Penguins

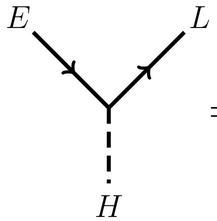
In terms of Weyl spinors, the leading order (in mass insertion) operator contributing to the $\mu \to e\gamma$ Weyl penguin[16] is

$$a_{k\ell}\frac{g_5}{16\pi^2} H \cdot \overline{L}_i \sigma^{\mu\nu}(y_{ik} {y^\dagger}_{k\ell} y_{\ell j}) E_j F_{\mu\nu}. \tag{6.34}$$

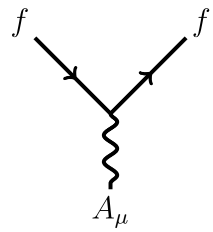
As discussed in our main paper, this is the only operator to contribute by gauge invariance. The corresponding diagrams contributing to this operator are shown in fig. 3.

[16] We only use this rather meaningless phrase to make fun of our colleague, D.C., who dishonors his German heritage by pronouncing Hermann Weyl's last name as 'whale.'

6.6 Feynman Rules, Schwinger Drools



$$= \left(\frac{R}{R'}\right)^3 Y_5 \equiv Y$$



$$= \left(\frac{R}{z}\right)^4 e\gamma^\mu$$

The first of these rules expresses how the effective Yukawa is warped down by a factor of $(R/R')^3$ by virtue of the Higgs being localized on the IR brane. The exact factor is determined by canonically normalizing the Higgs to be a 4D field. In detail,

$$S = \int d^4x \int_R^{R'} dz \left(\frac{R}{z}\right)^5 H_5 \cdot LY_5 E \frac{\delta(z-R')}{(R/z)} + \text{h.c} + \cdots , \tag{6.35}$$

where we recall that the covariant δ function is $\delta(\sqrt{g_{55}}z) = \delta(z)/\sqrt{g_{55}}$. We should now normalize the kinetic term and write the brane-localized 5D Higgs field H_5 explicitly as a fully 4D (e.g. dimension one) field H. Note the Higgs kinetic term takes the form

$$S = \int d^4x \int dz \left(\frac{R}{z}\right)^5 (D_M H_5)(D_N H_5) g^{MN} \frac{\delta(z-R')}{(R/z)} + \cdots \tag{6.36}$$

$$= \int d^4x \left(\frac{R}{R'}\right)^2 (D_\mu H_5)(D^\mu H_5) + \cdots . \tag{6.37}$$

Thus in order to canonically normalize with respect to a 4D Higgs, we ought to define

$$H = \frac{R}{R'} H_5. \tag{6.38}$$

Finally, we may rewrite the 5D Yukawa couplings in terms of these manifestly 4D Higgs fields as

$$S = \int d^4x \left(\frac{R}{R'}\right)^3 H \cdot LY_5 E + \text{h.c} + \cdots . \tag{6.39}$$

In order to relate these to the 4D Yukawa couplings between the zero-mode fermions, one may KK-decompose the 5D spinors and consider the profile of the zero mode. For example, inserting eqs. (5.22) and (5.31), for example, the Yukawa couplings of the zero modes can be written as

$$S = \int d^4x \left(\frac{R}{R'}\right)^3 H \cdot \left(\frac{1}{\sqrt{R'}}\left(\frac{R'}{R}\right)^2 f_{c_L} \chi_{0L}\right) Y_5 \left(\frac{1}{\sqrt{R'}}\left(\frac{R'}{R}\right)^2 f_{-c_R} \overline{\psi_{0E}}\right) + \text{h.c} + \cdots \tag{6.40}$$

$$= \int d^4x \; H \cdot \chi_{0L} \left(\frac{1}{R} f_{c_L} Y_5 f_{-c_R}\right) \overline{\psi_{0E}} + \text{h.c} + \cdots , \tag{6.41}$$

and thus we may identify

$$y_{\text{SM}} = \frac{1}{R} f_{c_L} Y_5 f_{-c_R}. \tag{6.42}$$

Next we discuss the couplings of the fermions to gauge bosons. The kinetic term takes the form

$$\int d^4x \int dz \left(\frac{R}{z}\right)^4 \overline{\Psi}\Gamma^M D_M \Psi, \tag{6.43}$$

where $D_M = \partial_M + g_5 A_M$. If we are interested in the coupling to a zero-mode external photon, as in $\mu \to e\gamma$, then we can just read off the result. This is because we know that the photon doesn't participate in electroweak symmetry breaking and so its zero mode profile is flat (i.e. z-independent). Thus one only has to keep track of powers of R and R'. A handy mnemonic, however, is to recall that the z integral for this coupling must include the 4D coupling so that

$$\int dz\, g_5 A^{(0)}(x,z) = eA^{(0)}(x). \tag{6.44}$$

Thus the relevant Feynman rule is just the 4D coupling e times the appropriate power of the warp factor. Using this Feynman rule we are implicitly writing the photon as a 4D zero mode field. More generally, for a bulk gauge field (e.g. for internal W diagrams) the coupling is the same as above but with the 4D coupling replaced by the 5D coupling whose value is

$$(g_5)^2 = R(g_4)^2, \tag{6.45}$$

which is fixed by considering the zero mode coupling.

6.7 Amplitude

We now write out the amplitude for each diagram, $\mathcal{M} = \epsilon_\mu \left(M^\mu_{(a)} + M^\mu_{(b)}\right)$. For the first diagram we have

$$\begin{aligned}\mathcal{M}^\mu_{(a)} = \int đ^4k \int_R^{R'} dz\, \overline{u}_{L_i}(p')\, Y_{ik}\, \Delta_{E_k}(k',R',z)\, e\left(\frac{R}{z}\right)^4 \gamma^\mu\, \Delta_{E_k}(k,z,R') \\ \frac{v}{\sqrt{2}} Y^\dagger_{kl}\, \Delta_{L_\ell}(k,R',R')\, Y_{\ell j}\, u_{E_j}(p)\, \Delta_H(k-p).\end{aligned} \tag{6.46}$$

The external state spinors u and $\overline{u}$ are implicitly 5D, i.e. we mean $u = u_{4D} f_0(R')$. We can separate this out to make it more human-readable,

$$\mathcal{M}^\mu_{(a)} = \frac{ev}{\sqrt{2}} Y_{ik} Y^\dagger_{k\ell} Y_{\ell j} \int đ^4k \int_R^{R'} dz \left(\frac{R}{z}\right)^4 \overline{u}_{L_i}(p') \left[G^\mu_{(a)}\right]_{k\ell} u_{E_j}(p)\, \Delta_H(k-p), \tag{6.47}$$

where we've defined the 'gamma matrix structure,'

$$\left[G^\mu_{(a)}\right]_{k\ell} = \Delta_{E_k}(k',R',z)\, \gamma^\mu\, \Delta_{E_k}(k,z,R')\, \Delta_{L_\ell}(k,R',R'). \tag{6.48}$$

Similarly for the second diagram,

$$\mathcal{M}^\mu_{(b)} = \frac{ev}{\sqrt{2}} Y_{ik} Y^\dagger_{k\ell} Y_{\ell j} \int đ^4k \int_R^{R'} dz \left(\frac{R}{z}\right)^4 \overline{u}_{L_i}(p') \left[G^\mu_{(b)}\right]_{k\ell} u_{E_j}(p)\, \Delta_H(k-p), \tag{6.49}$$

with the corresponding gamma matrix structure

$$\left[G^\mu_{(b)}\right]_{k\ell} = \Delta_{E_k}(k', R', R')\ \Delta_{L_\ell}(k, R', z)\ \gamma^\mu\ \Delta_{L_\ell}(k, z, R'). \tag{6.50}$$

We will summarize the common prefactor as

$$A_{ik\ell j} = \frac{ev}{\sqrt{2}} Y_{ik} Y^\dagger_{k\ell} Y_{\ell j}. \tag{6.51}$$

6.8 Simplifying to Weyl Amplitudes

We can simplify the Dirac structure by recalling the structure of the fermion propagators 2×2 Weyl basis, eqs. (4.37-4.38). For example,

$$\begin{aligned}
G^\mu_{(a)} &= \Delta_{E_k}(k', R', z)\ \gamma^\mu\ \Delta_{E_k}(k, z, R')\ \Delta_{L_\ell}(k, R', R') && (6.52)\\
&= \begin{pmatrix} 0 & 0 \\ \Delta^{21}_{E_k} & \Delta^{22}_{E_k} \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \overline{\sigma}^\mu & \end{pmatrix} \begin{pmatrix} \Delta^{11}_{E_k} & 0 \\ \Delta^{21}_{E_k} & 0 \end{pmatrix} \begin{pmatrix} 0 & \Delta^{12}_{L_\ell} \\ 0 & 0 \end{pmatrix} && (6.53)\\
&= \begin{pmatrix} 0 & 0 \\ 0 & G^{22}_{(a)} \end{pmatrix}, && (6.54)
\end{aligned}$$

where we have now packaged the Dirac structure into Weyl structure,

$$\begin{aligned}
G^{22}_{(a)} = \ & \Delta^{22}_{E_k}(k', R', z)\ \overline{\sigma}^\mu\ \Delta^{11}_{E_k}(k, z, R')\ \Delta^{12}_{L_\ell}(k, R', R') \\
& + \Delta^{21}_{E_k}(k', R', z)\ \sigma^\mu\ \Delta^{21}_{E_k}(k, z, R')\ \Delta^{12}_{L_\ell}(k, R', R').
\end{aligned} \tag{6.55}$$

Does this make sense? As a sanity check, we check that the amplitude goes as

$$\mathcal{M}^\mu_{(a)} \sim \begin{pmatrix} \psi & \overline{\chi} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & G^{22}_{(a)} \end{pmatrix} \begin{pmatrix} \chi \\ \overline{\psi} \end{pmatrix} \sim \overline{\chi}\, G^{22}_{(a)}\, \overline{\psi} \sim \overline{L}\, G^{22}_{(a)}\, E, \tag{6.56}$$

which is indeed the correct form we wrote in eq. (6.34).

Let us do the same thing for $G^\mu_{(b)}$,

$$\begin{aligned}
G^\mu_{(b)} &= \Delta_{E_k}(k', R', R')\ \Delta_{L_\ell}(k, R', z)\ \gamma^\mu\ \Delta_{L_\ell}(k, z, R') && (6.57)\\
&= \begin{pmatrix} 0 & 0 \\ \Delta^{21}_{E_k} & 0 \end{pmatrix} \begin{pmatrix} \Delta^{11}_{L_\ell} & \Delta^{12}_{L_\ell} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \overline{\sigma}^\mu & \end{pmatrix} \begin{pmatrix} 0 & \Delta^{12}_{L_\ell} \\ 0 & \Delta^{22}_{L_\ell} \end{pmatrix} && (6.58)\\
&= \begin{pmatrix} 0 & 0 \\ 0 & G^{22}_{(b)} \end{pmatrix}, && (6.59)
\end{aligned}$$

where we have again packaged the Dirac structure into Weyl structure,

$$\begin{aligned}
G^{22}_{(b)} = \ & \Delta^{21}_{E_k}(k', R', R')\ \Delta^{11}_{L_\ell}(k', R', z)\ \sigma^\mu\ \Delta^{22}_{L_\ell}(k, z, R') \\
& + \Delta^{21}_{E_k}(k', R', R')\ \Delta^{12}_{L_\ell}(k', R', z)\ \overline{\sigma}^\mu\ \Delta^{12}_{L_\ell}(k, z, R').
\end{aligned} \tag{6.60}$$

6.9 Separating the Scalar Parts

We will now simplify the G^{22} functions. Recalling eq. (4.31),

$$\Delta = \mathcal{D}^* \begin{pmatrix} F- & \\ & F_+ \end{pmatrix} = \begin{pmatrix} \partial_5 - \frac{2}{z} - \frac{c}{z} & \not{k} \\ \bar{\not{k}} & \frac{2}{z} - \partial_5 - \frac{c}{z} \end{pmatrix} \begin{pmatrix} F_- & \\ & F_+ \end{pmatrix} \equiv \begin{pmatrix} D_-F_- & \not{k}F_+ \\ \bar{\not{k}}F_- & D_+F_+ \end{pmatrix}. \tag{6.61}$$

Now we can write out the G^22s as (using slightly modified notation for the Fs),

$$\begin{aligned} G^{22}_{(a)} = &\ \left[D_+F^{+>}_{E_k}(k', R', z)\right] \left[D_-F^{-<}_{E_k}(k, z, R')\right] \left[F^{+}_{L_\ell}(k, R', R')\right] \bar{\sigma}^\mu \not{k} \\ &+ \left[F^{->}_{E_k}(k', R', z)\right] \left[F^{-<}_{E_k}(k, z, R')\right] \left[F^{+}_{L_\ell}(k, R', R')\right] \bar{\not{k}}' \sigma^\mu \bar{\not{k}} \not{k} \end{aligned} \tag{6.62}$$

$$\begin{aligned} G^{22}_{(b)} = &\ \left[F^{-}_{E_k}(k', R', R')\right] \left[D_-F^{->}_{L_\ell}(k', R', z)\right] \left[D_+F^{+<}_{L_\ell}(k, z, R')\right] \bar{\not{k}} \sigma^\mu \\ &+ \left[F^{-}_{E_k}(k', R', R')\right] \left[F^{+>}_{L_\ell}(k', R', z)\right] \left[F^{+<}_{L_\ell}(k, z, R')\right] \bar{\not{k}}' \not{k}' \bar{\sigma}^\mu \not{k}. \end{aligned} \tag{6.63}$$

Recalling the Clifford algebra for Pauli matrices,

$$\not{p}\bar{\not{p}} = \bar{\not{p}}\not{p} = p^2 \mathbb{1}_{2\times 2}. \tag{6.64}$$

Let us thus write the G^{22}s in terms of purely scalar functions of the Fs,

$$G^{22}_{(a)} = \overline{g}_{(a)} \bar{\sigma}^\mu \not{k} + \hat{g}_{(a)} k^2 \bar{\not{k}}' \sigma^\mu \tag{6.65}$$

$$G^{22}_{(b)} = g_{(b)} \bar{\not{k}}' \sigma^\mu + \hat{\overline{g}}_b (k')^2 \bar{\sigma}^\mu \not{k}, \tag{6.66}$$

where we recall that $k^2 = \chi^2_k$ and $(k')^2 = \chi^2_{k+q}$. To write these in a more useful form, we can absorb these into the $\hat{g}$ and $\hat{\overline{g}}$ functions by defining $g_{(a)} = \hat{g}_{(a)} k^2$ and $\overline{g}_{(b)} = \hat{\overline{g}}_{(b)} (k')^2$,

$$G^{22}_{(a)} = \overline{g}_{(a)} \bar{\sigma}^\mu \not{k} + g_{(a)} \bar{\not{k}}' \sigma^\mu \tag{6.67}$$

$$G^{22}_{(b)} = g_{(b)} \bar{\not{k}}' \sigma^\mu + \overline{g}_{(b)} \bar{\sigma}^\mu \not{k}. \tag{6.68}$$

For concreteness let us write out our shorthand notation explicitly,

$$\overline{g}_{(a)} = \left[D_+F^{+>}_{E_k}(k', R', z)\right] \left[D_-F^{-<}_{E_k}(k, z, R')\right] \left[F^{+}_{L_\ell}(k, R', R')\right] \tag{6.69}$$

$$g_{(a)} = \chi^2_k \left[F^{->}_{E_k}(k', R', z)\right] \left[F^{-<}_{E_k}(k, z, R')\right] \left[F^{+}_{L_\ell}(k, R', R')\right] \tag{6.70}$$

$$g_{(b)} = \left[F^{-}_{E_k}(k', R', R')\right] \left[D_-F^{->}_{L_\ell}(k', R', z)\right] \left[D_+F^{+<}_{L_\ell}(k, z, R')\right] \tag{6.71}$$

$$\overline{g}_{(b)} = (\chi_{k+q})^2 \left[F^{-}_{E_k}(k', R', R')\right] \left[F^{+>}_{L_\ell}(k', R', z)\right] \left[F^{+<}_{L_\ell}(k, z, R')\right] \tag{6.72}$$

To really summarize our progress so far, let's write out the amplitudes for each diagram again,

$$\mathcal{M}^\mu_{(a)} = A_{ik\ell j} \int đ^4k \int dz \left(\frac{R}{z}\right)^4 \overline{u}_{L_i} \begin{pmatrix} 0 & 0 \\ 0 & \overline{g}_{(a)} \bar{\sigma}^\mu \not{k} + g_{(a)} \bar{\not{k}}' \sigma^\mu \end{pmatrix} u_{E_j}\ \Delta_H \tag{6.73}$$

$$\mathcal{M}^\mu_{(b)} = A_{ik\ell j} \int đ^4k \int dz \left(\frac{R}{z}\right)^4 \overline{u}_{L_i} \begin{pmatrix} 0 & 0 \\ 0 & g_{(b)} \bar{\not{k}}' \sigma^\mu + \overline{g}_{(a)} \overline{\sigma^\mu} \not{k} \end{pmatrix} u_{E_j}\ \Delta_H, \tag{6.74}$$

where we've dropped arguments and written the overall constants as $A_{ik\ell j}$. An alternate way to write this amplitude is to promote the Pauli structure back into Dirac structure and impose a projection operator,

$$\mathcal{M}^{\mu}_{(a)} = A_{ik\ell j} \int đ^4k \int dz \left(\frac{R}{z}\right)^4 \; \overline{u}_{L_i} \left(\overline{g}_{(a)}\gamma^{\mu}\not{k} + g_{(a)}\not{k}'\gamma^{\mu}\right) P_{,}u_{E_j} \;\; \Delta_H \tag{6.75}$$

$$\mathcal{M}^{\mu}_{(b)} = A_{ik\ell j} \int đ^4k \int dz \left(\frac{R}{z}\right)^4 \; \overline{u}_{L_i} \left(\overline{g}_{(b)}\gamma^{\mu}\not{k} + g_{(b)}\not{k}'\gamma^{\mu}\right) P_R \, u_{E_j} \;\; \Delta_H. \tag{6.76}$$

Even more succinctly, we can write these amplitudes completely in terms of Weyl spinors,

$$\mathcal{M}^{\mu}_{(a)} = A_{ik\ell j} \int đ^4k \int dz \left(\frac{R}{z}\right)^4 \overline{\chi}_{Li}(p') \left[\overline{H}_{(a)} + H_{(a)}\right] \overline{\psi}_{E_j}(p) \tag{6.77}$$

$$\mathcal{M}^{\mu}_{(b)} = A_{ik\ell j} \int đ^4k \int dz \left(\frac{R}{z}\right)^4 \overline{\chi}_{Li}(p') \left[H_{(b)} + \overline{H}_{(b)}\right] \overline{\psi}_{E_j}(p) \tag{6.78}$$

where the Weyl structure takes the form

$$\overline{H}_{(a)} = \overline{g}_{(a)}(\chi_{k+q})\, \Delta_H(k-p)\, \overline{\sigma}^{\mu}\not{k} \tag{6.79}$$

$$H_{(a)} = g_{(a)}(\chi_{k+q})\, \Delta_H(k-p) \; \left(\overline{\not{k}} + \overline{\not{q}}\right) \sigma^{\mu} \tag{6.80}$$

$$H_{(b)} = g_{(b)}(\chi_{k+q})\, \Delta_H(k-p) \; \left(\overline{\not{k}} + \overline{\not{q}}\right) \sigma^{\mu} \tag{6.81}$$

$$\overline{H}_{(b)} = \overline{g}_{(b)}(\chi_{k+q})\, \Delta_H(k-p)\, \overline{\sigma}^{\mu}\not{k}. \tag{6.82}$$

$$\tag{6.83}$$

6.10 Taylor Expansion in p and q

While these manipulations greatly simplify the amplitude into compartmentalized pieces, it is still quite a mess to evaluate even numerically. In particular, the dependence on the integration variable k is not manifestly even so that we cannot do our usual 4D trick of doing a spherical integral.

The next step is to expand these expressions in p and q, the so-called 'Yuhsin's q-expansion[17]'. As explained in our main paper, the general strategy is to use the equations of motion on the external spinor wavefunctions,

$$\overline{u}_e(p')\not{p}' = m_e \overline{u}_e(p') \tag{6.84}$$

$$\not{p} u_{\mu}(p) = m_{\mu} u(p). \tag{6.85}$$

We use $p' \equiv p + q$ to convert all of our momenta into the integration variable k and the external fermion momenta p, p'. By using the Clifford algebra,

$$\not{p}\overline{\sigma}^{\mu} = -\sigma^{\mu}\overline{\not{p}} + 2p^{\mu} \tag{6.86}$$

$$\overline{\not{p}}\sigma^{\mu} = -\overline{\sigma}^{\mu}\not{p} + 2p^{\mu}, \tag{6.87}$$

[17]Actually, only the authors call it that. And even then, only one of the authors calls it that since it would be silly for Yuhsin to refer to himself in the third person.

to commute the various $\not{p}$ and $\not{p}'$ factors toward their appropriate external state spinors we are left over with terms which go as p^μ and p'^μ. The coefficient of these terms completely determines the $\mu \to e\gamma$ amplitude. For concreteness we only need to determine the p^μ coefficient.

In order to do this we want to pull out the q-dependence of the various terms in our expressions. These all come from $k' = k + q$. In particular, they come from the factor

$$\chi_{k'} = \sqrt{k^2 + 2k \cdot q}. \tag{6.88}$$

We can thus pull out all of the qs by Taylor expanding in the q. Note that this will generically give us factors of $(k \cdot q)$. Upon doing the loop integral, $đ^4 k$, however, these turn into contractions among the qs and the γ^μ. For example, the familiar symmetry relation

$$k_\alpha k_\beta = \frac{1}{4}\eta_{\alpha\beta} k^2 \tag{6.89}$$

leads to manipulations such as $(k \cdot q)\not{k} = \frac{1}{4}k^2 \not{q}$. (Note that we've used the Minkowski metric $\eta_{\mu\nu}$ for these purely 4D loops.) By the on-shellness of the external photon we see that this expansion terminates since $q^2 = 0$. The relevant formula to note in this expansion is

$$\frac{\partial g(\chi_{k'}, \cdots)}{\partial q^\mu} = \frac{\partial g}{\partial \chi_{k'}}(\chi_k, \cdots) \cdot \frac{\partial \chi_{k'}}{\partial q^\mu} \tag{6.90}$$

$$= \frac{\partial g}{\partial \chi_{k'}}(\chi_k, \cdots) \frac{k_\mu}{\chi_k}. \tag{6.91}$$

Thus we can expand our g functions as

$$g(\chi_{k'}) = g(\chi_k) + \frac{k \cdot q}{\chi_k} \frac{\partial g}{\chi_{k'}}(\chi_k) + \cdots , \tag{6.92}$$

where the higher order terms are naively of order $q^2 = 0$.

We also need to expand the Higgs propagator in terms of the muon momentum. Writing the Higgs mass as M,

$$\Delta_H(k-p) = \frac{1}{(k-p)^2 - M^2} \tag{6.93}$$

$$= \Delta_H(k) - 2\Delta'_H(k) k \cdot p + \mathcal{O}(m_\mu^2) \tag{6.94}$$

$$= \Delta_H(k)\left[1 - 2(k \cdot p)\Delta_H(k) + \mathcal{O}(m_\mu^2)\right], \tag{6.95}$$

where we can safely drop the terms of order m_μ^2 since it is small compared to all other scales in the problem. Note that now upon integration of $đ^4 k$ we are left with factors p^μ which end up contracting with the γ^μ and the factors of q^μ from the q-expansion. Thus the q-expansion terminates for a fixed order expansion in p^μ, namely it terminates after the term proportional to $(p \cdot q)^n \not{q}$ since further terms are proportional to q^2.

After doing these expansions we are left with an amplitude whose k-integral is manifestly even, i.e. all dependence on k can be written in terms of $\chi_k = \sqrt{k^2}$. This can now be done at any order in m_μ^2. Practically we are only interested in the leading order term, dropping all terms of order

m_μ^2. Recall that this also drops terms of order $\mathcal{O}(p\cdot q)$, since such a term is proportional to $m_\mu E_\gamma \sim m_\mu^2$. Practically this means that we only have to expand each of p and q to first order.

Now let us start expanding the relevant terms, dropping those that are odd in the integration variable k.

$$\overline{H}_{(a)} = \left(\overline{g}_{(a)} + \overline{g}'_{(a)}\frac{k\cdot q}{\chi_k}\right)\left(\Delta_H + 2k\cdot p\Delta_H^2\right)\overline{\sigma}^\mu \not{k} \tag{6.96}$$

$$= \left(2(k\cdot p)\overline{g}_{(a)}\Delta_H^2 + \overline{g}'_{(a)}\frac{k\cdot q}{\chi_k}\Delta_H\right)\overline{\sigma}^\mu \not{k} \tag{6.97}$$

$$= \frac{1}{2}k^2\overline{g}_{(a)}\Delta_H^2\overline{\sigma}^\mu\not{p} + \frac{1}{4}\overline{g}'_{(a)}\Delta_H\sigma^\mu\not{q}, \tag{6.98}$$

where all terms on the right-hand side are evaluated at $k' = k$ and we've dropped terms of order m_μ^2. We proceed similarly for the $g_{(a)}$ term,

$$H_{(a)} = \left(g_{(a)} + g'_{(a)}\frac{k\cdot q}{\chi_k}\right)\left(\Delta_H + 2(k\cdot p)\Delta_H^2\right)\left(\overline{\not{k}} + \overline{\not{q}}\right)\sigma^\mu \tag{6.99}$$

$$= \left(2(k\cdot p)g_{(a)}\Delta_H^2 + \frac{k\cdot q}{\chi_k}g'_{(a)}\Delta_H\right)\overline{\not{k}}\sigma^\mu + g_{(a)}\Delta_H\overline{\not{q}}\sigma^\mu. \tag{6.100}$$

Fortunately the $g_{(b)}$ terms have exactly the same structure and can be read off analogously,

$$H_{(b)} = \left(2(k\cdot p)g_{(b)}\Delta_H^2 + \frac{k\cdot q}{\chi_k}g'_{(b)}\Delta_H\right)\overline{\not{k}}\sigma^\mu + g_{(b)}\Delta_H\overline{\not{q}}\sigma^\mu \tag{6.101}$$

$$\overline{H}_{(b)} = \frac{1}{2}k^2\overline{g}_{(b)}\Delta_H^2\overline{\sigma}^\mu\not{p} + \frac{1}{4}\overline{g}'_{(b)}\Delta_H\sigma^\mu\not{q}. \tag{6.102}$$

6.11 Identifying the p^μ coefficient

We now would like to use the Clifford algebra to use the equations of motion to completely eliminate all $\not{p}$ and $\not{p}'$ terms. Upon contraction with the appropriate spinors these become vector operators proportional to the external fermion masses, which we neglect. Whenever we have to interchange a $\not{p}$ with a σ^μ matrix, however, we pick up a desired term proportional to p^μ.

Since the (a) and (b) values of the H expressions are completely analogous, it is sufficient just to look at $H_{(a)}$ and $\overline{H}_{(a)}$. The (b) values are obtained trivially from $(a)\to(b)$. The first term is given by

$$\overline{H}_{(a)} = \underbrace{\frac{1}{2}\chi_k^2\overline{g}_{(a)}\Delta_H^2\overline{\sigma}^\mu\not{p}}_{\sim m_\mu} + \underbrace{\frac{1}{4}\chi_k\overline{g}'_{(a)}\Delta_H\overline{\sigma}^\mu\left(\not{p}' - \not{p}\right)}_{\sim 2p'^\mu - m_e - m_\mu}, \tag{6.103}$$

from which we see there are no contributions to the p^μ term. The next contribution is

$$H_{(a)} = \frac{1}{2}\chi_k^2 g_{(a)}\Delta_H^2\overline{\not{p}}\sigma^\mu + \frac{1}{4}\chi_k g'_{(a)}\Delta_H\left(\overline{\not{p}}' - \overline{\not{p}}\right)\sigma^\mu + g_{(a)}\Delta_H\left(\overline{\not{p}}' - \overline{\not{p}}\right)\sigma^\mu. \tag{6.104}$$

Dropping all of the p' terms and keeping only the p^μ terms upon using the Clifford algebra to commute $\overline{\not{p}}$ across σ^μ, we find that the relevant contribution is

$$H_{(a)} = \left(\chi_k^2 g_a \Delta_H^2 - \frac{1}{2}\chi_k g'_{(a)}\Delta_H - 2g_{(a)}\Delta_H\right) p^\mu. \tag{6.105}$$

This is what is left to compute in our integral,

$$\mathcal{M}^\mu_{(a)} = A_{ik\ell j} \int đ^4 k \int dz \left(\frac{R}{z}\right)^4 \overline{\chi}_{Li}(p')\ H_{(a)}\ \overline{\psi}_{E_j}(p) \tag{6.106}$$

$$\mathcal{M}^\mu_{(b)} = A_{ik\ell j} \int đ^4 k \int dz \left(\frac{R}{z}\right)^4 \overline{\chi}_{Li}(p')\ H_{(b)}\ \overline{\psi}_{E_j}(p). \tag{6.107}$$

7 Performing the Numerical Integration

Now all that's left to be done is the actual integral. Actual integration is done using *Mathematica*, but some steps need to be taken to massage the integral into a *Mathematica*ble form. The first rule of doing integrals on mathematica is to always use dimensionless intergation variables. Motivated by the arguments of our Bessel functions, we choose

$$x \equiv \chi_k z \tag{7.1}$$

$$y \equiv \chi_k R'. \tag{7.2}$$

The choice to define y with respect to R' rather than the 'fundamental' scale R amounts to choosing to let *Mathematica* work with exponentially small numbers rather than exponentially large ones[18]. For practical purposes we may measure all of our dimensionful variables in units of $R' \sim$ TeV, i.e. we may set $R' = 1$. We remind ourselves of the following expressions,

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} = \chi_k \frac{\partial}{\partial x} = \frac{y}{R'}\frac{\partial}{\partial x}; \qquad \frac{1}{z} = \frac{\chi_k}{x} = \frac{y}{R'}\frac{1}{x}. \tag{7.3}$$

The $D_\pm F_\pm$ expressions defined in Eq. (6.61) take the form

$$D_\pm F_\pm = \left[\pm\left(\frac{\partial}{\partial z} - \frac{2}{z}\right) - \frac{c}{z}\right] F_\pm = \frac{y}{R'}\left[\pm\left(\frac{\partial}{\partial x} - \frac{2}{x}\right) - \frac{c}{x}\right] F_\pm. \tag{7.4}$$

It is now straightforward to express the F functions in terms of x and y rather than z and χ_p. (Indeed, it turns out to be much nicer to derive the Fs in this basis.) One can now plug this into the $g_{(a)}$ and $g_{(b)}$ functions in our amplitude. Refer to our *Mathematica* file for an explicit calculation. There are a few important remarks.

- Note that x and y both have k dependence. The $\partial/\partial k'$ in the amplitude must be handled carefully to make sure one is taking the appropriate derivative on the appropriate variables.

[18] It appears that *Mathematica* prefers working with very small numbers rather than very big ones, which it tends to interpret as divergences

- Variables should be Wick-rotated. This amounts to taking $x, y \to ix_E, iy_E$.
- Do the x integral first. We found that it can be useful to plot the subsequent integrand in y. In particular, one should be able to see that the y integrand approaches zero faster than $1/y^2$ so that the integral really is finite.
- Don't forget that

$$\begin{aligned} dz\, d^4k &= dx \frac{dz}{dz}\, dy \frac{d\chi_k}{dy} \left(\frac{y}{R'}\right) d\Omega_3 \\ &= dx\, dy\, d\Omega_3 y^2 (R')^{-3} = dx\, dy\, (2\pi) y^2 (R')^{-3} \end{aligned}$$

- Second, *Mathematica* doesn't like very big or very small overall values. It is best to remove all overall warp factors $R/R' \equiv w \sim 10^{-16}$ from the numerical integration. (There will be warp factors in the arguments of the Bessel functions that cannot be removed.)

One should check explicitly that all factors of w cancel and that the amplitude's dimensions are carried by an overall factor of $(R')^2$. To see this, we just have to check the w and R' dependence of each part of the amplitude:

- **Bulk fermion propagator**. This goes as $(zz')^5/R^4$, which gives us a dependence of R'/w^4. There are three such propagators, so we get a contribution of $(R')^3 w^{-12}$.
- **Yukawa coupling**. These go as $(R/R')^3 Y_5$, where $Y_5 = RY_4$. This gives a dependence of $R'w^4$. With three Yukawa couplings in the amplitude (don't forget the Higgs-induced mass insertion), this contributes a factor of $(R')^3 w^1 2$.
- **Gauge coupling**. This gives $(R/z)^4 e_5$ with $e_5 = Re_4$. The dependence is thus $w^5 R'$.
- **External wavefunctions**. The wavefunctions for the external zero-mode fermions and photon come from recalling that

$$\begin{aligned} f_c &\sim \frac{1}{\sqrt{R'}} \left(\frac{z}{R}\right)^2 \left(\frac{z}{R'}\right)^{-c} \Bigg|_{z=R'} \\ A^{(0)} &\sim \frac{1}{R} A_4^{(0)}. \end{aligned}$$

  A useful mnemonic for the photon and gauge dependence comes from noting that $e_5 A_5^{(0)} = e_4 A_4^{(0)}$. Thus the external wavefunctions give an overall factor of $(R')^2 w^5$.
- **Measure**. The integration measure, as written above, gives a factor of $(R')^{-3}$.

Multiplying everything together we indeed get an amplitude that is proportional to $(R')^2$ and with no additional overall factors of the warp factor w.

8 Relation to $\mu \to (3)e$

The amplitude for $\mu \to e\gamma$ provides an upper bound on the RS anarchic Yukawa scale. Agashe, Blechman, and Petriello showed that the processes $\mu \to 3e$ and $\mu \to e$ provide a *lower* bound on this scale. This can heuristically be understood in terms of the profile of the gauge bosons.

$\mu \to 3e$ is mediated by tree-level zero- and KK-mode Z exchange. The flavor changing process occurs due to the kink in the otherwise-flat Z boson profile near the IR brane with electroweak symmetry breaking causes a small feature. As we increase the Yukawa couplings, the fermion zero-mode profiles must move towards the UV brane to maintain the observed mass relations. This causes them to have a smaller overlap with this region of flavor-changing in the Z profile and would thus *lower* the amplitude for $\mu \to 3e$. Thus the upper limits on $\mu \to 3e$ impose lower limits on the anarchic Yukawa scale.

For a fixed KK gauge boson mass M_{KK}, limits on $\mu \to 3e$ and $\mu \to e$ conversion provide the strongest *upper* bounds on the anarchic Yukawa scale Y_*. These tree-level processes are parameterized by 4-Fermi operators generated by Z and Z' exchange. The effective Lagrangian for these lepton flavor violating 4-Fermi operators are traditionally parameterized as [22]

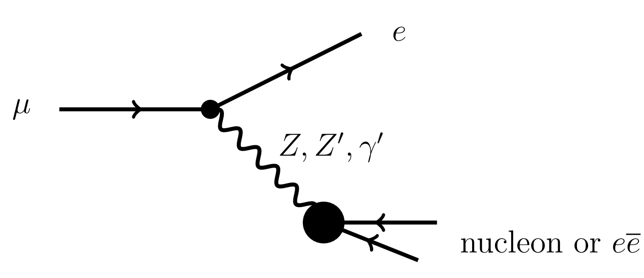
$$\begin{aligned}
\mathcal{L} \;=\; & \frac{4G_F}{\sqrt{2}}\left[g_3(\bar{e}_R\gamma^\mu\mu_R)(\bar{e}_R\gamma_\mu e_R) + g_4(\bar{e}_L\gamma^\mu\mu_L)(\bar{e}_L\gamma^\mu e_L) + g_5(\bar{e}_R\gamma^\mu\mu_R)(\bar{e}_L\gamma_\mu e_L)\right. \\
& \left. + g_6(\bar{e}_L\gamma^\mu\mu_L)(\bar{e}_R\gamma_\mu e_R)\right] + \frac{G_F}{\sqrt{2}}\bar{e}\gamma^\mu(v - a\gamma_5)\mu\sum_q \bar{q}\gamma_\mu(v^q - a^q\gamma_5)q, \qquad (8.1)
\end{aligned}$$

where we have only introduced the terms that are non-vanishing in the RS set up, and $v^q = T_3^q - 2Q^q\sin^2\theta$. The $g_{3,4,5,6}$ are responsible for the $\mu \to 3e$ decay, while the v, a are responsible for $\mu \to e$ conversion in nuclei. The rates are given by (with the conversion rate normalized to the muon capture rate):

$$\begin{aligned}
\text{Br}(\mu \to 3e) &= 2(g_3^2 + g_4^2) + g_5^2 + g_6^2\ , \\
\text{Br}(\mu \to e) &= \frac{p_e E_e G_F^2 F_p^2 m_\mu^3 \alpha^3 Z_{eff}^4}{2\pi^2 Z\Gamma_{capt}} Q_N^2 2(v^2 + a^2), \qquad (8.2)
\end{aligned}$$

where the parameters for the conversion depend on the nucleus and are calculated in the Feinberg-Weinberg approximation [23]. The most sensitive experiment is for ${}^{48}_{22}\text{Ti}$, for which $E_e \sim p_e \sim m_\mu, F_p \sim 0.55, Z_{\text{eff}} \sim 17.61, \Gamma_{\text{capt}} \sim 2.6 \cdot 10^6 \frac{1}{s}$, and $Q_N = v^u(2Z + N) + v^d(2N + Z)$.

In order to calculate the coefficients in the effective Lagrangian (8.1), we need to calculation the flavor violating couplings of the lightest neutral gauge bosons in the theory and compute the relevant tree-level contributions. In the basis of physical KK states all lepton flavor violating couplings are the consequence of the non-uniformity of the gauge boson wave functions.



8.1 The effect of the SM Z boson

Let us first consider the effect of the ordinary Z boson, whose wave function is approximately.

$$h^{(0)}(z) = \frac{1}{\sqrt{R \log \frac{R'}{R}}} \left[1 + \frac{M_Z^2}{4}\left(z^2 - 2z^2 \log \frac{z}{R}\right)\right].$$

Here we use the approximation (2.19) in Csáki, Erlich, and Terning [24] with a prefactor for canonical normalization and dropping some terms of $\mathcal{O}(M_Z^2 R'^2)$ which came from a silly normalization used in their paper. The generic solution for the n^{th} KK gauge boson wavefunction can be found by following the same techniques that we used to solve the scalar differential equation. By prodigal use of boundary conditions one can include the effects of electroweak symmetry breaking. For examples see [24] and [15]. The solution is

$$h^{(n)} = \mathcal{N} z \left[Y_0\left(M_{\text{KK}}^{(n)} R\right) J_1\left(M_{\text{KK}}^{(n)} z\right) - J_0\left(M_{\text{KK}}^{(n)} R\right) Y_1\left(M_{\text{KK}}^{(n)} z\right)\right], \tag{8.3}$$

up to some normalization $\mathcal{N}$ that is determined by the boundary conditions.For the zero mode we can Taylor expand with respect to the small arguments of the Bessel functions,

$$h^{(0)}(z) = \mathcal{N}\left[1 + \frac{M_Z^2}{4}\left(z^2 - 2z^2 \log \frac{z}{R}\right)\right]. \tag{8.4}$$

We want to determine $\mathcal{N}$ from the canonical normalization of the 4D Lagrangian. Let's look at the kinetic term:

$$\int d^4x \int_R^{R'} dz \left(\frac{R}{z}\right)^5 F_{MN}^{(5)} F_{PQ}^{(5)} g^{MP} g^{NQ} = \int d^4x \int_R^{R'} dz \frac{R}{z} F_{MN}^{(5)} F_{PQ}^{(5)} \eta^{MP} \eta^{NQ} \tag{8.5}$$

$$= \int d^4x \int_R^{R'} dz \frac{R}{z} F_{\mu\nu}^{(0)} F^{(0)\mu\nu} \left(h^{(0)}(z)\right)^2 + \cdots \tag{8.6}$$

We can ignore the z-dependent part of $h^{(0)}(z)$ since it is small (proportional to $(M_Z z)^2$) and normalize with respect to the leading term, $\mathcal{N}$. The integral gives an overall prefactor of

$$\mathcal{N}^2 R \log \frac{R'}{R}, \tag{8.7}$$

so that the canonical normalization is

$$\mathcal{N} = \frac{1}{\sqrt{R \log R'/R}}. \tag{8.8}$$

We would like to determine the flavor-changing coupling of the Z, i.e. the nonuniversal coupling. Recall that the profile of a zero-mode fermion with bulk mass $|c|$ is given by Eq. (5.22),

$$\Psi_c^{(0)}(x,z) = \frac{1}{\sqrt{R'}}\left(\frac{z}{R}\right)^2 \left(\frac{z}{R'}\right)^{-c} f_c \, \Psi^{(0)}(x), \tag{8.9}$$

where

$$f_c = \sqrt{\frac{1-2c}{1-(R/R')^{1-2c}}}. \tag{8.10}$$

Left-handed modes have $c > 0$ and right-handed modes have $c < 0$. The effective 4D coupling is determined by calculating the overlap of the 5D wavefunction profiles,

$$g_{4D}^{Zff} Z_\mu^{(0)} \overline{\Psi}_c^{(0)} \gamma^\mu \Psi_c^{(0)} = \int dz \left(\frac{R}{z}\right)^5 g_5^{Zff} Z_M(x,z) \overline{\Psi}_c^{(0)}(x,z) \Gamma^M \Psi_c^{(0)}(x,z) - \cdots , \tag{8.11}$$

where we've written g_{Zff} to mean the coupling constant with the quantum numbers of the fermions incorporated. For example,

$$g^{Zff_5} = g_5 c_Q T_3 - g_5' s_Q Y. \tag{8.12}$$

Note that we are still in the 5D basis where the cs are diagonal so that the fermions here are assumed to be of the same flavor (in this basis) and have the same c. In the physical KK basis (the 4D mass eigenbasis) we will rotate the fermions and this will lead to flavor-changing neutral currents. Writing out the overlap integral explicitly we get

$$\frac{g_5^{Zff}}{\sqrt{R \log R'/R}} \int_R^{R'} dz \left(\frac{R}{z}\right)^4 \left[\frac{1}{\sqrt{R}} \left(\frac{z}{R}\right)^2 \left(\frac{z}{R'}\right)^{-c} f_c\right]^2 \left[1 + \frac{M_Z}{4}(\cdots)\right] Z_\mu^{(0)} \overline{\chi}_c^{(0)} \overline{\sigma}^\mu \chi_c^{(0)}. \tag{8.13}$$

The $(\cdots)$ are the terms in the Z profile,

$$(\cdots) = 1 + \frac{M_Z}{4}\left(z^2 - 2z^2 \log\frac{z}{R}\right). \tag{8.14}$$

The leading term is the universal part of the Z profile while the $M_Z/4$ term has a non-trivial z-dependence and is the non-universal part. It is the latter that will contribute to flavor-changing when we rotate to the KK mass basis.

As a warm-up, let's remind ourselves of the universal part that gives us the relation of the Standard Model coupling to the 5D parameters. This is given by

$$g_{\text{SM}}^{Zff} = g_5^{Zff} \int_R^{R'} dz\, \frac{f_c^2 (R')^2 c}{R'\sqrt{R \log R'/R}} z^{-2c} \tag{8.15}$$

$$= \frac{g_5^{Zff} f_c^2}{\sqrt{R \log R'/R}} \left(1 - \left(\frac{R}{R'}\right)^{1-2c}\right) \frac{1}{1-2c} \tag{8.16}$$

$$= \frac{g_5^{Zff}}{\sqrt{R \log R'/R}}. \tag{8.17}$$

This is an important result to have handy.

Moving on let's consider the non-universal part. The non-universal coupling is

$$g_{\text{NU}}^{Zff} = \frac{g_5 f_c^2}{R'\sqrt{R \log R'/R}} \frac{M_Z}{4} \int_R^{R'} dz \left(\frac{z}{R'}\right)^{-2c} z^2 \left(1 - 2\log\frac{z}{R}\right). \tag{8.18}$$

Let's change variables to $y = z/R$ and perform the relevant integral... analytically! Yes, we're going to be calculus studs[19]. The integral is

$$\left(\frac{R}{R'}\right)^{-2c}\int_1^{R'/R} R dy\, y^{-2c}(Ry)^2(1-2\log y) = R^3\left(\frac{R}{R'}\right)^{-2c}\int_1^{R'/R} dy\,\left(y^{2-2c} - 2y^{2-2c}\log y\right).$$

The first term is easy:

$$\int_1^{R'/R} dy\, y^{2-2c} = \frac{1}{3-2c}\left[y^{3-2c}\right]_1^{R'/R}. \tag{8.19}$$

The second term can be computed using integration by parts. The trick is to use

$$\int dy\, y^a \log y = \frac{1}{a+1} y^{a+1}\log y - \int dy\, \frac{1}{a+1} y^a. \tag{8.20}$$

Thus

$$-2\int_1^{R'/R} dy\, y^{2-2c}\log y = -\frac{2}{3-2c}\left[y^{4-2c}\right]_1^{R'/R} + 2\int dy\,\frac{3-2c^{2-2c}}{y} \tag{8.21}$$

$$= -\frac{2}{3-2c}\left[y^{4-2c}\right]_1^{R'/R} + \frac{2}{(3-2c)^2}\left[y^{3-2c}\right]_1^{R'/R}. \tag{8.22}$$

Plugging everything in carefully, we get

$$\int_R^{R'} dz\,\left(\frac{z}{R'}\right)^{-2c} z^2\left(1-2\log\frac{z}{R}\right) = -R^3\left(\frac{R}{R'}\right)^{-2c}\frac{2}{3-2c}\left(\frac{R'}{R}\right)^{3-2c}\log\frac{R'}{R}$$
$$+ R^3\left(\frac{R}{R'}\right)^{-2c}\frac{5-2c}{(3-2c)^2}\left((R'/R)^{3-2c} - 1\right). \tag{8.23}$$

The second line is subleading in (R'/R) so that we can drop it. Though a slightly better approximation would include the $(R'/R)^{3-2c}$ term since this is only suppressed relative to the leading term by $\log R'/R \approx 37$. Simplifying everything, we get

$$g^{Zff}_{NU} = -\left(\frac{g_5^{Zff}}{\sqrt{R\log R'/R}}\right) f_c^2\frac{(M_Z R')^2}{2(3-2c)}\log\frac{R'}{R} \tag{8.24}$$

$$= -g^{Zff}_{SM} f_c^2 \frac{(M_Z R')^2}{2(3-2c)}\log\frac{R'}{R}. \tag{8.25}$$

Thus the full coupling of the Z to fermions, including both the universal and non-universal parts is

$$g^{Zff}_{\text{4D}} = g^{Zff}_{\text{SM}}\left(1 - \frac{(M_Z R')^2\log R'/R}{2(3-2c)} f_c^2\right) = g^{Zff}_{\text{SM}}(1-\Delta). \tag{8.26}$$

[19]We are completely aware that high school freshmen can perform these integrals in their sleep.

We see that in this c-basis the coupling is diagonal, but the Δ term is not proportional to the identity. As explained above, this means when we rotate to the KK mass basis we get off diagonal terms, i.e. flavor mixing. This is related to the so-called **RS-GIM mechanism**. In Appendix C we show that the rotation between flavors i and j go like f_i/f_j. Thus the effect of a flavor rotation (e.g. to the KK basis) is

$$g_{4\mathrm{D}}^{Z\mu e} = \left(U_L^\dagger g_{4\mathrm{D}}^{Zff} U_L\right)_{\mu e} \tag{8.27}$$

$$\approx \frac{f_e}{f_\mu}\frac{1}{2}\left(\frac{f_\mu^2}{3-2c_\mu} - \frac{f_3^2}{3-c_e}\right)(M_Z R')^2 \log\frac{R'}{R} g_{\mathrm{SM}}^{Zff} \tag{8.28}$$

$$\approx g_{\mathrm{SM}}^{Zff} \frac{(M_Z R')^2}{2(3-2c_\mu)\log\frac{R'}{R} f_\mu f_e} \equiv g_{\mathrm{SM}}^{Zff}\Delta_{\mu e}, \tag{8.29}$$

where we've used the fact that g_{SM}^{Zff} is flavor-independent and we've dropped the $f_e^2 \ll f_\mu^2$ term. If you're unhappy with the rotation here, consider a simple two dimensional case, the matrix

$$\begin{pmatrix} f_1 f_1 & f_1 f_2 \\ f_2 f_1 & f_2 f_2 \end{pmatrix} = f_1^2 \begin{pmatrix} 1 & \theta \\ \theta & \theta^2 \end{pmatrix}, \tag{8.30}$$

where $\theta = f_2/f_1$. We note that this matrix is diagonalized by

$$\begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}\begin{pmatrix} 1 & \theta \\ \theta & \theta^2 \end{pmatrix}\begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(\theta^2) \tag{8.31}$$

If we apply this same rotation to a diagonal matrix that is not proportional to the identity, we get

$$\begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \theta^2 \end{pmatrix}\begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix} = \begin{pmatrix} a & \theta(b-a) \\ \theta(b-a) & b \end{pmatrix} + \mathcal{O}(\theta^2). \tag{8.32}$$

Thus the off-diagonal term is indeed what we wrote above.

8.2 The effect of the Z'

The Z' is the first KK excitation of the Z boson, $Z' = Z^{(1)}$. To be more precise, boundary terms (e.g. EWSB) induce mixing between the KK modes, but to good approximation we can ignore this mixing.

As before, we need the wavefunction of the Z' so that we can calculate the effective 4D coupling (via the overlap integral). We can start again with our general solution, Eq. (8.3). This gives

$$h^{(1)}(z) \propto z\left[Y_0(M_{\mathrm{KK}}R)J_1(M_{\mathrm{KK}}z) - J_0(M_{\mathrm{KK}}R)Y_1(M_{\mathrm{KK}}z)\right]. \tag{8.33}$$

Ugly! We would desperately like to simplify this. We can't really Taylor expand the way we did for the zero mode gauge field because the argument is now no longer manifestly small. More than that, we are discouraged because it looks like we have to do some work to determine the KK mass. Fortunately, there's still some slick moves we can make to avoid any real heavy lifting.

Because we're considering the KK excitation of the Z, we know that this is a gauge field whose boundary conditions admit a zero mode. This means that the field has Neumann boundary conditions. You can check from the derivative formulae for Bessel functions that the boundary condition at $z = R'$ is

$$Y_0(M_{\rm KK}R)J_0(M_{\rm KK}R') = J_0(M_{\rm KK}R)Y_0(M_{\rm KK}R'), \tag{8.34}$$

where we remember that $M_{\rm KK} \sim 1/R'$ and $R \lll R'$ so that $M_{\rm KK}R \approx 0$. Now we only need to remember some properties of J_0 and Y_0:

1. $J_0(0) = 1$ and $J_0(x > 0)$ is under control, i.e. $|J_0(x > 0)| < 1$.
2. $Y_0(0) = -\infty$ and $Y_0(x > y_1)$ is under control where y_1 is the first zero, $Y_0(y_1) = 0$.

[**Work:** This would be a good place to have a sketch of these functions.] What does this tell us? On the left-hand side of Eq. (8.34), $Y_0(M_{\rm KK}R)$ is *very* large (and negative) while on the right-hand side both terms are $\mathcal{O}(1)$ or less. This means that $J_0(M_{\rm KK}R') \approx 0$, in other words,

$$(M_{\rm KK}R') = x_1 \approx 2.405, \tag{8.35}$$

where x_1 is the first zero of J_0. We now have a handy formula for the mass of the KK gauge boson as a function of the radius of compactification. We hardly had to work for it, too. In fact, this trick works for all KK modes. The spacing for the KK gauge boson tower is determined by the spacing of the zeroes of J_0. As one goes to larger KK number, this spacing becomes regular since J_0 becomes more sinusoidal. This is exactly what one would expect since at large KK number we are probing higher energies which become increasingly insensitive to the AdS curvature when $M_{\rm KK}^{(n)} \gg 1/R$.

Let's go back to the 5D wavefunction of the first KK mode. We now know that the coefficient of the $J_1(M_{\rm KK}z)$ term is much larger than that of the $Y_1(M_{\rm KK}z)$ term, while the functions themselves are well behaved[20] (i.e. $\mathcal{O}(1)$). Thus we can approximate

$$h^{(1)}(z) \propto zJ_1(M_{\rm kk}z). \tag{8.36}$$

All that's left to check is the normalization. The procedure follows as before, just canonically normalize the 4D field. The result is (as you can check[21])

$$h^{(1)}(z) = \sqrt{\frac{2}{R}}\frac{z}{J_1(x_1)R'}J_1(x_1 z/R'). \tag{8.37}$$

We can now proceed to do the analogous overlap integrals to determine the 4D coupling. The relevant integral is

$$g_5^{Zff}\int_R^{R'} dz\,\left(\frac{R}{z}\right)^4\left[\frac{1}{\sqrt{R'}}\left(\frac{z}{R}\right)^2\left(\frac{z}{R'}\right)^{-c}f_c\right]^2\sqrt{\frac{2}{R}}\frac{z}{J_1(x_1)R'}J_1\left(x_1\frac{z}{R'}\right).$$

[20] Okay, the Y_ns also diverge at the origin, but Y_0 has the strongest divergence so that $Y_0(M_{\rm KK}R)J_1(M_{\rm KK}z)$ will always be larger than in magnitude than $J_0(M_{\rm KK}R)Y_1(M_{\rm KK}z)$ over the RS domain.

[21] Plugging into *Mathematica*:

$$\int_0^1 dx\,\frac{1}{x}\left[\frac{\sqrt{2}x}{J_1(x_1)}J_1(x_1x)\right]^2 = 1 - \frac{J_0(x_1)J_2(x_1)}{J_1(x_1)^2} = 1.$$

Simplifying and performing a handy change of variable we get

$$g_5^{Zff} f_c^2 \sqrt{\frac{1}{R}} \frac{\sqrt{2}}{J_1(x_1)} \int_0^1 dx\, x^{1-2x} J_1(x_1 x).$$

For simplicity, let's define the "flavor function"

$$\gamma_c \equiv \frac{\sqrt{2}}{J_1(x_1)} \int_0^1 dx\, x^{1-2x} J_1(x_1 x) \approx \frac{\sqrt{2}}{J_1(x_1)} \frac{0.7}{2(3-2c)} \left(1 + e^{c/2}\right) \approx \frac{\sqrt{2}}{J_1(x_1)} \frac{0.7 x_1}{2(3-2c)}. \tag{8.38}$$

Typically one can further simplify $2(3-2c) \approx 4$, but for 'real' calculations one should just perform the integral. Finally, then, we get the coupling

$$g_{4D}^{Z'e\mu} = g_5^{Zff} \frac{1}{\sqrt{R}} f_e f_\mu \gamma_c, \tag{8.39}$$

where we've taken the liberty to rotate into the KK basis as we did above. In terms of the usual Standard Model coupling, this is

$$g_{4D}^{Z'e\mu} = g_{\mathrm{SM}}^{Zff} \sqrt{\log \frac{R'}{R}} f_e f_\mu \gamma_c. \tag{8.40}$$

8.3 Matching to the Effective Lagrangian

As a reminder, recall the Z-mediated 4-Fermi effective Lagrangian in the Standard Model (see, e.g. Peskin p. 709 [25]),

$$\Delta\mathcal{L} = \frac{g^2}{2M_Z^2} J_Z^\mu J_{Z\mu} = \frac{4G_F}{\sqrt{2}} \left(\sum_f \overline{f} \gamma \left(T^3 - s_W^2 Q\right) f \right). \tag{8.41}$$

Recall that

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} = \frac{g^2}{8c_W^2 M_Z^2}. \tag{8.42}$$

To set our notation, let us write out the Z coupling to the neutral leptonic current,

$$\begin{aligned} \Delta\mathcal{L} &= g Z_\mu J_Z^\mu \qquad (8.43)\\ &= \frac{g}{c_W} Z_\mu \left[\overline{\ell}_L \gamma^\mu \left(-\frac{1}{2} + s_W^2 \right) \ell_L + \overline{\ell}_R \gamma^\mu \left(s_W^2\right) \ell_R \right] \qquad (8.44)\\ &= \frac{g}{c_W} Z_\mu \left[g_L \overline{\ell}_L \gamma^\mu \ell_L + g_R \overline{\ell}_R \gamma^\mu \ell_R \right]. \qquad (8.45) \end{aligned}$$

Great. Now let's consider the effect of the non-universal couplings of the Z boson. We'll ignore the KK Z since it will be a small $\mathcal{O}(10\%)$ correction. Don't worry, we didn't go through all that work for nothing: we'll use that later.

The whole point is that we'd like to calculate the tree-level diagrams contributing to $\mu \to (3)e$ and match those coefficients to those of the effective Lagrangian in Eq. (8.2). Let's consider the g_3

term. I apologize that there are so many gs floating around, maybe you should pause a moment to make sure you've got them all sorted out. The relevant tree-level diagram gives us

$$\mathcal{M} = g_{4\text{D}}^{Ze_R\mu_R} \frac{1}{M_Z^2} g_{4\text{D}}^{Ze_Re_R} \left(\bar{e}_R\gamma\mu_R\right)\left(\bar{e}_R\gamma e_R\right). \tag{8.46}$$

We now know from our analysis of the non-universal coupling of the Z that

$$g_{4\text{D}}^{Ze_R\mu_R} = g_{\text{SM}}^{Ze_Re_R}\Delta_{\mu e} = g_{\text{SM}}^{Ze_Re_R}\frac{(M_ZR')^2}{2(3-2c_\mu)}\log\frac{R'}{R}f_\mu f_e. \tag{8.47}$$

Matching to the effective Lagrangian gives

$$\frac{4}{\sqrt{2}}G_F g_3 = \left(g_{\text{SM}}^{Ze_Re_R}\right)^2\frac{1}{M_Z^2}\Delta_{e\mu}. \tag{8.48}$$

Plugging in Eqs. (8.42) and (8.45), we get

$$\frac{g^2}{2c_W^2M_Z^2}g_3 = \left(\frac{g}{c_W}g_R\right)^2\frac{1}{M_Z^2}\Delta_{e\mu}. \tag{8.49}$$

Again sorry about all the different gs floating around. As a quick sanity check, g is the Standard Model $SU(2)_L$ coupling, g_3 is an effective coupling that we're solving for, g_R are the quantum numbers with which the right-handed leptons couple to the Z, and $g_{\text{SM}}^{Zf_Rf_R} = g_R(g/c_W)$ is the full coupling of the Z to the right-handed leptons: i.e. $\Delta L = gZ_\mu\bar{e}_R\gamma^\mu e_R$. This gives us

$$g_3 = 2g_R^2\Delta_{e\mu}. \tag{8.50}$$

Completely analogous calculations give

$$g_4 = 2g_L^2\Delta_{e\mu} \tag{8.51}$$

$$g_{5,6} = 2g_Lg_R\Delta_{e\mu}. \tag{8.52}$$

Now we have everything we need to crunch numbers. It's advisable to do this in a computer algebra system like *Mathematica* since it's easy to modify calculations after you find mistakes. We note that it's convenient to approximate $f_{\ell_R} = f_{\ell_L}$ so that

$$f_\ell = \sqrt{\frac{\lambda_\ell}{Y_*}} \approx \sqrt{\frac{m_\ell}{Y_*m_t}}, \tag{8.53}$$

which makes it easier to plug in actual numbers.

Now let's quickly go over the effective Lagrangian for $\mu \to e$ conversion. The effective Lagrangian is written in this strange way,

$$\Delta\mathcal{L} = \frac{G_F}{\sqrt{2}}\bar{e}\gamma(v - a\gamma^5)\mu\cdot[\text{hadronic}], \tag{8.54}$$

where I've written [hadronic] to mean stuff that I don't care about, but noting that I've pulled out the factor of g/c_W from the hadronic vertex in order to form the G_F. One more note: whomever wrote down this effective Lagrangian first (probably Feinberg and Weinberg [23]) used a stupid normalization of their quantum numbers. In particular, they wrote $g_{L,R}^{\text{stupid}} = T_3 + 2Qs_W^2$ so that $|T_3| = 1$ and so $g_{L,R}^{\text{stupid}} = 2g_{L,R}$.

Our first task is to write the effective Lagrangian in terms of $P_{L,R} = \frac{1}{2}(1 \mp \gamma^5)$, so that $\Delta\mathcal{L} \sim A\bar{e}\gamma P_L e + B\bar{e}\gamma P_R e$.

$$v - a\gamma^5 = \frac{1}{2}A(1-\gamma^5) + \frac{1}{2}B(1+\gamma^5), \tag{8.55}$$

so that

$$A = v + a \tag{8.56}$$
$$B = v - a. \tag{8.57}$$

Good, now we can match these coefficients to the tree-level diagram,

$$\frac{G_F}{\sqrt{2}}(v \pm a) = g_{L,R}\frac{1}{M_Z^2}\Delta_{e\mu}\left(\frac{g}{c_W^2}\right)^2\frac{1}{2}, \tag{8.58}$$

where the extra factor of 1/2 comes from the stupid normalization. We didn't bother matching the hadronic parts because they cancel on both sides. We end up with

$$v \pm a = 4g_{L,R}\Delta_{e\mu}. \tag{8.59}$$

9 Custodial protection of $\mu \to (3)e$

An novel feature that was not discussed by Agashe, Blechmann, and Petrillo [26] is the use of custodial symmetry to protect the $\mu \to (3)e$ amplitude. The use of custodial symmetry to suppress anomalous contributions to the Zbb coupling was first proposed by Agashe *et al.* [27] and was applied to heavy quark FCNCs in the Randall-Sundrum model by the Munich flavor physics factory [28] (and see references therein). This can also be used analogously to protect the $Z\ell\ell'$ couplings that mediate $\mu \to (3)e$.

Bulk, gauged custodial symmetry is typically imposed on the RS model with bulk fields in order to protect the T-parameter, which would otherwise tightly constrain the KK scale. The custodial symmetry is broken on the UV brane, 'far away' from electroweak symmetry breaking on the IR brane. The cute realization was that this symmetry also automatically protects against some of the tree-level FCNCs.

The basic idea is that if a Standard Model field (e.g. the b) and any new physics operators that it couples to ($\mathcal{O}_{\text{BSM}}$) respect the custodial $O(3) = SU(2)_V \times P_{LR}$ symmetry (where P_{LR} is a discrete symmetry), then the Zbb coupling must be protected. This is because the Z coupling takes the form

$$\frac{g}{\cos\theta_W}\left[Q_L^3 - Q_{\text{EM}}\sin^2\theta_W\right]Z\overline{\Psi}\gamma\Psi. \tag{9.1}$$

The electric charge $Q_{\rm EM}$ is conserved so that this cannot be affected by new physics. Any contribution to new physics would then modify the $SU(2)_L$ diagonal charge Q^3_L. By the $SU(2)_V$, symmetry, however,

$$Q_V = Q^3_L + Q^3_R \tag{9.2}$$

is protected and so the modification from new physics must satisfy $\delta Q^3_L = -\delta Q^3_R$. On the other hand, the $P_{LR} : SU(2)_L \leftrightarrow SU(2)_R$ symmetry enforces $Q_L = Q_R$ and hence the Q_L cannot be affected by new physics.

This only holds when the Standard Model field respects the P_{LR} symmetry, which depends on the embedding of the field into $SU(2)_L \times SU(2)_R \times U(1)_X$. In particular, in the lepton sector we have fields $L = (\mathbf{2}, \mathbf{2})_0$, $N = (1, 1)_0$, $E_r = (1, \mathbf{3})_0$, $E_\ell = (\mathbf{3}, 1)_0$. The right-handed electron lives in the representation $E = E_r \times E_\ell$. These are rather large representations with lots of extra fields. We may decouple these fields from the low-energy spectrum by imposing appropriate boundary conditions[22]. One can see that the left-handed electron $\ell \in L$ is a P_{LR} eigenstate and so should be protected by the above argument.

[**Should still write up:** I should write more details about the custodial protection in RS, but most of the details are worked out nicely in the Munich paper, [28].]

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A Conventions

General (e.g. 5D) spacetime indices are written with capital Roman letters from the middle of the alphabet, $M, N, \cdots$. 4D Minkowski indices are written with lower-case Greek letters from the middle of the alphabet, $\mu, \nu, \cdots$. Tangent space indices are written in Roman letters from the beginning of the alphabet, $a, b, \cdots$. We will occasionally write x to mean a generic coordinate on the spacetime, e.g. $F(x) = F(x^\mu, z)$ in our warped 5D compactification. We use the particle physics (West Coast, mostly-minus) metric for Minkowski space, $ds^2 = (+, -, -, -, (-))$. Flavor indices are denoted by i, j, k, ℓ and Kaluza-Klein number by $(a), (b), \cdots$. Occasionally we will have to 'overload' these conventions to label other quantities, but we hope that these exceptional cases should be easily identified based on context.

[22] The boundary conditions in these models appear, at first glance, rather arbitrary, with fields within a given multiplet having different BCs. The point, however, is that these can all be obtained 'naturally' from brane-localized Higgs fields charged under the $SU(2)_R \times U(1)_X$ which get 'infinite' vevs. This breaks the $SU(2)_R \times U(1)_X$ symmetry and allows the component fields of $SU(2)_R$ multiplets to get different boundary conditions

We choose conformally flat coordinates for the Randall-Sundum background,

$$ds^2 = \left(\frac{R}{z}\right)^2 \left(dx^\mu dx^\nu \eta_{\mu\nu} - dz^2\right),$$

where R is the AdS radius of curvature and with a UV brane at $z = R$ and an IR brane at $z = R'$ such that $R^{-1} \sim M_{\text{Pl}}$ and $R'^{-1} \sim$ TeV. These coordinates are related to the exponential formulation of the original RS paper by

$$z = Re^{ky} \qquad\qquad k = 1/R.$$

Fermions are bulk Dirac spinors $\Psi(x,z)$ and are decomposed into left- and right-handed Weyl spinors via,

$$\Psi(x,z) = \begin{pmatrix} \chi(x,z) \\ \overline{\psi}(x,z) \end{pmatrix}, \tag{A.1}$$

subject to orbifold boundary conditions

$$\psi_L(x^\mu, R) = \psi_L(x^\mu, R') = 0 \qquad\qquad \chi_R(x^\mu, R) = \chi_R(x^\mu, R') = 0, \tag{A.2}$$

where the L and R refer to $SU(2)_L$ doublets and singlets respectively. This imposes that the zero modes of the 5D spinors are chiral. Fermion bulk masses are given by c/R where c is a dimensionless parameter controlling the localization of the 5D profile. Our gamma matrices are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \qquad\qquad \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{A.3}$$

where σ^μ are the usual Pauli matrices and $\overline{\sigma}^\mu \equiv (\sigma^0, -\sigma^i)$. This normalization of γ^5 is necessary to satisfy the Clifford algebra but differs from that used in, e.g. Peskin and Schroeder [25]. We use the standard convention $\sigma^0 = \mathbb{1}$, which has the opposite sign as Csáki et al., [2] and [13]. We shall use the usual Feynman slash notation only for the 4D Minkowski directions $\not{p} = p_\mu\gamma^\mu$ so that $p_M\gamma^M = \not{p} + p_5\gamma^5$.

The 5D Yukawa matrices (3×3 parameters in the Lagrangian) are written as Y or Y_5. The average anarchic value of this matrix is written as Y_*. The 4D zero-Yukawa matrix (to be identified with the Standard Model Yukawa) is written

$$y_{ij} = f_{cL_i} Y_{ij} f_{-cR_j} = \lambda_{ij}^{SM}. \tag{A.4}$$

Bulk 5D parameters such as gauge couplings are written with a subscript 5, e.g. e_5, and are related to the 4D couplings via $e_5 = Re$. We perform all calculations in the unitary gauge where the Goldstone bosons decouple. Further we perform our 5D calculation in the basis where the kinetic terms are diagonal (i.e. the 'c-basis') and all flavor-changing effects come from the Yukawa interaction, including mass insertions. This is analogous to performing Standard Model calculations without diagonalizing the Yukawas, i.e. without the CKM rotation.

For shorthand we define the barred differential operator, $\bar{d} = d/2\pi$.

B Flat XD fermion propagator

Here we derive the chiral fermion propagator in a flat interval extra dimension as a model calculation using the methods described in this note. The set up for our simplified model is as follows. We begin with a flat extra dimensional interval with coordinate $z \in [0, L]$ where one may take $L = \pi R$ to match with standard orbifold conventions.

A complete set of propagators for flat 5D intervals was derived by Puchwein and Kunzst [16] using finite temperature field theory techniques. Our derivation directly calculates Green's functions using simpler methods. The propagator from a fixed point x' to a given point x is given by the two point Green's function of the 5D Dirac operator,

$$\mathcal{D}\,\Delta(x,x') \equiv i\gamma^M\partial_M + m = i\delta^{(5)}(x-x'). \tag{B.1}$$

In mixed position-momentum space where the noncompact dimensions dimensions are treated in momentum space while the finite dimension is treated in position space the Green's function equation is given by

$$\left(-\not{p} + i\partial_5\gamma^5 - m\right)\Delta(p,z,z') = i\delta(z-z'). \tag{B.2}$$

This is a first-order differential equation with non-trivial Dirac structure. As a trick to solve this equation we will 'square' this operator into one that is second-order and diagonal on the space of Weyl spinors. We define a pseudo-conjugate[23] Dirac operator,

$$\mathcal{D}^* = -i\gamma^M\partial_M + m. \tag{B.3}$$

With this operator one can square the Dirac equation into the usual 5D Klein-Gordon equation,

$$\mathcal{D}\mathcal{D}^* = \begin{pmatrix} \partial^2 - \partial_5^2 + m^2 & \\ & \partial^2 - \partial_5^2 + m^2 \end{pmatrix}, \tag{B.4}$$

where we've explicitly written out the diagonal chiral structure on Weyl spinors. Each element in the Klein-Gordon operator is a 2×2 matrix acting on a Weyl spinor. We may now look for Green's functions $F(p,z,z')$ for this $\mathcal{D}\mathcal{D}^*$ operator in mixed position-momentum space,

$$\mathcal{D}\mathcal{D}^* F(p,z,z') = \begin{pmatrix} -p^2 - \partial_5^2 + m^2 & \\ & -p^2 - \partial_5^2 + m^2 \end{pmatrix}\begin{pmatrix} F_+ & \\ & F_- \end{pmatrix} = i\delta(z-z'). \tag{B.5}$$

We now see that solving this simpler equation allows us to trivially construct a solution for the Dirac Green's function which satisfies Eq. B.1,

$$\Delta(p,z,z') \equiv \mathcal{D}^* F(p,z,z') = \begin{pmatrix} (\partial_5 + m)\,F_+ & \sigma^\mu p_\mu F_- \\ \overline{\sigma}^\mu p_\mu F_+ & (-\partial_5 + m)\,F_- \end{pmatrix}. \tag{B.6}$$

We solve this by separating $F_\pm(z)$ into pieces

$$F_\pm(p,z,z') = \begin{cases} F_\pm^<(p,z,z') & \text{if } z < z' \\ F_\pm^>(p,z,z') & \text{if } z > z' \end{cases} \tag{B.7}$$

[23] We call this a 'pseudo-conjugate' because this is neither a complex nor Hermitian conjugate but an operator where only explicit factors i are conjugated.

and then solving the homogeneous Klein-Gordon equations for each $F^<$ and $F^>$. The general solution is

$$F_\pm^{<,>}(p,z,z') = A_\pm^{<,>}\cos(\chi_p z) + B_\pm^{<,>}\sin(\chi_p z), \tag{B.8}$$

where the eight coefficients $A_\pm^{<,>}$ and $B_\pm^{<,>}$ are determined by the boundary conditions at $0, L$ and z'. The factor χ_p is the magnitude of the 5-momentum in the z-direction defined by

$$p_M p^M = m^2 = p^2 - \chi_p^2. \tag{B.9}$$

We impose matching boundary conditions at $z = z'$. By integrating the Green's function equation B.5 over a sliver $z = [z' - \epsilon, z' + \epsilon]$ we obtain the conditions

$$\partial_5 F_\pm^>(z') - \partial_5 F_\pm^<(z') = i, \tag{B.10}$$

$$F_\pm^>(z') - F_\pm^<(z') = 0. \tag{B.11}$$

These are a total of four equations. The remaining four equations on the branes at $z = 0, L$ impose the chirality of the fermion zero mode and are the equivalent of treating the interval as an orbifold. We will denote the left-chiral zero mode states with a superscript L and the right-chiral zero mode states with a superscript R. We impose that the Green's function vanishes if a wrong-chirality state propagates to either brane,

$$P_R\,\Delta^L(p,z,z')\big|_{z=0,L} = P_R\mathcal{D}^*\,F^L(p,z,z')\big|_{z=0,L} = 0, \tag{B.12}$$

$$P_L\,\Delta^R(p,z,z')\big|_{z=0,L} = P_L\mathcal{D}^*\,F^R(p,z,z')\big|_{z=0,L} = 0, \tag{B.13}$$

where $P_{L,R} = \frac{1}{2}(1 \mp i\gamma^5)$ are the usual projection operators. Note from Eq. B.6 that each of these equations is actually a set of two boundary conditions on each brane. For example, the left-handed boundary conditions may be written explicitly as

$$F_+^L(p,z,z')\big|_{z=0,L} = 0, \tag{B.14}$$

$$\partial_5 F_-^R(p,z,z')\big|_{z=0,L} = 0, \tag{B.15}$$

where we've used that p_μ is arbitrary and $m = 0$. Note that Csakí et al. [13] emphasized that only one boundary condition for a Dirac fermion needs to be imposed in order not to overconstrain the first-order Dirac equation since the bulk equations of motion converted boundary conditions for χ into boundary conditions for ψ. In this case, however, we are dealing with a *second*-order Klein-Gordon equation that does not mix χ and ψ. Thus the appearance and necessity of two boundary conditions per brane for a chiral fermion is not surprising, we are only converting the single boundary condition on $\Delta(p,z,z')$ into two boundary conditions for $F(p,z,z')$.

Solving for the coefficients $A_\pm^{<,>}(p,z)$ and $B_\pm^{<,>}(p,z)$ for each type of fermion (left- or right-chiral zero modes) one finds the results of table 1. Using trigonometric identities one may combine the $z < z'$ and $z > z'$ results to obtain

$$F_\pm^X = \frac{-i\cos\chi_p\,(L - |z - z'|) + \gamma^5\wp(X)\cos\chi_p\,(L - (z + z'))}{2\chi_p\sin\chi_p L}, \tag{B.16}$$

where $X = \{L, R\}$ with $\wp(L) = +1$ and $\wp(R) = -1$. The fermion Green's function can then be obtained trivially from Eq. B.6.

$A_+^{L<} = \mathrm{s}_p(L-z')\mathrm{s}_pL$	$A_+^{L>} = \mathrm{s}_pz'\mathrm{c}_pL$	$A_+^{R<} = 0$	$A_+^{R>} = -\mathrm{c}_pz'\mathrm{s}_pL$
$B_+^{L<} = 0$	$B_+^{L>} = \mathrm{s}_pz'\mathrm{s}_pL$	$B_+^{R<} = -\mathrm{c}_p(L-z')$	$B_+^{R>} = -\mathrm{c}_pz'\mathrm{c}_pL$
$A_-^{L<} = 0$	$A_-^{L>} = -\mathrm{c}_pz'\mathrm{s}_pL$	$A_-^{R<} = \mathrm{s}_p(L-z')$	$A_-^{R>} = -\mathrm{s}_pz'\mathrm{c}_pL$
$B_-^{L<} = \mathrm{c}_p(L-z')$	$B_-^{L>} = -\mathrm{c}_pz'\mathrm{c}_pL$	$B_-^{R<} = 0$	$B_-^{R>} = \mathrm{s}_pz'\mathrm{s}_pL$

Table 1: Flat case coefficients in (B.8) upon solving with the boundary conditions (B.10-B.13). We have used the notation $\mathrm{c}_px = \cos\chi_px$ and $\mathrm{s}_px = \sin\chi_px$.

C Diagonalization of Anarchic Yukawas

In this appendix we make some notes about the order of magnitude of the off-diagonal elements of the rotation matrix which diagonalizes the zero mode (SM) effective Yukawa coming from an anarchic 5D Yukawa matrix. In the c-basis, i.e. the basis where the bulk masses are diagonalized, the zero-mode Yukawas look like

$$\begin{pmatrix} f_1c_{11}f_1 & f_1c_{12}f_2 & f_1c_{13}f_3 \\ f_2c_{21}f_2 & f_1c_{22}f_2 & f_2c_{23}f_3 \\ f_3c_{31}f_3 & f_1c_{32}f_2 & f_3c_{33}f_3 \end{pmatrix} \tag{C.1}$$

where all of the c_{ij}s are $\mathcal{O}(1)$ (i.e. we factor out a Y_*). The fs, which represent the fermion wavefunctions on the brane, generate the observed mass hierarchies. To make this manifest, let us define

$$\delta_1^2 = f_1/f_3 \qquad\qquad \delta_2 = f_2/f_3 \tag{C.2}$$

so that $\delta_1 \sim \delta_2 \sim \delta \ll 1$. We can then write the SM Yukawa as

$$f_3^2 \begin{pmatrix} \delta_1^4c_{11} & \delta_1^2\delta_2c_{12} & \delta_1^2c_{13} \\ \delta_1^2\delta_2c_{21} & \delta_2^2c_{22} & \delta_2c_{23} \\ \delta_1^2c_{31} & \delta_2c_{32} & c_{33} \end{pmatrix}. \tag{C.3}$$

First we claim that the eigenvalues are given by $\mathcal{O}(f_1^2, f_2^2, f_3^2)$, i.e. that upon diagonalization we indeed get a realistic hierarchy assuming generic non-hierarchical c_{ij}s. This is important because we then know that the rotation matrix will have terms proportional to the δs on their off-diagonal elements.

A cute way to prove this claim is to use perturbation theory in the hierarchies of the δs. The eigenvalues are given by solutions to

$$\det(\lambda - \lambda_i) = (1-\lambda_i)(\delta_2^2 - \lambda_i)(\delta_1^4 - \lambda_i) + \mathcal{O}(\delta_1^4\delta_2^2) = 0. \tag{C.4}$$

Consider the largest eigenvalue, λ_3. We may write

$$(1-\lambda_3) = \frac{\mathcal{O}(\delta^6)}{(\delta_2^2-\lambda_3)(\delta_1^4-\lambda_3)}. \tag{C.5}$$

This is generically solved by $\lambda_3 \sim \mathcal{O}(1)$. As a sanity check, we would have expected that λ_3 be on the order of the largest element in such a hierarchical matrix so that the right-hand side is $\mathcal{O}(\delta^6) \ll 1$ since the denominator should be $\mathcal{O}(1)$. One can repeat this argument for the other λs to show that they are respectively of order δ^2 and δ^4.

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