Supersymmetry and Extra Dimensions

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A pedagogical set of notes based on lectures by Fernando Quevedo, Adrian Signer, and Csaba Csáki, as well as various books and reviews.

Last updated: January 9, 2009

Abstract

This is a set of combined lecture notes on supersymmetry and extra dimensions based on various lectures, textbooks, and review articles. The core of these notes come from Professor Fernando Quevedo's 2006-2007 Lent Part III lecture course of the same name [1].

Acknowledgements

Inspiration to write up these notes in LaTeX came from Steffen Gielen's excellent notes from the Part III Advanced Quantum Field Theory course and the 2008 ICTP Introductory School on the Gauge Gravity Correspondence. Notes from Professor Quevedo's 2005-2006 Part III Supersymmetry and Extra Dimensions course exist in TeXform due to Oliver Schlotterer. The present set of notes were written up independently, but simliarities are unavoidable. It is my hope that these notes will provide a broader pedagogical introduction supersymmetry and extra dimensions.

Preface

These are lecture notes. Version 1 of these notes are based on Fernando Quevedo's lecture notes and structure. I've also incorporated some relevant topics from my research that I think are important to round-out the course. Version 2 of these notes will also incorporate Csaba Csáki's Advanced Particle Physics notes.

Framed text. Throughout these notes framed text will include parenthetical discussions that may be omitted on a first reading. They are meant to provide a broader picture or highlight particular applications that are not central to the main purpose of the chapter.

The wise men of physics leave behind notes and lectures that are able to convey insight with fantastic economy. In contrast, for the oaf who put together these notes (but certainly not the teachers from whom he learned the subject), subtlety would come at the cost of clarity. Thus I apologize in advance for erring on the side of loquaciousness in an attempt to overcome my own tentative grasp of the subjects henceforth.

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"The naming of sparticles is a difficult thought
It isn't just one of your grad student games
You may think at first I'm mad as a crackpot
When I tell you, a sparticle has three different names.

First of all, there's the name we physicists use daily
Such as stop, selectron, photino (twiddle A)
Such as higgsino, chargino, sdown, or the LSP,
Each of them a sensible physicsy name.

There are fancier names if you think they sound neato,
Some are quite playful, some are quite lame:
Such as CP-odd Higgs, sneutrino, stau, gravitino
But all of them sensible physicsy names

But I tell you, a field needs a name that's particular
A name that's peculiar, and more dignified,
Else how can it make its gauge representation much clearer
than to write out its indices, dotting the i's

Of the names of this kind, I can give you a lot,
Such as H-up-j, B-nu, or q-LH-i,
Such as g-alpha-sigma, or else twiddle-chi-nought
Names that would make many-an-undergrad cry.

But above and beyond there's still one name left over,
The name that would make even your adviser impressed,
The name that no physics research can discover -
But the sparticle itself knows, and will never confess.

When you detect a field in profound propagation,
There's only one thing to do that's worth mention,
Time-ordered product, two-point correlation;
And compute, and compute, and compute the cross section.

That symmetrically super, supersymmetric,
Deep inelastic nonsingular cross section."
— The Naming of Sparticles (Apologies to TS Eliot)

Chapter 1

Introduction and History

"Supersymmetry is nearly thirty years old. It seems that now we are approaching the fourth supersymmetry revolution which will demonstrate its relevance to nature."

— G.L. Kane and M. Shifman [2]

Here we go over the basics.

Why SUSY and XD? Both extensions to the SM that evade Coleman-Mandula. Also they both come together in dualities, e.g. AdS/CFT. Though we won't get to the AdS or the CFT sides, we hope to present enough foundational material for SUSY and XD.

1.1 Prerequisite knowledge

1.2 Heuristic motivation

1.3 Experimental prospects

1.4 Theoretical prospects

1.5 The plan

We'll start with SUSY then do XD. If there's time I'd like to add on some technicolor and little Higgs stuff later as well.

I should include a broad picture of the program. SUSY requires that we establish some mathematical machinery before hand, so we'll start with that. We will first develop the SUSY algebra as an extension of the Poincare group. Then we will find representations for this algebra and introduce the superfield notation. Then we'll do real stuff.

*** I should say something about the general path. The first few chapters will seem to be rather abstract and won't have much connection to the model building that one might be used to from QFT or SM courses. But these build the necessary formalism to do SUSY.

Chapter 2

The Poincaré Algebra and its Representations

"I explained the fermion work to my colleague Don Weingarten, and I remember his answer for he said I was 'set for life'!"
— P. Ramond [2]

We will see in subsequent lectures that supersymmetry is inherently connected to the symmetries of spacetime. Here we briefly review the Poincaré group and its spinor representations. See Appendix D for a more detailed treatment of the Poincaré group.

2.1 Poincaré Symmetry and Spinors

The **Poincaré group** is given by transformations of Minkowski space of the form

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{2.1}$$

Here a^μ parameterizes translations and $\Lambda^\mu{}_\nu$ parameterizes transformations of the Lorentz group containing rotations and boosts. These latter matrices satisfy the relation

$$\mathbf{\Lambda^T} \eta \mathbf{\Lambda} = \eta, \tag{2.2}$$

where $\eta = \text{diag}(+,-,-,-)$ is the usual Minkowski metric used by particle physicists. Recall that the Poincaré group has four disconnected parts. We specialize to the subgroup $SO(3,1)^{\uparrow}$, i.e. the **orthochronous Lorentz group**, $SO(3,1)^{\uparrow}$ which further satisfies the constraints

$$\det \mathbf{\Lambda} = +1 \tag{2.3}$$

$$\Lambda^0_0 \geq 1. \tag{2.4}$$

This is the part of the Lorentz group that is connected to the identity. Other parts of the Lorentz group can be obtained from $SO(3,1)^{\uparrow}$ by applying the transformations

$$\Lambda_P = \text{diag}(+,-,-,-) \tag{2.5}$$

$$\Lambda_T = \text{diag}(-,+,+,+). \tag{2.6}$$

Here Λ_P and Λ_T respectively refer to parity and time-reversal transformations. It is worth noting that the fact that the Lorentz group is not simply connected is related to the existence of a 'physical' spinor representation, as we will mention below.

2.2 Properties of the Poincaré Group

Let's review a few important properties of the Poincaré group.

2.2.1 Algebra of the Poincaré Group

Locally the Poincaré group is represented by the algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}) \tag{2.7}$$

$$[P^{\mu}, P^{\nu}] = 0 \tag{2.8}$$

$$[M^{\mu\nu}, P^{\sigma}] = i(P^{\mu}\eta^{\nu\sigma} - P^{\nu}\eta^{\mu\sigma}). \tag{2.9}$$

The $\mathbf{M}$ are the antisymmetric generators of the Lorentz group,

$$(M^{\mu\nu})_{\rho\sigma} = i(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}), \tag{2.10}$$

and the **P** are the generators of translations. As a 'sanity check,' one should be able to recognize in equation (2.7) the usual Euclidean symmetry $O(3)$ by taking $\mu, \nu, \rho, \sigma \in \{1, 2, 3\}$ and noting that at most only one term on the right-hand side survives. equation (2.8) says that translations commute, while equation (2.9) says that the generators of translations transform as vectors under the Lorentz group. This is, of course, expected since the generators of translations are precisely the four-momenta. The factors of i should also be clear since we're taking the generators **P** and **M** to be Hermitian.

The 'translation' part of the Poincaré algebra is generally boring. It is the Lorentz algebra that yields the interesting features of our fields under Poincaré transformations.

2.2.2 The Lorentz Group is related to $SU(2) \times SU(2)$

Locally the Lorentz group is related to the group $SU(2) \times SU(2)$, i.e. one might suggestively write

$$SO(3, 1) \approx SU(2) \times SU(2). \tag{2.11}$$

Let's flesh this out a bit. One can explicitly separate the Lorentz generators $M^{\mu\nu}$ into the generators of rotations, J_i, and boosts, K_i:

$$J_i = \frac{1}{2} \epsilon_{ijk} M_{jk} \tag{2.12}$$

$$K_i = M_{0i}, \tag{2.13}$$

where ϵ_{ijk} is the usual antisymmetric Levi-Civita tensor. We can now define 'nice' combinations of these two sets of generators,

$$A_i = \frac{1}{2}(J_i + iK_i) \tag{2.14}$$

$$B_i = \frac{1}{2}(J_i - iK_i). \tag{2.15}$$

This may seem like a very arbitrary thing to do, and indeed it's *a priori* unmotivated. However, we can now consider the commutators of these generators,

$$[A_i, A_j] = i\,\epsilon_{ijk} A_k \tag{2.16}$$

$$[B_i, B_j] = i\,\epsilon_{ijk} B_k \tag{2.17}$$

$$[A_i, B_j] = 0. \tag{2.18}$$

Magic! The **A** and **B** generators form *decoupled* representations of the $SU(2)$ algebra. Note, however, will note that these generators are *not Hermitian* . Thus we were careful above not to say that $SU(3,1)$ *equals* $SU(2)\times SU(2)$, where 'equals' usually means either isomorphic or homomorphic. Further, the Lorentz group is not compact (because of boosts) while $SU(2)\times SU(2)$ is. Anyway, we needn't worry about the precise sense in which $SU(3,1)$ and $SU(2)\times SU(2)$ are related, the point is that we may label representations of $SU(3,1)$ by the quantum numbers of $SU(2)\times SU(2)$, (A,B). For example, a Dirac spinor is in the $(\frac{1}{2},\frac{1}{2}) = (\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation, i.e. the direct sum of two Weyl reps. (More on this in Section 2.3.) To connect back to reality, the physical meaning of all this is that we may write the spin of a representation as $J = A + B$.

So how are $SO(3,1)$ and $SU(2)\times SU(2)$ *actually* related? We've been deliberately vague about the exact relationship between the Lorentz group and $SU(2)\times SU(2)$. The precise relationship between the two groups are that the *complex* linear combinations of the generators of the Lorentz algebra are isomorphic to the *complex* linear combinations of the Lie *algebra* of $SU(2)\times SU(2)$.

$$\mathcal{L}_{\mathbb{C}}(SO(3,1)) \cong \mathcal{L}_{\mathbb{C}}(SU(2)\times SU(2)) \tag{2.19}$$

Be careful not to say that the Lie algebras of the two groups are identical, it is important to emphasize that only the *complexified* algebras are identifiable.

2.2.3 The Lorentz group is isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2$

While the Lorentz group and $SU(2)\times SU(2)$ were not related by either a isomorphism or homomorphism, we *can* relate the Lorentz group more concretely to $SL(2,\mathbb{C})$. More precisely, the Lorentz group is isomorphic to the coset space $SL(2,\mathbb{C})/\mathbb{Z}_2$

$$SO(3,1) \cong SL(2,\mathbb{C})/\mathbb{Z}_2 \tag{2.20}$$

Recall that we may represent four-vectors in Minkowski space as complex Hermitian 2×2 matrices via $V^\mu \to V_\mu \sigma^\mu$, where the σ^μ are the usual Pauli matrices,

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.21}$$

To be explicit, we may associate a vector $\mathbf{x}$ with either a vector in Minkowski space $\mathbb{M}^4$ spanned by the unit vectors e^μ,

$$\mathbf{x} = x^\mu e_\mu, \tag{2.22}$$

or with a matrix in $SL(2, \mathbb{C})$,

$$\mathbf{x} = x_\mu \sigma^\mu. \tag{2.23}$$

For the Minkowski four-vectors, we already understand how a Lorentz transformation $\mathbf{\Lambda}$ acts on a [covariant] vector x^μ while preserving the vector norm[1],

$$|\mathbf{x}|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{2.24}$$

For Hermitian matrices, there is an analogous transformation by the action of the group of invertible complex matrices of unitary determinant, $SL(2, \mathbb{C})$. For $\mathbf{N} \in SL(2, \mathbb{C})$, $\mathbf{N}^\dagger \mathbf{x} \mathbf{N}$ is also in the space of Hermitian 2×2 matrices. Such transformations preserve the determinant of $\mathbf{x}$,

$$\det \mathbf{x} = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{2.25}$$

The equivalence of the right-hand sides of equations (2.24) and (2.25) are very suggestive of an identification between the Lorentz group $SO(3, 1)$ and $SL(2, \mathbb{C})$. Indeed, equation (2.25) implies that for each $SL(2, \mathbb{C})$ matrix $\mathbf{N}$, there exists a Lorentz transformation Λ such that

$$\mathbf{N}^\dagger x^\mu \sigma_\mu \mathbf{N} = (\Lambda x)^\mu \sigma_\mu. \tag{2.26}$$

[1]This is the content of equation (2.2), which defines the Lorentz group.

We discuss this in more detail in Appendix D, but a very important feature should already be apparent: the map from $SL(2,\mathbb{C}) \to SO(3,1)$ is 2-1. This is clear since the matrices $\mathbf{N}$ and $-\mathbf{N}$ yield the *same* Lorentz transformation, $\Lambda^\mu{}_\nu$. Hence it is not $SO(3,1)$ and $SL(2,\mathbb{C})$ that are isomorphic, but rather $SO(3,1)$ and $SL(2,\mathbb{C})/\mathbb{Z}_2$.

The point that we should glean from this is that one will miss something if on only looks at representations of $SO(3,1)$ and not the representations of $SL(2,\mathbb{C})$. This 'something' is the spinor representation. How should we have known that $SL(2,\mathbb{C})$ is the important group? One way of seeing this is noting that $SL(2,\mathbb{C})$ is **simply connected** as a group manifold.

By the polar decomposition for matrices, any $g \in SL(2,\mathbb{C})$ can be written as the product of a unitary matrix U times the exponentiation of a traceless Hermitian matrix h,

$$g = Ue^h. \tag{2.27}$$

We may write these matrices explicitly in terms of real parameters $a, \cdots, g$;

$$h = \begin{pmatrix} c & a - ib \\ a + ib & -c \end{pmatrix} \tag{2.28}$$

$$U = \begin{pmatrix} d + ie & f + ig \\ -f + ig & d - ie \end{pmatrix}. \tag{2.29}$$

Here a, b, c are unconstrained while $d, \cdots, g$ must satisfy

$$d^2 + e^2 + f^2 + g^2 = 1. \tag{2.30}$$

Thus the space of 2×2 traceless Hermitian matrices $\{h\}$ is topologically identical to $\mathbb{R}^3$ while the space of unit determinant 2×2 unitary matrices $\{U\}$ is topologically identical to the three-sphere, S_3. Thus we have

$$SL(2,\mathbb{C}) = \mathbb{R}^3 \times S_3. \tag{2.31}$$

As both of the spaces on the right-hand side are simply connected, their product, $SL(2,\mathbb{C})$, is also simply connected. This is a 'nice' property because we can write down any element of the group by exponentiating its generators at the identity. But even further, since $SL(2,\mathbb{C})$ is simply connected, its quotient space $SL(2,\mathbb{C})/\mathbb{Z}_2 = SO(3,1)$ is *not* simply connected. We already mentioned this when we introduced the orthochronous Lorentz group, but the point is that we would like to use simply connected groups to construct our representations (more on this in the box below). Thus we shall use $SL(2,\mathbb{C})$, not $SO(3,1)$, for our representations of the Lorentz part of the Poincaré group. $SL(2,\mathbb{C})$ is called the **universal covering group** of $SO(3,1)$, meaning that it is the 'minimal' simply connected group homeomorphic to $SO(3,1)$. This universal covering group is often referred to as $Spin(3,1)$.

Projective representations and universal covering groups. For the uninitiated, it may not be clear why the above rigamarole is necessary or even interesting. Here we would like to approach the topic from a different direction to answer, in words, the question of what the spinor representation is and why it is physical.

A typical "representation theory for physicists" course goes into detail about constructing the usual tensor representations of groups but only mentions the spinor representation of the Lorentz group in passing. Students 'inoculated' with a quantum field theory course will not bat an eyelid at this, since they're already used to the technical manipulation of spinors. But where does the spinor representation come from if all of the 'usual' representations we're used to are tensors?

The answer lies in quantum mechanics. Recall that when we write representations U of a group G, we have $U(g_1)U(g_2) = U(g_1g_2)$ for $g_1, g_2 \in G$. In quantum physics, however, physical states are invariant under phases, so we have the freedom to be more general with our multiplication rule for representations: $U(g_1)U(g_2) = U(g_1g_2)\exp(i\phi(g_1,g_2))$. Such 'representations' are called **projective representations**. In other worse, quantum mechanics allows us to use projective representations rather than ordinary representations.

It turns out that not every group admits 'inherently' projective representations. In cases where such reps are not allowed, a representation that one *tries* to construct to be projective can have its generators redefined to reveal that it is actually an ordinary non-projective representation. It turns out that groups that are *not* simply connected, such as the Lorentz group, admit inherently projective representations.

In particular, the Lorentz group is *doubly* connected, i.e. going over any loop *twice* will allow it to be contracted to a point. This means that the phase in the projective representation must be ± 1. One can consider taking a loop in the Lorentz group that corresponds to rotating by 2π along the $\hat{z}$-axis. Representations with a projective phase $+1$ will return to their original state after a single rotation, these are the particles with integer spin. Representations with a projective phase -1 will return to their original state only after *two* rotations, and these correspond to spin-1/2 particles, or spinors.

There is an excellent discussion of this in Weinberg, Volume I. We reproduce the main parts of Weinberg's argument in Appendix D. More on the representation theory of the Poincaré group and its SUSY extension can be found in Buchbinder and Kuzenko [3]. Further pedagogical discussion of spinors can be found in [4].

2.3 Representations of $SL(2,\mathbb{C})$

The representations of the universal cover of the Lorentz group, $SL(2,\mathbb{C})$, are spinors. Most standard quantum field theory texts do calculations in terms of four-component Dirac spinors. This has the benefit of representing all the degrees of freedom of a typical Standard Model massive fermion into a single object. In SUSY, on the other hand, it will turn out to be natural to work with two-component spinors. For example, a complex scalar field has two real degrees of freedom. In order to have a supersymmetry between complex scalars and fermions, we require that the number of degrees of freedom match for both types objects. A Dirac spinor, however, has four real degrees of freedom (2×4 complex degrees of freedom - 4 from the Dirac equation). Thus we argue that it is more useful to consider Weyl (and later Majorana) spinors with the same number of degrees of freedom as the complex scalar field that they mix with under SUSY. For a comprehensive guide to calculating with two-component spinors, see the review by Dreiner, Haber, and Martin [5].

Let us start by defining the **fundamental** and **conjugate** (or **antifundamental**) representations of $SL(2,\mathbb{C})$. These are just the matrices $N_\alpha{}^\beta$ and $(N^*)_{\dot{\alpha}}{}^{\dot{\beta}}$. Don't be startled by the dots on the indices, they're just a book-keeping device to keep the fundamental and the conjugate indices from getting confused. One cannot contract a dotted with an undotted $SL(2,\mathbb{C})$ index; this would be like trying to contract spinor indices (α or $\dot{\alpha}$) with vector indices (μ): they index two totally different representations[2].

We are particularly interested in the objects that these matrices act on. Let us thus define **left-handed Weyl spinors**, ψ, as those acted upon by the fundamental rep and **right-handed Weyl spinors**, $\overline{\chi}$, as those that are acted upon by the conjugate rep. Again, do not be startled with the extra jewelry that our spinors display. The bar on the right-handed spinor just serves to distinguish it from the left-handed spinor. To be clear, they're both spinors, but they're different types of spinors that have different types of indices and that transform under different representations of $SL(2,\mathbb{C})$. Explicitly,

$$\psi'_\alpha = N_\alpha{}^\beta \psi_\beta \tag{2.32}$$

$$\overline{\chi}'_{\dot{\alpha}} = (N^*)_{\dot{\alpha}}{}^{\dot{\beta}} \overline{\chi}_{\dot{\beta}}. \tag{2.33}$$

2.4 Invariant Tensors

We know that $\eta_{\mu\nu}$ is invariant under $SO(3,1)$ and can be used (along with the inverse metric) to raise and lower $SO(3,1)$ indices. For $SL(2,\mathbb{C})$, we can build an analogous tensor, the unimodular antisymmetric tensor

$$\epsilon^{\alpha\beta} = i(\sigma^2)_{\alpha\beta} \tag{2.34}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.35}$$

Unimodularity (unit determinant) and antisymmetry uniquely define the above form up to an overall sign. The choice of sign is a convention. This tensor is invariant under $SL(2,\mathbb{C})$ since

$$\epsilon'^{\alpha\beta} = \epsilon^{\rho\sigma} N_\rho{}^\alpha N_\sigma{}^\beta \tag{2.36}$$

$$= \epsilon^{\alpha\beta} \det N \tag{2.37}$$

$$= \epsilon^{\alpha\beta}. \tag{2.38}$$

[2] This doesn't mean that we can't swap between different types of indices. In fact, this is exactly what we did in equations (2.22) and (2.23). We'll get to the role of the σ matrices very shortly.

We can now use this tensor to raise undotted $SL(2\mathbb{C})$ indices:

$$\psi^\alpha \equiv \epsilon^{\alpha\beta}\psi_\alpha. \tag{2.39}$$

To lower indices we can use an analogous unimodular antisymmetric tensor with two lower indices. For consistency, the overall sign of the lowered-indices tensor must be defined as

$$\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}. \tag{2.40}$$

This is to ensure that the upper- and lower-indices tensors are inverses, i.e. so that the combined operation of raising then lowering an index does not introduce a sign. Dotted indices indicate the complex conjugate representation, $\epsilon^*_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta}$. Since ϵ is real we thus use the same sign convention for dotted indices as undotted indices,

$$\epsilon^{\dot 1\dot 2} = \epsilon^{12} = -\epsilon_{\dot 1\dot 2} = -\epsilon_{12}. \tag{2.41}$$

So we may raise dotted indices in exactly the same way:

$$\overline{\chi}^{\dot\alpha} \equiv \epsilon^{\dot\alpha\dot\beta}\overline{\chi}_{\dot\alpha}. \tag{2.42}$$

2.5 Contravariant representations

Now that we're familiar with the ϵ tensor, we should tie up a loose end from Section 2.3. There we introduced the fundamental and conjugate representations of $SL(2,\mathbb{C})$. What happened to the **contravariant** representations that transform under the inverse matrices N^{-1} and N^{*-1}?

It turns out that these representations are equivalent (in the group theoretical sense) to the fundamental and conjugate representations presented above. Using the antisymmetric tensor $\epsilon_{\alpha\beta}$ ($\epsilon^{12} = 1$) and the unimodularity of $N \in SL(2,\mathbb{C})$,

$$\epsilon_{\alpha\beta} N^\alpha{}_\gamma N^\beta{}_\delta = \epsilon_{\gamma\delta} \det N \tag{2.43}$$

$$\epsilon_{\alpha\beta} N^\alpha{}_\gamma N^\beta{}_\delta = \epsilon_{\gamma\delta} \tag{2.44}$$

$$\left(N^T\right)_\gamma{}^\alpha \epsilon_{\alpha\beta} N^\beta{}_\delta = \epsilon_{\gamma\delta} \tag{2.45}$$

$$\epsilon_{\alpha\beta} N^\beta{}_\delta = \left[\left(N^T\right)^{-1}\right]_\alpha{}^\gamma \epsilon_{\gamma\delta} \tag{2.46}$$

And hence by Schur's Lemma N and $(N^T)^{-1}$ are equivalent. Similarly, N^* and $(N^\dagger)^{-1}$ are equivalent. This is not surprising, of course, since we already knew that the antisymmetric tensor, ϵ, is used to raise and lower indices in $SL(2,\mathbb{C})$. Thus the equivalence of these representations is no more 'surprising' than the fact that Lorentz vectors with upper indices are equivalent to Lorentz vectors with lower indices. Explicitly, then, the contravariant representations transform as

$$\psi'^{\alpha} = \psi^{\beta}(N^{-1})_{\beta}{}^{\alpha} \tag{2.47}$$
$$\overline{\chi}'^{\dot{\alpha}} = \overline{\chi}^{\dot{\beta}}(N^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}}. \tag{2.48}$$

To summarize, our two-component spinor representations are

$$\psi'_{\alpha} = N_{\alpha}{}^{\beta}\psi_{\beta} \tag{2.49}$$
$$\overline{\chi}'_{\dot{\alpha}} = (N^*)_{\dot{\alpha}}{}^{\dot{\beta}}\overline{\chi}_{\dot{\beta}} \tag{2.50}$$
$$\psi'^{\alpha} = \psi^{\beta}(N^{-1})_{\beta}{}^{\alpha} \tag{2.51}$$
$$\overline{\chi}'^{\dot{\alpha}} = \overline{\chi}^{\dot{\beta}}(N^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}}. \tag{2.52}$$

Occasionally one will see equations (2.50) and (2.52) written in terms of Hermitian conjugates,

$$\overline{\chi}'_{\dot{\alpha}} = \overline{\chi}_{\dot{\beta}}(N^\dagger)^{\dot{\beta}}{}_{\dot{\alpha}} \tag{2.53}$$
$$\overline{\chi}'^{\dot{\alpha}} = (N^{\dagger -1})^{\dot{\alpha}}{}_{\dot{\beta}}\overline{\chi}^{\dot{\beta}}. \tag{2.54}$$

We will not advocate this notation, however, since Hermitian conjugates are a bit delicate notationally in quantum field theories.

Stars and daggers. Let us clarify some notation. When dealing with classical fields, the complex conjugate representation is the usual complex conjugate of the field; i.e. $\psi \to \psi^*$. When dealing with *quantum* fields, on the other hand, it is conventional to write a Hermitian conjugate; i.e. $\psi \to \psi^\dagger$. This is because the quantum field contains creation and annihilation *operators*. This is the same reason why Lagrangians are often written $\mathcal{L}$ = term + h.c. The classical Lagrangian is a scalar quantity, so in that case one could have just written 'c.c.' (complex conjugate) rather than 'h.c.' (Hermitian conjugate). In QFT, however, since the terms in the

Lagrangian are composed of quantum fields—which are operators—it is necessary for them to have a Hermitian conjugate.

It is worth making one further note about notation. Sometimes authors will write

$$\overline{\psi}_{\dot{\alpha}} = \psi^{\dagger}_{\alpha}. \tag{2.55}$$

This is technically correct, but it can be a bit misleading since one shouldn't get into the habit of thinking of the bar as some kind of operator. The bar and its dotted index are notation to distinguish the right-handed representation from the left-handed representation. The content of the above equation is the statement that the conjugate of a left-handed spinor transforms as a right-handed spinor.

In light of our previous info box, one might feel like we ought to be very explicit if the right-hand side of the above equation should have a dagger or a star. Actually, after spending all that time being pedantic, it doesn't matter. We know that under a Lorentz transformation, $\overline{\psi}_{\dot{\alpha}} \to (N^*)_{\dot{\alpha}}{}^{\dot{\beta}}\overline{\psi}_{\dot{\beta}}$. This seems awkward if we want to associate $\overline{\psi}$ with $\psi^{\dagger}$. Recall, however, that $N \in \mathcal{L}(SL(2,\mathbb{C}))$. Elements of the *group* $SL(2,\mathbb{C})$ have unit determinant, so elements of the *algebra* $\mathcal{L}(SL(2,\mathbb{C}))$ have the property $N = N^T$. Thus we may swap N^* with $N^{\dagger}$ and we may say either $\overline{\psi} = \psi^{\dagger}$ consistently.

2.6 Lorentz-Invariant Spinor Products

Now that we're armed with a metric to raise and lower indices, we can also define the inner product of spinors as the contraction of upper and lower indices. Note that in order to form inner products that are actually Lorentz-invariant, one cannot contract dotted and undotted indices.

There is a very nice short-hand that is commonly used in supersymmetry that allows us to drop contracted indices. Since it's important to distinguish between left- and right-handed Weyl spinors, we have to be careful that dropping indices doesn't introduce an ambiguity. This is why right-handed spinors are barred in addition to having dotted indices. Let us now define the contractions

$$\psi\chi \equiv \psi^\alpha \chi_\alpha \tag{2.56}$$

$$\overline{\psi}\overline{\chi} \equiv \overline{\psi}_{\dot{\alpha}} \overline{\chi}^{\dot{\alpha}}. \tag{2.57}$$

Note that the contractions are different for the left- and right-handed spinors. This is a choice of convention that has been chosen such that

$$(\psi\chi)^\dagger \equiv (\psi^\alpha \chi_\alpha)^\dagger = \overline{\chi}_{\dot{\alpha}} \overline{\psi}^{\dot{\alpha}} \equiv \overline{\chi}\overline{\psi} = \overline{\psi}\overline{\chi}. \tag{2.58}$$

The second equality is worth explaining. Why is it that $(\psi^\alpha \chi_\alpha)^\dagger = \overline{\chi}_{\dot{\alpha}} \overline{\psi}^{\dot{\alpha}}$? Recall from that the Hermitian conjugation acts on the creation and annihilation operators in the quantum fields ψ and χ. The Hermitian conjugate of the product of two Hermitian operators AB is given by $B^\dagger A^\dagger$. The coefficients of these operators in the quantum fields are just c-numbers ('commuting' numbers), so the conjugate of $\psi^\alpha \chi_\alpha$ is $\left(\chi^\dagger\right)_{\dot{\alpha}} \left(\psi^\dagger\right)^{\dot{\alpha}}$.

Now let's get back to our contraction convention. Recall that quantum spinor fields are Grassmann, i.e. they anticommute. Thus we show that with our contraction convention, the order of the contracted fields don't matter:

$$\psi\chi = \psi^\alpha \chi_\alpha = -\psi_\alpha \chi^\alpha = \chi^\alpha \psi_\alpha = \chi\psi \tag{2.59}$$

$$\overline{\psi}\overline{\chi} = \overline{\psi}_{\dot{\alpha}} \overline{\chi}^{\dot{\alpha}} = -\overline{\psi}^{\dot{\alpha}} \overline{\chi}_{\dot{\alpha}} = \overline{\chi}_{\dot{\alpha}} \overline{\psi}^{\dot{\alpha}} = \overline{\chi}\overline{\psi}. \tag{2.60}$$

It is actually rather important that quantum spinors anticommute. If the ψ were *commuting* objects, then

$$\psi^2 = \psi\psi = \epsilon^{\alpha\beta} \psi_\beta \psi_\alpha = \psi_2 \psi_1 - \psi_1 \psi_2 = 0. \tag{2.61}$$

Thus we must have ψ such that

$$\psi_1 \psi_2 = -\psi_2 \psi_1, \tag{2.62}$$

i.e. the components of the Weyl spinor must be Grassmann. So one way of understanding why spinors are anticommuting is that metric that raises and lowers the indices are antisymmetric. (We know, of course, that from another perspective this anticommutativity comes from the quantum creation and annihilation operators.)

Finally, we note a handy equality that stems from spinor antisymmetry:

$$\psi_\alpha \psi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\psi\psi. \tag{2.63}$$

2.7 Vector-like Spinor Products

Notice that the Pauli matrices give a natural way to go between $SO(3,1)$ and $SL(2,\mathbb{C})$ indices. Using equation (2.26),

$$(x_\mu \sigma^\mu)_{\alpha\dot{\alpha}} \to N_\alpha{}^\beta (x_\nu \sigma^\nu)_{\beta\dot{\gamma}} N^*{}_{\dot{\alpha}}{}^{\dot{\gamma}} \tag{2.64}$$

$$= (\Lambda_\mu{}^\nu x_\nu)\sigma^\mu{}_{\alpha\dot{\alpha}}. \tag{2.65}$$

Then we have

$$(\sigma^\mu)_{\alpha\dot{\alpha}} = N_\alpha{}^\beta (\sigma^\nu)_{\beta\dot{\gamma}} (\Lambda^{-1})^\mu{}_\nu N^*{}_{\dot{\alpha}}{}^{\dot{\gamma}}. \tag{2.66}$$

One could, for example, swap between the vector and spinor indices by writing

$$V_\mu \to V_{\alpha\dot{\beta}} \equiv V_\mu (\sigma^\mu)_{\alpha\dot{\beta}}. \tag{2.67}$$

We can define a 'raised index' σ matrix,

$$(\overline{\sigma}^\mu)^{\dot{\alpha}\alpha} \equiv \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^\mu)_{\beta\dot{\beta}} \tag{2.68}$$

$$= (\sigma^\mu)^\dagger \tag{2.69}$$

$$= (\mathbb{1}, -\overrightarrow{\boldsymbol{\sigma}}). \tag{2.70}$$

Note the bar and the reversed order of the dotted and undotted indices. The bar is just notation to indicate the index structure, similarly to the bars on the right-handed spinors. How do we understand the indices? Let us go back to the matrix form of the Pauli matrices (2.21) and the upper-indices epsilon tensor (2.35). One may use $\epsilon = i\sigma^2$ and to directly verify that

$$\epsilon \overline{\sigma}_m u = \sigma_\mu^T \epsilon, \tag{2.71}$$

and hence

$$\overline{\sigma}_\mu = \epsilon \sigma_\mu^T \epsilon^T. \tag{2.72}$$

Restoring indices on the right-hand side,

$$\epsilon \sigma_\mu^T \epsilon^T \to \epsilon^{\alpha\beta}(\sigma^{\mu T})_{\beta\dot{\beta}}(\epsilon^T)^{\dot{\beta}\dot{\alpha}} \tag{2.73}$$

$$\to \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^\mu)_{\dot{\beta}\beta}. \tag{2.74}$$

Thus we see that the $\overline{\sigma}^\mu$ have a dotted-then-undotted index structure. A further consistency check comes from looking at the structure of the γ matrices as applied to the Dirac spinors formed using Weyl spinors with our index convention. We do this in Section 2.9.

2.8 Generators of $SL(2,\mathbb{C})$

How do Lorentz transformations act on Weyl spinors? We should already have a hint from the generators of Lorentz transformations on Dirac spinors. (Go ahead and review this section of your favorite QFT textbook.) The objects that obey the Lorentz algebra, equation (2.7), and generate the desired transformations are given by the matrices,

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)_\alpha{}^\beta \tag{2.75}$$

$$(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{4}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{2.76}$$

The assignment of dotted and undotted indices are deliberate; they tell us which generator corresponds to the fundamental versus the conjugate representation. (The choice of *which* one is fundamental versus conjugate, of course, is arbitrary.) Thus the left and right-handed Weyl spinors transform as

$$\psi_\alpha \to \left(e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}\right)_\alpha{}^\beta \psi_\beta \tag{2.77}$$

$$\overline{\chi}^{\dot{\alpha}} \to \left(e^{-\frac{i}{2}\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}}\right)^{\dot{\alpha}}{}_{\dot{\beta}} \overline{\chi}^{\dot{\beta}}. \tag{2.78}$$

We can invoke the $SU(2)\times SU(2)$ 'representation' (and we use that word *very* loosely) of the Lorentz group from equations (2.12) and (2.13) to write the $\sigma^{\mu\nu}$ generators as

$$J_i = \frac{1}{2}\epsilon_{ijk}\sigma_{jk} = \frac{1}{2}\sigma_i \tag{2.79}$$

$$K_i = \sigma_{0i} = -\frac{1}{2}\sigma_i, \tag{2.80}$$

where one then finds

$$A_i = \frac{1}{2}(J_i + iK_i) = \frac{1}{2}\sigma_i \tag{2.81}$$

$$B_i = \frac{1}{2}(J_i - iK_i) = 0. \tag{2.82}$$

Thus the left-handed Weyl spinors ψ_α are $(\frac{1}{2}, 0)$ spinor representations Similarly, one finds that the right-handed Weyl spinors $\overline{\chi}^{\dot{\alpha}}$ are $(0, \frac{1}{2})$ spinor representations.

2.9 Chirality

Now let's get back to a point of nomenclature. Why do we call them left- and right-handed spinors? The Dirac equation tells us[3]

$$p_\mu \sigma^\mu \psi = m\psi \tag{2.83}$$

$$p_\mu \overline{\sigma}^\mu \overline{\chi} = m\overline{\chi}. \tag{2.84}$$

Equation (2.84) follows from equation (2.83) via Hermitian conjugation, as appropriate for the conjugate representation.

In the massless limit, then, $p^0 \to |\mathbf{p}|$ and hence

$$\left(\frac{\sigma \cdot \mathbf{p}}{|\mathbf{p}|}\psi\right) = \psi \tag{2.85}$$

$$\left(\frac{\sigma \cdot \mathbf{p}}{|\mathbf{p}|}\overline{\chi}\right) = -\overline{\chi}. \tag{2.86}$$

[3]To be clear, there's some arbitrariness here. How do we know which 'Dirac equation' (i.e. with σ or $\overline{\sigma}$) to apply to ψ (the fundamental rep) versus $\overline{\chi}$ (the conjugate rep)? This is convention, 'by the interchangeability of the fundamental and conjugate reps' and 'the interchangeability of σ and $\overline{\sigma}$' if you wish. Once we have chosen the convention of equation (2.83), then equation (2.84) follows from Hermitian conjugation. In other words, once we've chosen that the fundamental representation goes with the 'σ' Dirac equation (2.83), we know that the conjugate representation goes with the '$\sigma^\dagger = \overline{\sigma}$' Dirac equation (2.84). If you ever get confused, check the index structure of σ and $\overline{\sigma}$ and make sure they are contracting honestly.

We recognize the quantity in parenthesis as the helicity operator, and hence ψ has helicity +1 (left-handed) and $\overline{\chi}$ has helicity -1 (right-handed). Non-zero masses complicate things, of course. In fact, they complicate things differently depending on whether the masses are Dirac or Majorana. We'll get to this in due course, but the point is that even though ψ and $\overline{\chi}$ are no longer helicity eigenstates, they are *chirality* eigenstates:

$$\gamma_5 \begin{pmatrix} \psi \\ 0 \end{pmatrix} = \begin{pmatrix} \psi \\ 0 \end{pmatrix} \tag{2.87}$$

$$\gamma_5 \begin{pmatrix} 0 \\ \overline{\chi} \end{pmatrix} = - \begin{pmatrix} 0 \\ \overline{\chi} \end{pmatrix}, \tag{2.88}$$

where we've put the Weyl spinors into four-component Dirac spinors in the usual way so that we may apply the chirality operator, γ_5. (See Section 2.11.)

Chirality. Keeping the broad program in mind, let us take a moment to note that chirality will play an important role in whatever new physics we might find at the Terascale. The Standard Model is a chiral theory (e.g. q_L and q_R are in different gauge representations), so whatever Terascale completion supersedes it must also be chiral. This is no problem in SUSY where we may place chiral fields into different supermultiplets ('superfields'). In XD, however, we run into the problem that there is no chirality operator in five dimensions. This leads to a lot of subtlety in model-building that we shall discuss in the second-half of this document.

It is assumed that the reader can distinguish between helicity and chirality. If not, then s/he is kindly requested to review this for posterity's sake.

2.10 Fierz Rearrangement

Fierz identities are useful for rewriting spinor operators by swapping the way indices are contracted. For example,

$$(\chi\psi)(\chi\psi) = -\frac{1}{2}(\psi\psi)(\chi\chi). \tag{2.89}$$

One can understand these Fierz identities as an expression of the decomposition of tensor products in group theory. For example, we could consider the decomposition $(\frac{1}{2},0)\otimes(0,\frac{1}{2})=(\frac{1}{2},\frac{1}{2})$:

$$\psi_\alpha\overline{\chi}_{\dot{\alpha}}=\frac{1}{2}(\psi\sigma_\mu\overline{\chi})\sigma^\mu{}_{\alpha\dot{\alpha}}, \tag{2.90}$$

where, on the right-hand side, the object in the parenthesis is a vector in the same sense as equation (2.67). The factor of $\frac{1}{2}$ is, if you want, a Clebsch-Gordan coefficient.

Another example is the decomposition for $(\frac{1}{2},0)\otimes(\frac{1}{2},0)=(0,0)+(1,0)$:

$$\psi_\alpha\chi_\beta=\frac{1}{2}\epsilon_{\alpha\beta}(\psi\chi)+\frac{1}{2}(\sigma^{\mu\nu}\epsilon^T)_{\alpha\beta}(\psi\sigma_{\mu\nu}\chi). \tag{2.91}$$

Note that the $(1,0)$ rep is the antisymmetric tensor representation. All higher dimensional representations can be obtained from products of spinors. Explicit calculations can be found in the lecture notes by Müller-Kirsten and Wiedemann [6].

A set of Fierz identities are listed in Section C.3.

2.11 Connection to Dirac Spinors

We would now like to explicitly connect the machinery of two-component Weyl spinors to the four-component Dirac spinors that we (unfortunately) teach our children.

Let us define

$$\gamma^\mu\equiv\begin{pmatrix}0 & \sigma^\mu\\ \overline{\sigma}^\mu & 0\end{pmatrix}. \tag{2.92}$$

This, one can check, gives us the Clifford algebra

$$\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}\cdot\mathbb{1}. \tag{2.93}$$

We can further define the fifth γ-matrix, the four-dimensional chirality operator,

$$\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}-\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}. \tag{2.94}$$

A **Dirac spinor** is defined, as mentioned above, as the direct sum of left- and right-handed Weyl spinors, $\Psi_D = \psi \oplus \overline{\chi}$, or

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{2.95}$$

The choice of having a lower undotted index and an upper dotted index is convention and comes from how we defined our spinor contractions. The generator of Lorentz transformations takes the form

$$\Sigma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix}, \tag{2.96}$$

with spinors transforming as

$$\Psi_D \to e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}\Psi_D. \tag{2.97}$$

In our representation the action of the chirality operator is given by γ_5,

$$\gamma^5 \Psi_D = \begin{pmatrix} -\psi_\alpha \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{2.98}$$

We can then define left- and right-handed projection operators,

$$P_{L,R} = \frac{1}{2}\left(\mathbb{1} \mp \gamma^5\right). \tag{2.99}$$

Using the standard notation, we shall define a barred *Dirac* spinor as $\overline{\Psi}_D \equiv \Psi^\dagger \gamma^0$. Note that this bar has nothing to do with the bar on a Weyl spinor. We can then define a charge conjugation matrix C via $C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$ and the Dirac conjugate spinor ${\Psi_D}^c = C\overline{\Psi}_D{}^T$, or explicitly in our representation,

$${\Psi_D}^c = \begin{pmatrix} \chi_\alpha \\ \overline{\psi}^{\dot{\alpha}} \end{pmatrix}. \tag{2.100}$$

A **Majorana spinor** is defined to be a Dirac spinor that is its own conjugate, $\Psi_M = \Psi_M^c$. We can thus write a Majorana spinor in terms of a Weyl spinor,

$$\Psi_M = \begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot{\alpha}} \end{pmatrix}. \tag{2.101}$$

It is worth noting that in four dimensions there are no Majorana-Weyl spinors. This, however, is a dimension-dependent statement, as we will see in Section ***. A good treatment of this can be found in the appendix of Polchinksi's second volume [7].

Much ado about dots and bars. It's worth emphasizing once more that the dots and bars are just book-keeping tools. Essentially they are a result of not having enough alphabets available to write different kinds of objects. The bars can be especially confusing for beginning supersymmetry students since one is tempted to associate them with the barred Dirac spinors, $\overline{\Psi} = \Psi^\dagger \gamma_0$. *Do not make this mistake.* Weyl and Dirac spinors are different objects. The bar on a Weyl spinor has *nothing* to do with the bar on a Dirac spinor, and certainly has nothing to do with antiparticles. We see this explicitly when we construct Dirac spinors out of Weyl spinors (namely $\Psi = \psi \oplus \overline{\chi}$), but it's worth remembering because the notation can be misleading.

In principle ψ and $\overline{\psi}$ are totally different spinors in the same way that α and $\dot{\alpha}$ are totally different indices. Sometimes—as we have done above—we may also use the bar as an operation that converts an unbarred Weyl spinor into a barred Weyl spinor. That is to say that for a left-handed spinor ψ, we may define $\overline{\psi} = \psi^\dagger$. To avoid ambiguity it is customary—as we have also done—to write ψ for left-handed Weyl spinors, $\overline{\chi}$ for right-handed Weyl spinors, and $\overline{\psi}$ to for the right-handed Weyl spinor formed by taking the Hermitian conjugate of the left-handed spinor ψ.

To make things even trickier, much of the literature on extra dimensions use the convention that ψ and χ (unbarred) refer to left- and right-'chiral' *Dirac* spinors. Here 'chiral' means that they permit chiral zero modes, a non-trivial subtlety of extra dimensional models that we will get to in due course. For now we'll use the 'SUSY' convention that ψ and $\overline{\chi}$ are left- and right-handed Weyl spinors.

Chapter 3

The SUSY Algebra

"Supersymmetry is nearly thirty years old. It seems that now we are approaching the fourth supersymmetry revolution which will demonstrate its relevance to nature."

— G.L. Kane and M. Shifman [2]

3.1 The Supersymmetry Algebra

Around the same time that the Beatles released *Sgt. Pepper's Lonely Hearts Club Band*, Coleman and Mandula published their famous 'no-go' theorem which stated that the most general symmetry Lie group of an S-matrix in four dimensions is the direct product of the Poincaré group with an internal symmetry group[1]. In other words, there can be no mixing of spins within a symmetry multiplet.

Ignorance is bliss, however, and physicists continued to look for extensions of the Poincaré symmetry for some years without knowing about Coleman and Mandula's result. in particular, Golfand and Licktmann extended the Poincaré group using Grassmann operators, 'discovering' supersymmetry in physics. Independntly, Ramond, Neveu, Schwarz, Gervais, and Sakita where applying similar ideas in two dimensions to insert fermions into a budding theory of strings, hence developing (wait for it...) superstring theory.

[1] See Weinberg Vol III for a proof of the Coleman-Mandula theorem.

SUSY, then, is able to evade the Coleman-Mandula theorem by generalizing the symmetry from a Lie algebra to a **graded Lie algebra**. This has the property that if $\mathcal{O}_a$ are operators, then

$$\mathcal{O}_a\mathcal{O}_b - (-1)^{\eta_a\eta_b}\mathcal{O}_b\mathcal{O}_a = iC^e{}_{ab}\mathcal{O}_e, \tag{3.1}$$

where,

$$\eta_a = \begin{cases} 0 & \text{if } \mathcal{O}_a \text{ is bosonic} \\ 1 & \text{if } \mathcal{O}_a \text{ is fermionic} \end{cases} \tag{3.2}$$

The Poincaré generators P^μ, $M^{\mu\nu}$ are both bosonic generators with $(A,B) = (\frac{1}{2},\frac{1}{2}), (1,0)\oplus(0,1)$ respectively. In supersymmetry, on the other hand, we add *fermionic generators*, $Q^A_\alpha, \overline{Q}^B_{\dot\alpha}$. Here $A, B = 1, \cdots, \mathcal{N}$ label the number of **supercharges** (these are, of course, different from the (A,B) that label representations of the Lorentz algebra) and $\alpha, \dot\alpha = 1,2$ are Weyl spinor indices. We will primarily focus on **simple supersymmetry** where $\mathcal{N} = 1$. We call $\mathcal{N} > 1$ **extended supersymmetry**.

Haag, Lopouszanski, and Sohnius showed in 1974 that $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ are the only generators for supersymmetry. For example, it would be inconsistent to include generators $\tilde{Q}$ in the representation $(A, B = (\frac{1}{2},1))$. The general argument is that the product of two spinor generators has to be bosonic and the only bosonic generators are M and P. A further discussion of this can be found in Weinberg III [8].

Without further ado, let's write down the supersymmetry algebra.

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\nu}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}) \tag{3.3}$$
$$[P^\mu, P^\nu] = 0 \tag{3.4}$$
$$[M^{\mu\nu}, P^\sigma] = i(P^\mu\eta^{\nu\sigma} - P^\nu\eta^{\mu\sigma}) \tag{3.5}$$
$$[Q_\alpha, M^{\mu\nu}] = i(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \tag{3.6}$$
$$[Q_\alpha, P^\mu] = 0 \tag{3.7}$$
$$\{Q_\alpha, Q^\beta\} = 0 \tag{3.8}$$
$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu \tag{3.9}$$

We're already familiar with equations (3.3 - 3.5) as being the usual Poincaré algebra. It remains to discuss the remaining equations involving the new fermionic generators.

Before we do that, however, two important notes are in order. First, one should check that the assignment of commutators in the above equations matches our definition for a graded Lie algebra, equation (3.1). Second, one should note that up to overall constants, we should *almost* have been able to *guess* the form of the new equations by matching the index structure on the left- and right-hand sides of each equation, using only the SUSY algebra and the generators of $SL(2,\mathbb{C})$, equations (2.75) and (2.76).

3.1.1 $[Q_\alpha, M^{\mu\nu}] = i(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta$

Now consider equation (3.6). How do we understand this? First of all, because Q_α is a spinor, we may write down its transformation under an infinitesimal Lorentz transformation,

$$Q'_\alpha = (e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\sigma\nu}})_\alpha{}^\beta Q_\beta \tag{3.10}$$

$$\approx (\mathbb{1} - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta. \tag{3.11}$$

However, Q_α also leads a second life as an operator. Thus we know it *also* transforms as

$$Q'_\alpha = U^\dagger Q_\alpha U \tag{3.12}$$

$$U = e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}, \tag{3.13}$$

and hence,

$$Q'_\alpha \approx (\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})\, Q_\alpha\, (\mathbb{1} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}). \tag{3.14}$$

Setting equations (3.11) and (3.14) equal to one another,

$$Q_\alpha - \frac{1}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta = Q_\alpha - \frac{i}{2}\omega_{\mu\nu}(Q_\alpha M^{\mu\nu} - M^{\mu\nu}Q_\alpha) + \mathcal{O}(\omega^2), \tag{3.15}$$

from which we finally deduce equation (3.6)

$$[Q_\alpha, M^{\mu\nu}] = i(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta.$$

Note that the commutator for the right-handed representation corresponds to placing bars on this relation,

$$[\overline{Q}_{\dot{\alpha}}, M^{\mu\nu}] = i\epsilon_{\dot{\alpha}\dot{\delta}}(\overline{\sigma}^{\mu\nu})^{\dot{\delta}}{}_{\beta}\overline{Q}^{\dot{\beta}}. \tag{3.16}$$

This follows from the transformation law in equation (2.76).

3.1.2 $[Q_\alpha, P^\mu] = 0$

Equation (3.7) tells us that translations don't affect the fermionic transformations. This is a bit surprising since our mnemonic of looking at the index structure suggests that the right-hand side of this equation could be proportional to $(\sigma^\mu)_{\alpha\dot{\alpha}}\overline{Q}^{\dot{\alpha}}$. Indeed, let us assume this to find that the proportionality constant, c must be zero. Thus,

$$[Q_\alpha, P^\mu] = c(\sigma^\mu)_{\alpha\dot{\alpha}}\overline{Q}^{\dot{\alpha}}. \tag{3.17}$$

This is actually two equations since we can get a corresponding equation for $\overline{Q}$. Recall that taking the Hermitian conjugate of a left-handed spinor operator produces a right-handed spinor (and vice versa), so that ${Q_\alpha}^\dagger = \overline{Q}_{\dot{\alpha}}$. What about the σ matrix? From equation (2.68) we have $(\sigma^\mu)_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}(\overline{\sigma}^\mu)^{\dot{\beta}\beta}$. Putting this together and taking the Hermitian conjugate of equation (3.17),

$$[{Q_\alpha}^\dagger, P^\mu] = c^*(\sigma^\mu)_{\alpha\dot{\alpha}}\overline{Q}^{\dot{\alpha}\dagger} \tag{3.18}$$

$$[\overline{Q}_{\dot{\alpha}}, P^\mu] = c^*\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}(\overline{\sigma}^\mu)^{\dot{\beta}\beta}Q^\alpha \tag{3.19}$$

$$[\overline{Q}^{\dot{\alpha}}, P^\mu] = c^*(\overline{\sigma}^\mu)^{\dot{\beta}\alpha}Q_\alpha. \tag{3.20}$$

Note that the Hermitian conjugate acts only on the *operator* $\overline{Q}$, that is to say that there is no transpose of the σ matrix. Equations (3.17) and (3.20) are, by index structure (that is, by Lorentz covariance), the most general form of the commutators of Q and $\overline{Q}$ with P. To find c we invoke the Jacobi identity for P^μ, P^ν, and Q_α:

$$0 = [P^\mu, [P^\nu, Q_\alpha]] + [P^\nu, [Q_\alpha, P^\mu]] + [Q_\alpha, [P^\mu, P^\nu]] \tag{3.21}$$

$$= -c{\sigma^\nu}_{\alpha\dot{\alpha}}\left[P^\mu, \overline{Q}^{\dot{\alpha}}\right] + c{\sigma^\mu}_{\alpha\dot{\alpha}}\left[P^\nu, \overline{Q}^{\dot{\alpha}}\right] \tag{3.22}$$

$$= |c|^2\,{\sigma^\mu}_{\alpha\dot{\alpha}}\overline{\sigma}^{\nu\dot{\alpha}\beta}Q_\beta - |c|^2\,{\sigma^\nu}_{\alpha\dot{\alpha}}\overline{\sigma}^{\mu\dot{\alpha}\beta}Q_\beta \tag{3.23}$$

$$= |c|^2\,{(\sigma^{\mu\nu})_\alpha}^\beta Q_\beta. \tag{3.24}$$

From this we conclude that $c = 0$, hence proving our assertion.

3.1.3 $\{Q_\alpha, Q^\beta\} = 0$

Equation (3.8) comes from a similar argument. Again we may write the most general form of the anticommutator,

$$\left\{Q_\alpha, Q^\beta\right\} = k\left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta} M_{\mu\nu}. \tag{3.25}$$

Since $[Q, P] = 0$, the left-hand side of the above equation manifestly commutes with P. The right-hand side, however, manifestly does not commute with P from equation (3.5). In order for the above equation to be consistent, then, $k = 0$. Taking the Hermitian conjugate of the above equation of course also gives us

$$\left\{\overline{Q}^{\dot\alpha}, \overline{Q}_{\dot\beta}\right\} = 0. \tag{3.26}$$

3.1.4 $\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu$

Thus far none of the previous results have been particularly interesting. We saw that the spinor SUSY generator has a nontrivial commutator with with Lorentz transformations, but this is actually obvious because it is a nontrivial representation of the Lorentz group. The other (anti)commutators have been zero. By this point one might have become rather bored. Luckily, this anticommutator is the payoff for our patience.

Using the same index argument as we've been using, we may write the anticommutator as

$$\left\{Q_\alpha, \overline{Q}_{\dot\beta}\right\} = t(\sigma^\mu)_{\alpha\dot\beta} P_\mu. \tag{3.27}$$

This time, however, we cannot find an argument to set $t = 0$. By convention we set $t = 2$, though in principle we could have chosen any positive number. Since the right-hand side is the four-momentum operator, we require positivity to have positive energies.

Now let's step back for a moment. It is common to 'dress' this equation in words. A particularly nice description is to say that the supersymmetry generators are a kind of square root of the four-momentum. Another description is to say that combining two supersymmetry transformations (one of each helicity) gives a spacetime translation.

If $|F\rangle$ represents a fermionic state and $|B\rangle$ a bosonic state, then the SUSY algebra tells us that

$$Q|F\rangle = |B\rangle \tag{3.28}$$

$$Q|B\rangle = |F\rangle, \tag{3.29}$$

that is the SUSY generators turn bosons into fermions and vice-versa. However, the product of two generators preserves the spin of the particle,

$$Q\overline{Q}|B\rangle = |B\rangle, \tag{3.30}$$

but the particle is translated in spacetime. Thus the SUSY generators 'know' all about spacetime. This is starting to become interesting.

3.1.5 Commutators with Internal Symmetries

By the Coleman-Mandula theorem, we know that internal symmetry generators commute with the Poincaré generators. For example, the Standard Model gauge group commutes with the momentum, rotation, and boost operators. This carries over to the SUSY algebra. For an internal symmetry generator T_a,

$$[T_a, Q_\alpha] = 0. \tag{3.31}$$

This is true with one exception. The SUSY generators come equipped with their own internal symmetry, called **R-symmetry**. There exists an automorphism of the supersymmetry algebra,

$$Q_\alpha \to e^{i\gamma} Q_\alpha \tag{3.32}$$

$$\overline{Q}_{\dot{\alpha}} \to e^{-i\gamma} \overline{Q}_{\dot{\alpha}}. \tag{3.33}$$

This is a $U(1)$ internal symmetry. Applying this symmetry preserves the SUSY algebra. If R is the generator of this $U(1)$, then its action on the SUSY operators is given by

$$Q_\alpha \to e^{-iRt} Q_\alpha e^{iRt}, \tag{3.34}$$

where t is the transformation parameter. The corresponding algebra is

$$[Q_\alpha, R] = Q_\alpha \tag{3.35}$$

$$[\overline{Q}_{\dot\alpha}, R] = -\overline{Q}_{\dot\alpha}. \tag{3.36}$$

3.1.6 Extended Supersymmetry

The most general supersymmetry algebra contains an arbitrary number $\mathcal{N}$ of SUSY generators, which we may label with capital roman letters: $Q^A_\alpha, \overline{Q}^B_{\dot\alpha}$ where $A, B = 1, \cdots, \mathcal{N}$. The SUSY anticommutators take the general form

$$\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = 2(\sigma^\mu)_{\alpha\dot\beta}\delta^A{}_B P_\mu \tag{3.37}$$

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta}Z^{AB}. \tag{3.38}$$

There's nothing special about the upper or lower capital letters, they're just labels. The first equation is a little boring, the different generators don't mix to form the momentum generator. The second equation, however, starts to get more interesting. The Zs are called **central charges**. The antisymmetric tensor is the only object that has the right index structure. In order to be consistent with the symmetry of the left-hand-side, the central charge must be antisymmetric, $Z^{AB} = -Z^{BA}$.

Z is like an abelian generator of an internal symmetry group. The commutator of the central charges with the other elements of the algebra are all null:

$$[Z^{AB}, P^\mu] = [Z^{AB}, M^{\mu\nu}] = [Z^{AB}, Q^C_\alpha] = [Z^{AB}, Z^{BC}] = [Z^{AB}, T_a] = 0. \tag{3.39}$$

The central charges affect the R-symmetry described in the previous section. If the central charges all vanish $Z^{AB} = 0$, then the R-symmetry group is $U(\mathcal{N})$. If the charges do not all vanish, then the R-symmetry group is a subset of $U(\mathcal{N})$.

Central charges play an important role in the nonperturbative nature of supersymmetry. Additionally, they appear generically in the analysis of projective representations of a symmetry group.

Central Charges and Projective Representations. Recall that for a projective representation U of a symmetry group with elements T, T',

$$U(T)U(T') = e^{i\phi(T,T')}U(TT'), \tag{3.40}$$

where $\phi(T, T')$ is a phase that depends on the particular group elements being multiplied. Consistency requires that $\phi(T, 1) = \phi(1, T) = 0$ since the phase must vanish when multiplying by the identity. Parameterizing the group elements by α, we can Taylor expand

$$\phi(T(\alpha), T(\alpha')) = w_{ab}\alpha^a\alpha'^b + \cdots , \tag{3.41}$$

where the w are real constants. The effect of this phase on the algebra (with elements t, t') of the Lie group is that the commutator is modified to include a central charge, $z_{ab} = -w_{ab} + w_{ba}$:

$$[t_b, t_c] = iC^a{}_{bc}t_a + iz_{bc}\mathbb{1}. \tag{3.42}$$

Generally one can redefine the generators of the algebra to remove the central charges from the commutator. If this can be done, then it turns out that the group does not admit projective representations. Recall that we used an alternate topological argument to show that the Lorentz group admits projective representations.

Chapter 4

Representations of Supersymmetry

"I had to figure out whether less complex superalgebras existed and then to determine whether they had any relation to field theory or high energy physics. The first part didn't take much time — I wrote out fairly quickly all extensions of the algebra of generators of the Poincaré group by bispinor generators. It took significantly longer to put together the free field representations: one had to get used to the fact that in one multiplet were unified fields with both integer and half-integer spins."

— Evgeny Likhtman [2]

4.1 Representations of the Poincaré Group

As a quick refresher, let's briefly review the rotation group. The algebra is given by

$$[J_i, J_j] = i\epsilon_{ijk} J_k. \tag{4.1}$$

$SO(3)$ has one **Casimir operator**, i.e. an operator built out of the generators that commute with all of the generators. For $SO(3)$ this is

$$\mathbf{J}^2 = \sum J_i^2. \tag{4.2}$$

Each irreducible representation (irrep) takes a single value of the Casimir operator. For example, the eigenvalues of $\mathbf{J}^2$ are $j(j+1)$ where $j = 1, \frac{1}{2}, \cdots$. Thus each irrep is labelled by j. To label each element of the irrep, we pick eigenvalues of J_3 from the set $j_3 = -j, \cdots, j$. Thus each state is labelled as $|j; j_3\rangle$, identifying individual states with respect to their transformation properties under the symmetry. As Fernando Quevedo might say, "I'm sure you've known this since you were in primary school."

Let's do the analogous analysis for the Poincaré group. This requires a bit more machinery. Unfortunately a proper treatment of the construction of irreducible representations of the Poincaré group would be a lengthy diversion, so we shall only give a heuristic derivation. A proper derivation can be found in the appropriate chapters of Weinberg [9] or Gutowski [10] or Kuzenko and Buchbinder [3]. Let us define the **Pauli-Lubanski** vector,

$$W^\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}. \tag{4.3}$$

We can now define two Casimir operators,

$$C_1 \equiv P^\mu P_\mu \tag{4.4}$$

$$C_2 \equiv W^\mu W_\mu. \tag{4.5}$$

These can be checked explicitly with a bit of effort. The eigenvalue of C_1 is, of course, the particle mass. This is the Casimir operator we expect from the Lorentz group. We will get to the business of interpreting C_2 shortly. From these two we thus label Poincaré irreps by their mass, m, and the eigenvalue of C_2, which we call ω: $|m, \omega\rangle$.

To label elements within an irrep, we need to pick eigenvalues of generators that commute with each other. For example, the momentum operator P^μ,

$$P^\mu|m, \omega; p^\mu\rangle = p^\mu|m, \omega; p^\mu\rangle. \tag{4.6}$$

Are there more labels? Yes. To find these, we need to divide the cases in to massive and massless one-particle representations.

4.1.1 Massive Representations

For the case of massive particles one can always boost into a frame where

$$p^\mu = (m, 0, 0, 0). \tag{4.7}$$

We search for generators that leave $p^\mu = (m, 0, 0, 0)$ invariant. This is given by the generators of the rotation group, $SO(3)$. We say that $SO(3)$ is the **stability group** or the **little group**. This implies that we may use labels j and j^3 as we did before.

This sheds a little light on the nature of W_μ. We notice that $W_0 = 0$ and $W_i = mJ_i$. In the massive representation the Pauli-Lubanski vector does not contain any new information; ω is the same as, for example, j_3.

We may label elements within an irrep as $|m, j; p^\mu, j_3\rangle$. To be clear, this is *precisely* what we mean by a **one-particle state**, i.e. the definition of an elementary particle.

4.1.2 Massless Representations

For massless particles we are unable to boost into a rest frame. The best we can do is boost into a frame where

$$p^\mu = (E, 0, 0, -E). \tag{4.8}$$

Looking at this, we expect once again that the stability group is $SO(2)$. This is indeed correct, though a proper analysis is a lot trickier. Writing out each element of the Pauli-Lubanski vector, one finds

$$W_0 = EJ_3 \tag{4.9}$$

$$W_1 = E(-J_1 + K_2) \tag{4.10}$$

$$W_2 = E(J_2 - K_1) \tag{4.11}$$

$$W_3 = EJ_3, \tag{4.12}$$

from which one can write down the commutation relations

$$[W_1, W_2] = 0 \tag{4.13}$$

$$[W_3, W_1] = iW_2 \tag{4.14}$$

$$[W_3, W_2] = -iW_1. \tag{4.15}$$

This is the algebra for the two dimensional Euclidean group. Evidently the little group is more than just the $SO(2)$ group we originally expected. There is a problem with this, however. This group has infinite-dimensional representations and hence we get a continuum label for each of our massless states. This, in turn, is patently ridiculous since we don't see massless particles with a continuum of states. We thus restrict to finite dimensional representations by imposing

$$W_1 = W_2 = 0. \tag{4.16}$$

If you want you can consider this an 'experimental input[1].' The W_3 generates $O(2)$, as we wanted. Then

$$W^\mu = \lambda P^\mu, \tag{4.17}$$

with λ defining the helicity of the particle. Recalling that the algebra **(FLIP: Work this out ***)** requires $e^{4\pi i\lambda}|\lambda\rangle = |\lambda\rangle$, we know that $\lambda \in \pm\frac{1}{2}, 1\cdots$; i.e. it takes on the value of a half integer. In fact, for a field theory with massless fields in the representation (A, B), the helicity is given by $\lambda = B - A$. (See p. 253 of Weinberg.) Massless particle states can thus be labelled as

$$|0, j; p^\mu, \lambda\rangle. \tag{4.18}$$

4.2 $\mathcal{N} = 1$ SUSY

What happens when we now supersymmetrize our theory? $C_1 = P^2$ is still a Casimir operator, but now $C_2 = W^2$ is no longer a Casimir. This is rather intuitive since we saw that the Pauli-Lubanski vector had to do with spin and supersymmetry mixes particles of different spins into a single irreducible representation. This is, of course, how it evades the Coleman-Mandula theorem.

[1] This argument is certainly unsatisfactory, but it appears to be the best that we can do for the moment.

In place of C_2, we can define another Casimir operator, $\tilde{C}_2$, in a somewhat oblique way:

$$\tilde{C}_2 \equiv C_{\mu\nu} C^{\mu\nu} \tag{4.19}$$

$$C_{\mu\nu} \equiv B_\mu P_\nu - B_\nu P_\mu \tag{4.20}$$

$$B_\mu \equiv W_\mu - \frac{1}{4} \overline{Q}_{\dot{\alpha}} (\sigma_\mu)^{\dot{\alpha}\alpha} Q_\alpha. \tag{4.21}$$

Good students will check, with some pain, that $\tilde{C}_2$ is indeed a Casimir operator. Thus our irreducible representations still have two labels, but the second one isn't really related to spin any longer.

Finding Casimir operators. It is clear that the whole business of finding a complete set of Casimir operators for a spacetime symmetry is rather important. Here we've just written down the results for the Poincaré group and for SUSY. For compact, simple groups it is a bit more straightforward to formulaically determine the Casimirs. For more general groups, on the other hand, there is no clear systematic method. For our purposes we can leave the task of finding a complete set of Casimirs to mathematicians.

4.2.1 Massless Multiplets

As before we can boost into a frame where $p_\mu = (E, 0, 0, E)$. Explicit calculation shows that both Casimir operators vanish,

$$C_1 = \tilde{C}_2 = 0. \tag{4.22}$$

Now consider the now-familiar anticommutator of Q and $\overline{Q}$ and write it out explicitly as

$$\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu = 2E(\sigma^0 + \sigma^4)_{\alpha\dot{\beta}} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.23}$$

In components,

$$\{Q_1, \overline{Q}_{\dot{1}}\} = 4E \tag{4.24}$$

$$\{Q_2, \overline{Q}_{\dot{2}}\} = 0. \tag{4.25}$$

Recall that the $\overline{Q}$ is really short-hand for the complex conjugate of Q. Thus the product $\overline{Q}_{\dot{\alpha}}Q_\alpha$ for $\dot{\alpha} = \alpha$ is something like $|Q_\alpha|^2$ and is non-negative. Thus the second equation tells us that for any massless state $|p_\mu, \lambda\rangle$,

$$Q_2|p_\mu, \lambda\rangle = 0. \tag{4.26}$$

To be explicit, one can write

$$\begin{aligned}
0 &= \langle p_\mu, \lambda|\{Q_2, \overline{Q}_{\dot{2}}\}|p^\mu, \lambda\rangle && (4.27)\\
&= \langle p_\mu, \lambda|Q_2\overline{Q}_{\dot{2}} + \overline{Q}_{\dot{2}}Q_2|p^\mu, \lambda\rangle && (4.28)\\
&= \langle p_\mu, \lambda|Q_2\overline{Q}_{\dot{2}}|p^\mu, \lambda\rangle + \langle p^\mu, \lambda|\overline{Q}_{\dot{2}}Q_2|p^\mu, \lambda\rangle && (4.29)\\
&= \left|\overline{Q}_{\dot{2}}|p_\mu, \lambda\rangle\right|^2 + \left|Q_2|p_\mu, \lambda\rangle\right|^2, && (4.30)
\end{aligned}$$

from which each term on the right hand side must vanish and we get equation (4.26).

Using equation (4.24) we can define raising and lowering operators,

$$a \equiv \frac{Q_1}{2\sqrt{E}} \tag{4.31}$$

$$a^\dagger \equiv \frac{\overline{Q}_{\dot{1}}}{2\sqrt{E}}. \tag{4.32}$$

These satisfy the anticommutation relation $\{a, a^\dagger\} = 1$. We can now consider the spin of a massless state after acting with these operators.

$$\begin{aligned}
J^3 a|p_\mu, \lambda\rangle &= \left(aJ^3 - [a, J^3]\right)\, |p^\mu, \lambda\rangle && (4.33)\\
&= \left(aJ^3 - \frac{1}{2}a\right)\, |p_\mu, \lambda\rangle && (4.34)\\
&= \left(\lambda - \frac{1}{2}\right)\, a|p_\mu, \lambda\rangle. && (4.35)
\end{aligned}$$

In the second line we have used the fact that $[J^3, Q_{1,2}] = \mp\frac{1}{2}Q_{1,2}$. This is just a statement of the helicity of the SUSY generators. Thus if we start with a state $|p_\mu, \lambda\rangle$

of helicity λ, acting with $a \sim Q_1$ produces a state of helicity $(\lambda - \frac{1}{2})$. Similarly, because $[J^3, \overline{Q}_{\dot{1},\dot{2}}] = \pm\frac{1}{2}\overline{Q}_{\dot{1},\dot{2}}$, acting with $a^\dagger \sim \overline{Q}_{\dot{1}}$ produces a state of helicity $(\lambda + \frac{1}{2})$.

Since this is rather important, let's work through this explicitly:

$$[J_3, Q_\alpha] = [M_{12}, Q_\alpha] \tag{4.36}$$
$$= -i(\sigma^{12})_\alpha{}^\beta Q_\beta \tag{4.37}$$
$$= -\frac{i}{4}(\sigma^1\overline{\sigma}^2 - \sigma^2\overline{\sigma}^1)_\alpha{}^\beta Q_\beta \tag{4.38}$$
$$= \frac{i}{4}(\sigma^1\sigma^2 - \sigma^2\sigma^1)_\alpha{}^\beta Q_\beta \tag{4.39}$$
$$= \frac{i}{4} \cdot 2i(\sigma^3)_\alpha{}^\beta Q_\beta \tag{4.40}$$
$$= -\frac{1}{2}(\sigma^3)_\alpha{}^\beta Q_\beta. \tag{4.41}$$

Hence

$$[J^3, Q_{1,2}] = \mp\frac{1}{2}Q_{1,2} \tag{4.42}$$

and thus $a|p_\mu, \lambda\rangle$ has helicity $(\lambda - \frac{1}{2})$. The commutator for $\overline{Q}_{\dot{\alpha}}$ differs, as we saw in equation (3.16). In particular, the *lower* index right-handed generator has an ϵ in its commutator with the generators of Lorentz transformations. One could say that this is because the right-handed spinor index is 'naturally' and upper index in our convention. The result is that

$$[J_3, \overline{Q}_{\dot{\alpha}}] = [M_{12}, Q_{\dot{\alpha}}] \tag{4.43}$$
$$= -i\epsilon_{\dot{\alpha}\dot{\delta}}(\overline{\sigma}^{12})^{\dot{\delta}}{}_{\dot{\beta}}\overline{Q}^{\dot{\beta}} \tag{4.44}$$
$$= -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\delta}}(\sigma^3)^{\dot{\delta}}{}_{\dot{\beta}}Q^{\beta}. \tag{4.45}$$

The presence of the ϵ adds an additional sign, so that we have

$$[J^3, \overline{Q}_{\dot{1},\dot{2}}] = \pm\frac{1}{2}\overline{Q}_{\dot{1},\dot{2}} \tag{4.46}$$

and thus $a^\dagger|p_\mu, \lambda\rangle$ has helicity $(\lambda + \frac{1}{2})$.

Now we're cookin'. Let's build a (super)multiplet. We start with a state that is annihilated by the lowering operator, i.e. a state of minimum helicity $|\Omega\rangle = |p_\mu, \lambda\rangle$ such that $a|\Omega\rangle = 0$. The next state we can construct comes from acting on $|Omega\rangle$ with a creation operator,

$$a^\dagger|\Omega\rangle = |p_\mu, (\lambda + \frac{1}{2})\rangle. \tag{4.47}$$

What next? We could try acting with another creation operator, $a^\dagger a^\dagger|\Omega\rangle$, but $a^\dagger a^\dagger \equiv 0$ from the Grassmann nature of the SUSY generator. To exhaust our possibilities, $aa^\dagger|\Omega\rangle = (1 - a^\dagger a)|\Omega\rangle = |\Omega\rangle$. Thus our massless $\mathcal{N} = 1$ supersymmetry multiplet has only two states, $|p_\mu, \lambda\rangle$ and $|p_\mu, (\lambda + \frac{1}{2})\rangle$. We have paired a bosonic and a fermionic state, so we're happy that this is supersymmetric in an intuitive way. We haven't said anything about what the lowest helicity λ is, and in fact we are free to choose this.

Let us note here that nature respects the discrete CPT symmetry. Thus if we construct a model of a massless supermultiplet that is not CPT self-conjugate, then we are obliged to also add a partner CPT-conjugate multiplet as well. For example, if $\lambda = \frac{1}{2}$, then our construction yields a multiplet with a fermion of helicity $\lambda = \frac{1}{2}$ and a vector partner with helicity $\lambda = 1$. CPT invariance mandates that we must also have a fermion with helicity $\lambda = -\frac{1}{2}$ and a vector partner with helicity $\lambda = -1$. More generally, CPT compels us to fill in our massless multiplets with states $|p_\mu, \pm\lambda\rangle$ and $|p_\mu, \pm(\lambda + \frac{1}{2})\rangle$.

Let us go over some examples of massless supermultiplets.

- **Chiral multiplet**. If we take $\lambda = 0$ we have the multiplets for the Standard Model fermions. These are composed of the states $2|p_\mu, 0\rangle$ (i.e. two such states by CPT) and $|p_\mu, \pm\frac{1}{2}\rangle$. These could represent pairs of squarks and quarks, sleptons and leptons, or Higgses and Higgsinos[2]. One could pause and ask why these particles are massless supermultiplets when we know quarks, leptons, and the Higgs have mass (and their superparners ought to be even heavier to avoid detection) – but just as in the Standard Model, these massless multiplets obtain mass from electroweak symmetry breaking.
- **Gauge multiplet**. If we take $\lambda = \frac{1}{2}$ we have multiplets for the Standard Model gauge bosons. These are composed of the states $|p_\mu, \pm\frac{1}{2}\rangle$ and $|p_\mu, \pm 1\rangle$. These would then represent gauginos and their Standard Model gauge boson counterparts. Since this multiplet contains spin-$\frac{1}{2}$ and spin-1 particles, would it have been more economical to try to fit the entire Standard Model into gauge multiplets? While that would be tidy indeed, this is not possible since the gauge particles are in the adjoint

[2]The SUSY nomenclature should be clear. Scalar partners to Standard Model fermions have an 's-' prefix while fermionic partners to Standard Model bosons have an '-ino' suffix.

representation of the gauge group while the chiral fermions are in the fundamental and antifundamental representations. Further, the fact that the gauge multiplet is in the adjoint gauge representation allows the fermions in this multiplet to be Majorana. Why not pick $\lambda = 1$? We avoid this choice since there is no consistent way to couple spin-$\frac{1}{2}$ particles with spin-1.

- **Gravity multiplet**. We can also consider a supermultiplet containing a spin-2 particle, i.e. a graviton. For this we choose $\lambda = \frac{3}{2}$. We end up with a pair of gravitinos[3] $|p_\mu, \pm\frac{3}{2}\rangle$ and gravitons $|p_\mu, \pm 2\rangle$..

4.2.2 Massive Multiplets

Having fleshed out the massless supermultiplet, let's play the same game for the massive multiplets. In this case we can boost to a particle's rest frame,

$$p_\mu = (m, 0, 0, 0). \tag{4.48}$$

The Casimir operators are given by

$$C_1 = m^2 \tag{4.49}$$

$$\tilde{C}_2 = 2m^4 Y^i Y_i, \tag{4.50}$$

where $Y = J_i - \frac{1}{4m}\left(\overline{Q}\sigma_i Q\right)$ is the **superspin**. The nice feature of the superspin is that

$$[Y_i, Y_j] = i\epsilon_{ijk} Y_k, \tag{4.51}$$

that is they satisfy the same algebra as the angular momentum operators, J_i. Thus we can label a multiplet by its mass m and y, the root of the eigenvalue of Y^2. As before, we can work out the anticommutator of the SUSY generators acting on a state with $p_\mu = (m, 0, 0, 0)$:

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.52}$$

[3]This appears to be the correct pluralization of 'gravitino,' though 'gravitinii' is also acceptable.

We now have *two* sets of raising and lowering operators,

$$a_{1,2} = \frac{1}{\sqrt{2m}} Q_{1,2} \tag{4.53}$$

$$a^\dagger_{1,2} = \frac{1}{\sqrt{2m}} \overline{Q}_{\dot{1},\dot{2}}. \tag{4.54}$$

These satisfy the anticommutation relations

$$\{a_p, a^\dagger_q\} = \delta_{pq} \tag{4.55}$$

$$\{a_p, a_q\} = 0 \tag{4.56}$$

$$\{a^\dagger_p, a^\dagger_q\} = 0. \tag{4.57}$$

As before we define a ground state $|\Omega\rangle$ that is annihilated by both a_1 and a_2, $a_{1,2}|\Omega\rangle = 0$. It is important to note that for the ground state,

$$\mathbf{Y}|\Omega\rangle = \mathbf{J}|\Omega\rangle, \tag{4.58}$$

and so we can label the ground state by

$$|\Omega\rangle = |m, y = j; p_\mu, j_3\rangle. \tag{4.59}$$

The spin in the z-direction, j_3, takes values from $-y$ to y and so there are $(2y+1)$ ground states.

We can now act on $|\Omega\rangle$ with creation operators. Recalling equations (4.42) and (4.46), we see that the resulting states are

$$a^\dagger_1|\Omega\rangle = |m, j = y + \frac{1}{2}; p_\mu, j_3\rangle \tag{4.60}$$

$$a^\dagger_2|\Omega\rangle = |m, j = y - \frac{1}{2}; p_\mu, j_3\rangle. \tag{4.61}$$

We see that $a^\dagger_1|\Omega\rangle$ has $2(y+\frac{1}{2})+1 = 2y+2$ states while $a^\dagger_2|\Omega\rangle$ has $2(y-\frac{1}{2})+1 = 2y$ states. This can be understood group theoretically, since

$$\frac{1}{2} \otimes j = (j - \frac{1}{2}) \oplus (j + \frac{1}{2}) \tag{4.62}$$

We're going to want to keep track of these to make sure that our bosonic and fermionic degrees of freedom match.

Unlike the massless case, we can now form a state with two creation operators,

$$a_1^\dagger a_2^\dagger|\Omega\rangle = -a_2^\dagger a_1^\dagger|\Omega\rangle = |m, j = y; p_\mu, j_3\rangle = |\Omega'\rangle. \tag{4.63}$$

This state looks very similar to the base state Ω, but the two are not equivalent: $\Omega'\rangle$ is annihilated by the $a^\dagger$s rather than the as:

$$a_{1,2}^\dagger|\Omega'\rangle = 0 \tag{4.64}$$

$$a_{1,2}|\Omega\rangle = 0. \tag{4.65}$$

The $a_p^\dagger$ and a_p are related by a parity transformation:

$$\underbrace{a_{1,2}^\dagger}_{(0,\frac{1}{2})} \leftrightarrow \underbrace{a_{1,2}}_{(\frac{1}{2},0)}, \tag{4.66}$$

and so the above equation suggests that $|\Omega\rangle$ and $|\Omega'\rangle$ are also related by parity. Then we can define parity eigenstates

$$|\pm\rangle = |\Omega\rangle \pm |\Omega'\rangle. \tag{4.67}$$

For $y = 0$ the $|+\rangle$ is a scalar while $|-\rangle$ is a pseudoscalar.

Now we'd like to 'check the accounting' and make sure our fermionic and bosonic states have the same number of degrees of freedom. $|\Omega\rangle$ and $|\Omega'\rangle$ each have $2y+1$ states, while $a_{1,2}^\dagger|\Omega\rangle$ give $(2y+1)\pm 1$ states. Hence there sums are each $4y+2$, and hence the number of fermionic and bosonic states are equal.

In summary, for $y > 0$, we have the states

$$|\Omega\rangle = |m, j = y; p_\mu, j_3\rangle \tag{4.68}$$

$$|\Omega'\rangle = |m, j = y; p_\mu, j_3\rangle \tag{4.69}$$

$$a_1^\dagger|\Omega\rangle = |m, j = y + \frac{1}{2}; p_\mu, j_3\rangle \tag{4.70}$$

$$a_2^\dagger|\Omega\rangle = |m, j = y - \frac{1}{2}; p_\mu, j_3\rangle. \tag{4.71}$$

For $y = 0$, we have the states

$$|\Omega\rangle = |m, j = 0; p_\mu, j_3\rangle \tag{4.72}$$

$$|\Omega'\rangle = |m, j = 0; p_\mu, j_3\rangle \tag{4.73}$$

$$a_1^\dagger|\Omega\rangle = |m, j = \frac{1}{2}; p_\mu, j_3 = \pm\frac{1}{2}\rangle. \tag{4.74}$$

That's it for the representations of $\mathcal{N} = 1$ supersymmetry!

4.2.3 Equality of Fermionic and Bosonic States

Let us now prove a rather intuitive statement: In any SUSY multiplet, the number n_B of bosons equals the number n_F of fermions.

We shall make use of the operator $(-)^F$, which assigns a 'parity' to a state depending on whether it is a boson ($|B\rangle$) or fermion ($|F\rangle$):

$$(-)^F|B\rangle = |B\rangle \tag{4.75}$$

$$(-)^F|F\rangle = -|F\rangle. \tag{4.76}$$

This operator is sometimes written using less-elegant notation like $(-1)^{n_F}$.

We note that this operator anticommutes with SUSY generators since

$$(-)^F Q_\alpha|F\rangle = (-)^F|B\rangle = |B\rangle = Q_\alpha|F\rangle = -Q_\alpha(-)^F|F\rangle. \tag{4.77}$$

Let us now calculate the following curious-looking trace:

$$\mathrm{Tr}\left\{(-)^F\{Q_\alpha, \overline{Q}_{\dot\beta}\}\right\} = \mathrm{Tr}\left\{(-)^F Q_\alpha\overline{Q}_{\dot\beta} + (-)^F\overline{Q}_{\dot\beta}Q_\alpha\right\} \tag{4.78}$$

$$= \mathrm{Tr}\left\{ \underbrace{-Q_\alpha(-)^F\overline{Q}_{\dot\beta}}_{\text{Using anticommutator}} + \underbrace{Q_\alpha(-)^F\overline{Q}_{\dot\beta}}_{\text{Using cyclicity of trace}} \right\} \tag{4.79}$$

$$= 0. \tag{4.80}$$

But since $Q_\alpha, \overline{Q}_{\dot\beta} = 2(\sigma^{mu})_{\alpha\dot\beta}P_\mu$, the above trace is

$$\mathrm{Tr}\left\{(-)^F 2(\sigma^{mu})_{\alpha\dot\beta}P_\mu\right\} = 2(\sigma^{mu})_{\alpha\dot\beta}P_\mu\mathrm{Tr}\left((-)^F\right), \tag{4.81}$$

and hence $\text{Tr}\left((-)^F\right) = 0$. This trace is called the **Witten index** and will play a central role we study SUSY breaking in Chapter 6. The Witten index can be written more explicitly as a sum over bosonic and fermionic states,

$$\text{Tr}\left((-)^F\right) = \sum_B \langle B|(-)^F|B\rangle + \sum_F \langle F|(-)^F|F\rangle \tag{4.82}$$

$$= \sum_B \langle B|B\rangle - \sum_F \langle F|F\rangle \tag{4.83}$$

$$= n_B - n_F. \tag{4.84}$$

Thus the vanishing of the Witten index implies that $n_B = n_F$, or that there are an equal number of bosonic and fermionic states.

4.2.4 Massless $\mathcal{N} > 1$ Representations

Let's move on to $\mathcal{N} > 1$ representations. This is a bit outside the scope of a typical introductory SUSY course, but a lot of recent developments in field theory have come from looking at $\mathcal{N} > 1$ SUSY so we'll take some time to introduce it. The motivation, to be clear, is formal rather than phenomenological.

For massless representations, once again we can boost to a frame $p_\mu = (E, 0, 0, E)$ and the anticommutator acting on this state is the same as before with the addition of a δ function,

$$\{Q^A_\alpha, \overline{Q}^B_{\dot\beta}\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta^A_B. \tag{4.85}$$

Thus, by the same arguments as the $\mathcal{N} = 1$ massless representation, $Q^A_2 = \overline{Q}^A_{\dot 2} = 0$. But then recall the anticommutator for the central charge,

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{AB}. \tag{4.86}$$

Since $Q_2 = 0$ the right-hand side is always zero and the central charges play no role in the massless multiplet. We can now define $\mathcal{N}$ pairs of raising and lowering operators

$$a^A = \frac{1}{\sqrt{4E}} Q_1^A \tag{4.87}$$

$$a^{\dagger A} = \frac{1}{\sqrt{4E}} \overline{Q}_{\dot{1}}^A, \tag{4.88}$$

with the anticommutation relation

$$\{a^A, a_B^\dagger\} = \delta_B^A. \tag{4.89}$$

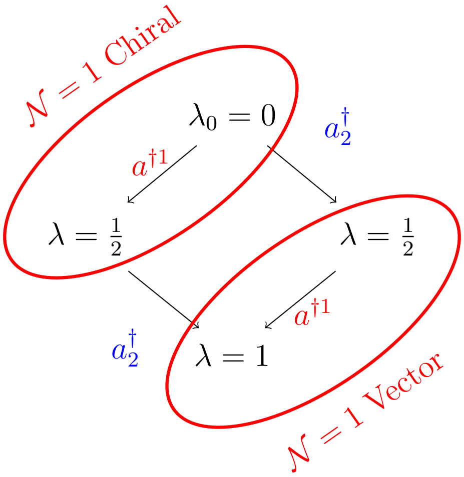
Recall that the positions of the A, B labels are irrelevant. By now you know what's coming. We define a base state $|\Omega\rangle$ such that $a^A|\Omega\rangle = 0$ and start building up our multiplet by acting with creation operators. With $\mathcal{N}$ different raising operators, counting states becomes an exercise in counting:

State	Helicity	Degrees of Freedom
$\lvert\Omega\rangle$	λ_0	$1 = \binom{\mathcal{N}}{0}$
$a_A^\dagger\lvert\Omega\rangle$	$\lambda_0 + \frac{1}{2}$	$\mathcal{N} = \binom{\mathcal{N}}{1}$
$a_A^\dagger a_B^\dagger\lvert\Omega\rangle$	$\lambda_0 + 1$	$\frac{1}{2}\mathcal{N}(\mathcal{N}-1) = \binom{\mathcal{N}}{2}$
$a_A^\dagger a_B^\dagger a_C^\dagger\lvert\Omega\rangle$	$\lambda_0 + \frac{3}{2}$	$\cdots = \binom{\mathcal{N}}{3}$
$\vdots$	$\vdots$	$\vdots$
$a_1^\dagger \cdots a_{\mathcal{N}}^\dagger\lvert\Omega\rangle$	$\lambda_0 + \frac{\mathcal{N}}{2}$	$1 = \binom{\mathcal{N}}{\mathcal{N}}$

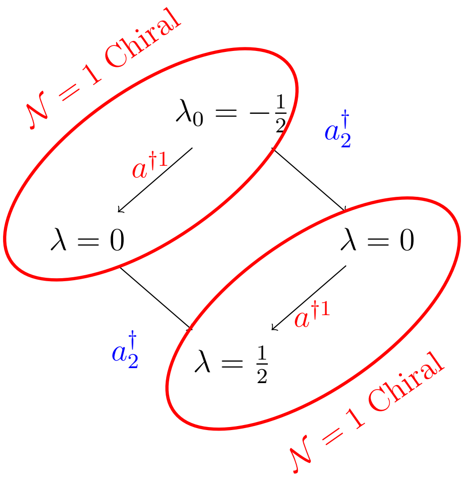
We see that the total number of states (number of degrees of freedom) is given by

$$\sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}}. \tag{4.90}$$

For $\mathcal{N} = 2$ we can chart the supermultiplet. For example, for the $\mathcal{N} = 2$ **vector multiplet** has $\lambda_0 = 0$, we have:



Notice that the $\mathcal{N} = 2$ vector multiplet is composed of an $\mathcal{N} = 1$ chiral multiplet and an $\mathcal{N} = 1$ vector multiplet. We can draw the analogous diagram for the $\mathcal{N} = 2$ **hypermultiplet**, which starts with $\lambda_0 = -\frac{1}{2}$.



This multiplet is composed of two $\mathcal{N} = 1$ chiral multiplets of opposite helicity, hence the hypermultiplet has the nice feature of being CPT self-conjugate.

Next we can write out the $\mathcal{N} = 4$ **vector multiplet**, which has a base helicity of $\lambda_0 = -1$. Let us write out the states:

	$\lambda = -1$	$\lambda = -\frac{1}{2}$	$\lambda = 0$	$\lambda = \frac{1}{2}$	$\lambda = 1$
# of States	1	4	6	4	1

This is rather special as it is the *only* multiplet for a renormalizable $\mathcal{N} = 4$ SUSY theory. What about $\mathcal{N} = 3$? The spectrum of $\mathcal{N} = 3$ SUSY (with its CPT conjugate) coincides exactly with the $\mathcal{N} = 4$ vector multiplet and hence the quantum field theories are identical.

This brings us to a natural point to make some general comments about extended SUSY multiplets.

- First of all, note that for every multiplet

$$\lambda_{\max} - \lambda_{\min} = \mathcal{N}/2. \tag{4.91}$$

  This is straightforward since each creation operator $a^{A\dagger}$ raises the helicity by $+\frac{1}{2}$.
- In quantum field theory, renormalizability imposes that the maximum helicity is $\lambda = 1$. Thus the maximum number of supersymmetries in a renormalizable theory is $\mathcal{N} = 4$. (This is why we said that $\mathcal{N} = 4$ is special.)
- We have a "strong belief" that there are no massless particles of helicity $|\lambda| > 2$. This is because there is no conserved current for such a particle to couple to. The general argument is that massless particles with $|\lambda| > \frac{1}{2}$ must couple at low momentum to conserved quantities. For example, $|\lambda| = 1$ couples to the electric or color currents j^μ. For $|\lambda| = 2$, the graviton can couple to the energy-momentum tensor. Beyond this there are no conserved currents for a higher-spin particle to couple to. A further discussion of this can be found in Weinberg I, Section 13.1 [11].
- We also strongly believe that the maximum number of supersymmetries is $\mathcal{N} = 8$, corresponding to one graviton and $\mathcal{N} = 8$ gravitinos. If $\mathcal{N} > 8$ then we would have an uncomfortably large number of gravitons. $\mathcal{N} = 8$ SUSY has the following states:

	$\|\lambda\| = 2$	$\|\lambda\| = -\frac{3}{2}$	$\|\lambda\| = 1$	$\|\lambda\| = \frac{1}{2}$	$\|\lambda\| = 0$
# of States	1	8	28	56	70

- Extended SUSY is usually not considered to be phenomenologically relevant at, say, the TeV scale since all $\mathcal{N} > 1$ theories are non-chiral and hence would have difficulty reproducing the chiral nature of the Standard Model at low energies.

4.2.5 Massive $\mathcal{N} > 1$ Representations with $Z^{AB} = 0$

By now we're old pros at building multiplets. For the case where there are no central charges, we may follow the same steps that we took for the massive $\mathcal{N} = 1$ multiplet, just being careful to account for the $\mathcal{N} > 1$ different sets of SUSY generators. We boost into a rest frame $p_\mu = (m, 0, 0, 0)$ and write out the anticommutation relation

$$\{Q^A_\alpha, \overline{Q}^B_{\dot\beta}\} = 2(\sigma^0)_{\alpha\dot\beta}\, m\, \delta^A_B = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta^A_B. \tag{4.92}$$

We find $2\mathcal{N}$ pairs of creation and annihilation operators,

$$a^A_{1,2} = \frac{1}{\sqrt{2m}} Q^A_{1,2} \tag{4.93}$$

$$a^{\dagger A}_{1,2} = \frac{1}{\sqrt{2m}} \overline{Q}^A_{\dot 1,\dot 2}. \tag{4.94}$$

We thus have $2^{2\mathcal{N}}$ states of a given superspin y, and hence a total of $2^{2\mathcal{N}} \times (2y+1)$ states. Be careful with the 1,2 indices: recall from equations (4.42) and (4.46) that these correspond to different helicities. In particular, $a^{\dagger A}_1$ will *raise* helicity by $\frac{1}{2}$ while $a^{\dagger A}_2$ will *lower* helicity by $\frac{1}{2}$.

Hence we can write out the example of the $\mathcal{N} = 2$ multiplet for with $y = 0$:

$\|\Omega\rangle$	1 state	spin-0
$a^{\dagger A}_{1,2}\|\Omega\rangle$	4 states	spin-$\frac{1}{2}$
$a^{\dagger A}_{1,2}a^{\dagger B}_{1,2}\|\Omega\rangle$	3 states	spin-0
	3 states	spin-1
$a^{\dagger A}_{1,2}a^{\dagger B}_{1,2}a^{\dagger C}_{1,2}\|\Omega\rangle$	4 states	spin-$\frac{1}{2}$
$a^{\dagger A}_{1,2}a^{\dagger B}_{1,2}a^{\dagger C}_{1,2}a^{\dagger D}_{1,2}\|\Omega\rangle$	1 states	spin-0

We end up with $16 = 2^4$ total states. Aside from being careful with the helicities being raised and lowered (as opposed to only raised), this follows straightforwardly from our previous analyses of the $\mathcal{N} = 1$ massive representations and the $\mathcal{N} > 1$ massless representations.

4.2.6 Massive $\mathcal{N} > 1$ Representations with $Z^{AB} \neq 0$

We get much more interesting properties in the case where there are central charges. Let us define the following objects

$$\begin{aligned} \mathcal{H} &\equiv (\overline{\sigma}^0)^{\dot{\beta}\alpha} \left\{ Q^A_\alpha - \Gamma^A_\alpha, \overline{Q}_{\dot{\beta}A} - \overline{\Gamma}_{\dot{\beta}A} \right\} && (4.95)\\ \Gamma^A_\alpha &\equiv \epsilon_{\alpha\beta}\, U^{AB}\, \overline{Q}_{\dot{\gamma}B} (\overline{\sigma}^0)^{\dot{\gamma}\beta}. && (4.96) \end{aligned}$$

Here U is a unitary matrix, $U^\dagger U = \mathbb{1}$. Thus Γ^A_α is essentially $\overline{Q}$ with objects contracted to change the index structure. Note further that $\mathcal{H} \geq 0$ since it is of the form $X\overline{X} = |X|^2$.

Now using the extended SUSY algebra, we can explicitly calculate

$$\mathcal{H} = \underbrace{8m\mathcal{N}}_{\text{from } \{Q,\overline{Q}\}} - \underbrace{2\mathrm{Tr}(ZU^\dagger + UZ^\dagger)}_{\text{from } \{Q,Q\} \text{ and } \{\overline{Q},\overline{Q}\}} \geq 0. \qquad (4.97)$$

We may now polar decompose the matrix $Z = HV$, with H Hermitian and V unitary. We choose $U = V$, so that

$$\mathcal{H} = 8m\mathcal{N} - 4\mathrm{Tr}H \geq 0, \qquad (4.98)$$

or in other words,

$$m \geq \frac{1}{2\mathcal{N}}\,\mathrm{Tr}H = \frac{1}{2\mathcal{N}}\,\mathrm{Tr}\sqrt{ZZ^\dagger}. \qquad (4.99)$$

This is the Bogomolnyi-Prasard-Sommerfeld (BPS) bound on the masses and is something you should remember for the rest of your life. If the BPS bound is saturated, i.e.

$$m = \frac{1}{2\mathcal{N}}\,\mathrm{Tr}\sqrt{ZZ^\dagger}, \qquad (4.100)$$

then the states satisfying this condition are called **BPS states**. For such states we have

$$\mathcal{H} = 0 \quad \Rightarrow \quad \{Q^A_\alpha - \Gamma^A_\alpha, \overline{Q}_{\dot{\beta}A} - \overline{\Gamma}_{\dot{\beta}A}\} = 0. \qquad (4.101)$$

Compare this to the massless multiplets we discussed earlier where we had $\{Q^A_2, \overline{Q}^A_{\dot{2}}\} = 0$ implying $Q^A_2 = 0$ and hence we had fewer creation operators and fewer states. The exact same effect is occurring here, where equation (4.101) is telling us that

$$Q^A_\alpha - \Gamma^A_\alpha = 0 \tag{4.102}$$

and hence we have a reduced number of creation and annihilation operators. In fact, we have $\mathcal{N}$ pairs of a and $a^\dagger$ operators, which means we have $2^{\mathcal{N}}$ states. Compare this to the massive multiplets with no central charges in which we found there are $2^{2\mathcal{N}}$ states. The lesson is that BPS multiplets are shorter than non-BPS multiplets.

For the case $\mathcal{N} = 2$, we may write

$$Z^{AB} = \begin{pmatrix} 0 & q_1 \\ -q_1 & 0 \end{pmatrix}, \tag{4.103}$$

from which we find the BPS bound

$$m \geq \frac{1}{2} q. \tag{4.104}$$

For $\mathcal{N} > 2$ with $\mathcal{N}$ even, we can block-diagonalize Z^{AB} and constrain our multiplets block-by-block.

$$Z^{AB} = \begin{pmatrix} 0 & q_1 & & & \\ -q_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & q_{\mathcal{N}/2} \\ & & & q_{\mathcal{N}/2} & 0 \end{pmatrix}, \tag{4.105}$$

from which we have the BPS bound

$$m \geq \frac{1}{2} q_i. \tag{4.106}$$

If k of the q_i satisfy $q_i = 2m$ then there exists $2\mathcal{N} - 2k$ creation and annihilation operators ($k < \mathcal{N}/2$), i.e. there are $2^{2(\mathcal{N}-k)}$ states.

Table 4.1 summarizes the representations of $\mathcal{N} > 1$ SUSY.

Mass	Condition	# States	Name
Massless		$2^{\mathcal{N}}$	Massless Multiplet
Massive	$Z^{AB} = 0$	$2^{2\mathcal{N}}$	Massive Multiplet
Massive	$k = 0$	$2^{2\mathcal{N}}$	Long Multiplet
Massive	$0 < k < \frac{\mathcal{N}}{2}$	$2^{2(\mathcal{N}-k)}$	Short Multiplet
Massive	$k = \frac{\mathcal{N}}{2}$	$2^{\mathcal{N}}$	Ultra-short Multiplet

Table 4.1: Representations of $\mathcal{N} > 1$ SUSY.

Some general remarks on BPS states are now in order to explain why all of this is important.

- BPS states and BPS bounds have their origin in soliton solutions of Yang-Mills systems. Solitons are nonperturbative field configurations that can be thought of as "classical" versions of particles.
- The BPS bound refers to an energy bound.
- BPS states are stable. They are the lightest objects that carry central charge.
- The equivalence of charge and mass (up to a factor of 2) in BPS states i reminiscent of charged black holes. In fact, extremal black holes are stable BPS solutions to supergravity theories.
- BPS states are important in strong/weak coupling dualities in string and field theory.
- In string theory, D-branes are BPS states.

Chapter 5

Superfields and Superspace

"So in supersymmetry, you have superfields and superpotentials and everything is 'super.' At some point this naming convention becomes rather ridiculous, doesn't it? Why not 'hyper'? I'll invent my own theory and call it 'hypersymmetry;' then everything will be 'hyper.'"
— Steffen Gielen, 2007 Mayhew Prize Recipient

So far we've been doing purely algebraic manipulations. We know the characters of the play, but we need a field theory to provide a script describing the dynamics of these objects. Superspace, developed by Strathdee and Salam in 1974 [12, 13], is a convenient way to do this.

Here we'll go over the necessary tools for $\mathcal{N} = 1$ (global) superspace. A much more general and thorough treatment can be found in DeWitt's text [14] while technical details for the mathematically-inclined can be found in the notes by Gieres [15]. It is critical to note that there are no (satisfactory) standard sign and phase conventions in the literature for the material in this chapter. We will be self-consistent, but it is unlikely that we will coincide with any other text[1]. A *very* useful convention-independent derivation is presented in Binetruy's text [17]. We reproduce parts of this derivation in Appendix B.4.

[1] It's also unlikely that any two texts will agree. In fact, there are even some texts, e.g. [16], whose conventions differ for different chapters!

5.1 Coset spaces

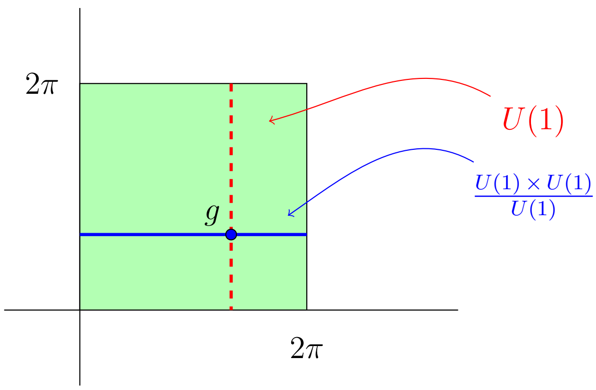
In many standard treatments of field theory, one begins by defining Minkowski space and then discussing its symmetries. We would like to turn this idea around and instead use symmetries to define a space.

One should already be familiar with the idea that Lie groups (i.e. continuous symmetries) are also manifolds[2]. For example, for the group $G = U(1)$, we may write $g = e^{i\alpha}$ with $\alpha \in [0, 2\pi]$. Thus the manifold associated with G is a circle, $M_{U(1)} = S^1$. Similarly, one finds that the manifold associated with $SU(2)$ is a 3-sphere, $M_{SU(2)} = S^3$.

Cosets, G/H (or "elements of G that aren't in H"), can be used to define more general manifolds. A coset is composed of equivalence classes,

$$g \equiv gh, \quad \forall h \in H. \tag{5.1}$$

This coset can be used to define submanifolds of G. For example S^2 is given by $SU(2)/U(1)$. We may draw this heuristically:



Here the x- and y-axes represent the transformation parameters for the $SU(2)$ generators. The manifold for $SU(2)$ is represented by the light green square. The dotted red line represents a section of $U(1)$ that we would like to identify as part of the equivalence class for a point g. The solid blue line represents the coset $SU(2)/U(1)$. More generally, we may write $S^n = SO(n+1)/SO(n)$.

We would like to use a cosets space to define superspace through supersymmetry (or 'super Poincaré' symmetry). As an illustrative example, we may define Minkowski space

[2]For a thorough refresher one is referred to the notes for Jan Gutowski's Part III course, *Symmetries and Particle Physics* [10].

as the coset space 'Poincaré/Lorentz', or $P/SO(3,1)^{\uparrow}$ where P is the Poincaré group[3]. This is an intuitive statement since one can map the generators of translations with points on Minkowski space. In slightly more rigor, the generators of the Poincaré group take the form

$$g_P = e^{i(\omega_{\mu\nu}M^{\mu\nu}+a_\mu P^\mu)}, \tag{5.2}$$

while the generators of the Lorentz group take the form

$$g_L = e^{i(\omega_{\mu\nu}M^{\mu\nu})}. \tag{5.3}$$

One can thus identify the coset manifold with the translation parameters,

$$M_{\text{Poincaré/Lorentz}} = \{a^\mu\}. \tag{5.4}$$

Multiplication of group elements correspond to successive translations on the Minkowski manifold. This is, of course, a bit of overkill for the rather trivial case of Minkowski space.

We now generalize this idea to an (arguably) non-trivial case: the coset space ($\mathcal{N} = 1$ super-Poincaré)/Lorentz, or $SP/SO(3,1)^{\uparrow}$. We call the resulting manifold $\mathcal{N} = 1$ **superspace**. The generators of the super-Poincaré group take the form

$$g_{SP} = e^{i(\omega_{\mu\nu}M^{\mu\nu}+a_\mu P^\mu+\theta^\alpha Q_\alpha+\overline{\theta}_{\dot\alpha}\overline{Q}^{\dot\alpha})}, \tag{5.5}$$

were $\omega_{\mu\nu}$ and a_μ are the usual c-number[4] parameters for the Poincaré group while θ and $\overline{\theta}$ are anticommuting Grassmann parameters. Thus we may write coordinates for $\mathcal{N} = 1$ superspace as

$$\{a^\mu, \theta^\alpha, \overline{\theta}_{\dot\alpha}\}. \tag{5.6}$$

In this sense, supersymmetry is a kind of *fermionic* extra dimension. The products θQ and $\overline{\theta Q}$ are commuting objects, and so we may write the SUSY algebra using commutators,

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2(\sigma^\mu)_{\alpha\dot\alpha}P_\mu \Rightarrow [\theta^\alpha Q_\alpha, \overline{\theta}^{\dot\beta}\overline{Q}_{\dot\beta}] = 2\theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}P_\mu. \tag{5.7}$$

[3]We form the coset using the part of the Lorentz group connected to the identity since this is the part of the Lorentz group included in Poincaré symmetry.
[4]Short for 'commuting' number.

This will allow us to apply useful results from non-graded Lie algebras, such as the Baker-Campbell-Hausdorff formula for the product of exponentiated generators.

> **Minkowski space and superspace as a coset**. For those who would prefer a slightly more rigorous treatment, one may follow the argument of Section 2.4.1 of Buchbinder and Kuzenko [3]. An even more formal mathematical treatment can be found in the first chapter of [15].

Armed with this spacetime extended by Grassmann coordinates, we may proceed to define **superfields** as a generalization of the usual fields that live on Minkowski space. We will see in Section 5.3 that these fields contain entire SUSY multiplets of component Minkowski-space fields. This will be the 'punchline' of what may presently seem like excessive formalism.

5.2 The Calculus of Grassmann Numbers

Now that we've generalized Minkowski space to superspace, we would like to write Lagrangian densities on superspace such that the action is given by an integration over $d^4x\, d^2\theta\, d^2\overline{\theta}$. In order to do this we'll have to familiarize ourselves with the calculus of Grassmann variables. For further references, see DeWitt [14] and Gie.

5.2.1 Scalar Grassmann Variable

We may expand a function of a single Grassmann variable, θ, by Taylor expanding,

$$f(\theta) = f_0 + f_1\theta + f_2\theta^2 + \cdots . \tag{5.8}$$

By the antisymmetry of θ, the f_2 term and all higher terms vanish. Hence the most general function of a single Grassmann variable can be written as

$$f(\theta) = f_0 + f_1\theta. \tag{5.9}$$

We now define calculus for Grassmann variables. With the expansion above, we can define differentiation with respect to θ in the natural way,

$$\frac{df}{d\theta} = f_1. \tag{5.10}$$

It is trickier to define integration over Grassmann variables. The integration operator on superspace is called the **Berezin integral**. To motivate this integration, we note that integration over $\mathbb{R}$ is translation invariant,

$$\int_{-\infty}^{\infty} dx\, f(x) = \int_{-\infty}^{\infty} dx\, f(x+a). \tag{5.11}$$

We would like to carry over this sense of translation-invariance for our $d\theta$ integral,

$$\int d\theta\, f(\theta) = \int d\theta\, f(\theta + \alpha). \tag{5.12}$$

Using the expansion of equation (5.9), this translates to

$$\int d\theta\, f_0 + f_1\theta = \int d\theta\, f_0 + f_1\theta + f_1\alpha, \tag{5.13}$$

from which we conclude

$$\int d\theta, f_1\alpha = 0. \tag{5.14}$$

Thus we define the integrals

$$\int d\theta = 0 \qquad\qquad \int d\theta\, \theta = 1. \tag{5.15}$$

If you want you can interpret the first equation to mean that the space spanned by θ has no boundary, while the second equation is an arbitrary normalization condition that we choose to be non-zero so that integration is a non-trivial operation. Thus we may summarize the Berezin integral by the rule

$$\int d\theta\, f(\theta) = \int d\theta\, (f_0 + f_1\theta) = f_1 = \frac{df}{d\theta}. \tag{5.16}$$

We see that derivatives and integrals of Grassmann variables are equivalent.

5.2.2 Spinor Grassmann Variables

Superspace extends Minkowski space with two spinor degrees of freedom, θ_α and $\theta_{\dot{\alpha}}$, so we ought to establish conventions for the calculus of Weyl spinor variables. We shall follow our previously defined convention for the contraction of left- and right-handed spinor coordinates,

$$\theta\theta \equiv \theta^\alpha\theta_\alpha \qquad \overline{\theta\theta} \equiv \overline{\theta}_{\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}. \tag{5.17}$$

Antisymmetry allows us to write out products of spinor components in terms of the antisymmetric tensor times the spinor contractions as follows,

$$\theta_\alpha\theta_\beta = -\frac{1}{2}\epsilon_{\alpha\beta}\theta\theta \qquad \overline{\theta}_{\dot{\alpha}}\overline{\theta}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\overline{\theta\theta}. \tag{5.18}$$

As before we may define differentiation in the usual way,

$$\frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta^\beta_\alpha \qquad \frac{\partial}{\partial\overline{\theta}_{\dot{\alpha}}}\overline{\theta}_{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}}. \tag{5.19}$$

Note that $\partial/\partial\theta^\alpha$ transforms as a lower-index left-handed spinor (i.e. ψ_α-type) and $\partial/\partial\overline{\theta}_{\dot{\alpha}}$ transforms as an upper-index right-handed spinor (i.e. $\overline{\chi}^{\dot{\alpha}}$-type). This is completely analogous to the case of vector derivatives where $\partial/\partial x^\mu$ transforms as a lower-index object. Following the convention of equation (5.19), however, we run into an immediate issue of consistency that requires some care. Suppose we naively defined the $\partial/\partial\theta_\alpha$ and $\partial/\partial\overline{\theta}^{\dot{\alpha}}$ partial derivatives in the same way. Then we'd run into problems since (ignoring the index height on the Kronecker δ),

$$\frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta^\beta_\alpha \stackrel{?}{=} \frac{\partial}{\partial\theta_\alpha}\theta_\beta, \tag{5.20}$$

while we also have, from equation (2.59),

$$\frac{\partial}{\partial\theta^\alpha}\theta^\beta = -\frac{\partial}{\partial\theta_\alpha}\theta_\beta. \tag{5.21}$$

The only way for equations (5.20) and (5.21) to be consistent is if both types of derivatives are identically zero. Thus we are led to the following definitions for the lower/upper-index spinor derivatives,

$$\frac{\partial}{\partial\theta_\alpha} = -\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta} \qquad\qquad \frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} = -\epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\overline{\theta}_{\dot{\beta}}}. \tag{5.22}$$

For consistency we must also define the complex conjugate relations

$$\left(\frac{\partial}{\partial\theta^\alpha}\right)^* = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} \qquad\qquad \left(\frac{\partial}{\partial\overline{\theta}_{\dot{\alpha}}}\right)^* = -\frac{\partial}{\partial\theta_\alpha} \tag{5.23}$$

We define the two-dimensional integral as follows,

$$\int d^2\theta \equiv \frac{1}{2}\int d\theta^1\, d\theta^2, \tag{5.24}$$

where the factor of $\frac{1}{2}$ comes from writing out

$$1 = \int d\theta^1 d\theta^2\, \theta^2\theta^1 = \frac{1}{2}\int d\theta^1 d\theta^2\, \theta\theta, \tag{5.25}$$

and thus with this normalization we have

$$\int d^2\theta\, (\theta\theta) = 1. \tag{5.26}$$

We use the same normalization for the right-handed superspace coordinates, and can thus write the integral over both θ and $\overline{\theta}$ as

$$\int d^2\theta \int d^2\overline{\theta}\, (\theta\theta)(\overline{\theta\theta}) = \int d^4\theta\, (\theta\theta)(\overline{\theta\theta}) = 1, \tag{5.27}$$

where we have defined measure $d^4\theta = d^2\theta\, d^2\overline{\theta}$.

It is, perhaps, worth emphasizing that the factor of $\frac{1}{2}$ above is a normalization condition on the Grassmann measure, and not some application of equations (5.18). If one wanted to use those expressions on the measure, then one could write

$$d^2\theta = -\frac{1}{4}d\theta^\alpha d\theta^\beta\epsilon_{\alpha\beta} \qquad\qquad d^2\overline{\theta} = \frac{1}{4}d\overline{\theta}^{\dot{\alpha}} d\overline{\theta}^{\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\beta}}. \tag{5.28}$$

The equivalence of the Berezin integral and the Grassmann derivative can be cast in the form

$$\int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} \qquad \int d^2\overline{\theta} = -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\overline{\theta}^{\dot{\beta}}}. \tag{5.29}$$

This will be rather useful as it will allow us to write out particular terms in the Taylor expansion of a function on superspace by performing superspace integrals.

Finally, we can introduce an inner product for superfields,

$$\langle F(x,\theta,\overline{\theta})\,,\,G(x,\theta,\overline{\theta})\rangle = \int d^4x\,d^4\theta\,F^*(x,\theta,\overline{\theta})\,G(x,\theta,\overline{\theta}). \tag{5.30}$$

This means that we can also define a superspace Hermitian conjugation operation, †. For example, using integration by parts the Hermitian conjugate of the (Minkowski) spacetime derivative behaves as

$$\partial_\mu{}^\dagger = -\partial_\mu. \tag{5.31}$$

This Hermitian conjugation is antilinear (i.e. it "represents an involutive anti-homomorphism"), for complex coefficients a, b and superfields ϕ, ψ,

$$(a\phi + b\psi)^\dagger = \phi^\dagger a^* + \psi^\dagger b^* \tag{5.32}$$

$$(\phi\psi)^\dagger = \psi^\dagger\phi^\dagger. \tag{5.33}$$

At this point I really have to apologize. I made a big deal on page 15 about stars and daggers being essentially the same thing: classical fields could get starred (complex conjugated) while quantum fields, being composed of operators, could get daggered (Hermitian conjugated). I tried to express that different books use different notation, but that we could afford to be nonchalant about this. Unfortunately, we've now defined yet another kind of dagger that is very different from the star. Note that the superspace Hermitian conjugation is defined with respect to the superspace inner product of (classical) superfields and is *completely* different from the Hermitian conjugation associated with quantum operators. If one wanted to one could write them separately as † and ‡, though this introduces a lot of clutter. Fortunately, we will use this superspace Hermitian conjugate in Section 5.3 and after that we can forget all of these little technical details.

∂_α and $\overline{\partial}^{\dot{\alpha}}$ Notation. Some references use the following shorthand notation:

$$\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} \qquad \partial^\alpha \equiv -\epsilon^{\alpha\beta}\partial_\beta \tag{5.34}$$

$$\overline{\partial}^{\dot{\alpha}} \equiv \frac{\partial}{\partial\overline{\theta}_{\dot{\alpha}}} \qquad \overline{\partial}_{\dot{\alpha}} \equiv -\epsilon_{\dot{\alpha}\dot{\beta}}\overline{\partial}^{\dot{\beta}}. \tag{5.35}$$

Thus we write an 'intuitive' relation between bars and stars,

$$\partial_\alpha{}^* = \overline{\partial}_{\dot{\alpha}} \qquad \overline{\partial}^{\dot{\alpha}*} = \partial^\alpha. \tag{5.36}$$

With this notation we can forget about the overall sign that appears when raising or lowering indices of differential operators. While this notation can be helpful, we will not implement it since it introduces an added layer of specialized notation that may make it more difficult to compare these notes to other references.

5.3 $\mathcal{N} = 1$ Superfields

Ok, so we've slogged through a lot of somewhat unusual formalism. Hang in there, we've almost arrived at the elegant part.

We can now define superfields as scalar functions of superspace. One could also define superfields of non-trivial spin, but this will not be necessary for our purposes and we will assume all superfields are spin-0. The novel feature of these superfields is that they are complete SUSY multiplets and so contain (Minkowski space) fields of different spins.

5.3.1 Expansion of $\mathcal{N} = 1$ Superfields

The key point is that we may Taylor expand a superfield $S(x^\mu, \theta^\alpha, \overline{\theta}_{\dot{\alpha}})$ in the Grassmann variables,

$$\begin{aligned} S(x^\mu, \theta^\alpha, \overline{\theta}_{\dot{\alpha}}) = \quad & a(x) + \theta^\alpha b_\alpha(x) + \overline{\theta}_{\dot{\alpha}}\overline{c}^{\dot{\alpha}}(x) + \theta\theta\, d(x) + \overline{\theta\theta}\, e(x) \\ & + \theta^\alpha f_{\alpha\dot{\beta}}(x)\overline{\theta}^{\dot{\beta}} + \theta\theta\,\overline{\theta}_{\dot{\alpha}}\overline{g}^{\dot{\alpha}}(x) + \overline{\theta\theta}\,\theta^\alpha h_\alpha(x) + \theta\theta\,\overline{\theta\theta}\, j(x). \end{aligned} \tag{5.37}$$

we've written out the components $a(x), b_\alpha(x), \cdots$ that are *normal* (not-super) fields on *Minkowski* space and we see that the Taylor expansion requires them to take on certain spin structures. In this way a single superfield contains a complete SUSY multiplet of different-spin fields. That the form of this expansion is completely general. Terms like $\theta^\alpha \, s_\alpha{}^\beta(x)\theta_\beta$ can be written as a contribution to $d(x)$ using, for example, relations like equations (5.18). Further, by equation (2.67), we may write the field $f_{\alpha\dot\beta}(x)$ as a vector,

$$f_{\alpha\dot\beta}(x) = V_\mu(x)(\sigma^\mu)_{\alpha\dot\beta}. \tag{5.38}$$

Thus let us rewrite our superfield expansion using the standard (historical) notation,

$$\begin{aligned} S(x^\mu, \theta^\alpha, \overline{\theta}_{\dot\alpha}) = \quad & \varphi(x) + \theta\psi(x) + \overline{\theta}\overline{\chi}(x) + \theta\theta\, M(x) + \overline{\theta\theta}\, N(x) \\ & + (\theta\sigma^\mu\overline{\theta})\, V_\mu(x) + \theta\theta\,\overline{\theta\lambda}(x) + \overline{\theta\theta}\,\theta\rho(x) + \theta\theta\,\overline{\theta\theta}\, D(x), \end{aligned} \tag{5.39}$$

where we have suppressed spinor indices using our convention for the contraction of those indices.

It is worth emphasizing once again that equation (5.39) is completely general. One might be concerned about the absence of terms like $\overline{\theta}\overline{\sigma}^\mu\theta$ or $\theta\sigma^{\mu\nu}\theta$. These, however, don't contribute anything new, since

$$\overline{\theta}\overline{\sigma}^\mu\theta = -\theta\sigma^\mu\overline{\theta} \tag{5.40}$$

$$\theta\sigma^{\mu\nu}\theta = 0 \tag{5.41}$$

$$\overline{\theta}\overline{\sigma}^{\mu\nu}\overline{\theta} = 0. \tag{5.42}$$

The first of these expressions comes from complex conjugation and the anticommutation properties of Grassmann variables, while the other two expressions follow from the antisymmetry of $\sigma^{\mu\nu}$ in its $SL(2,\mathbb{C})$ indices.

5.3.2 SUSY Differential Operators

To be a 'true' sueprfield, $S(x, \theta, \overline{\theta})$ must transform properly under SUSY. Let us refresh our memory with the transformation of non-SUSY fields on Minkowski space in non-supersymmetric field theory. Recall that a Minkowski-space field $\phi(x)$ transforms under translations,

$$\phi \to e^{-ia^\mu P_\mu}\,\phi\, e^{ia^\mu P\mu}, \tag{5.43}$$

where P_μ is the abstract generator of translations and ϕ is being thought of as an operator. (The convention for which exponential has the negative sign canb e thought of as the difference between active and passive transformations, or equivalently the difference between forward and backward transformations.) Alternately, we can think of ϕ as a function that transforms under translations via the differential operator[5] $\mathscr{P}_\mu$,

$$\phi(x) \to e^{ia^\mu \mathscr{P}_\mu}\phi(x) = \phi(x+a). \tag{5.44}$$

By comparing both transformations for infinitesimal parameter a, we find that

$$\delta\phi = i[\phi, a^\mu P_\mu] = ia^\mu \mathscr{P}_\mu \phi = a^\mu \partial_\mu \phi, \tag{5.45}$$

i.e. that we may write the differential operator as

$$\mathscr{P} = -i\frac{\partial}{\partial x^\mu}. \tag{5.46}$$

Now we'd like to do the same thing for the SUSY generators Q_α and $\overline{Q}_{\dot\alpha}$. As an operator, a superfield S transforms under infinitesimal parameters ϵ_α and $\overline{\epsilon}_{\dot\alpha}$ as

$$S(x,\theta,\overline{\theta}) \to e^{-i(\epsilon Q + \overline{\epsilon}\overline{Q})}\, S(x,\theta,\overline{\theta})\, e^{i(\epsilon Q + \overline{\epsilon}\overline{Q})}. \tag{5.47}$$

Alternately, we may define superspace differential operators $\mathscr{Q}$ and $\overline{\mathscr{Q}}$ so that the superfields transform as

$$S(x,\theta,\overline{\theta}) \to e^{i(\epsilon\mathscr{Q} + \overline{\epsilon}\overline{\mathscr{Q}})}\, S(x,\theta,\overline{\theta}) = S(x+\delta x, \theta+\epsilon, \overline{\theta}+\overline{\epsilon}). \tag{5.48}$$

We've written in a motion in Minkowski space, δx, with the foresight that supersymmetry transformations are a "square root" of translations so we ought to provide for the SUSY differential operators also having some Minkowski space component. The most general

[5] We will use the convention that differential operators will be written in Weinberg-esque script. Other references will denote differential operators from abstract operators with hats. Some will not explicitly differentiate betweet the two.

form that δx can take given the parameters ϵ_α and $\bar{\epsilon}_{\dot{\alpha}}$ is

$$\delta x^\mu = -ic(\epsilon\sigma^\mu\bar{\theta}) + ic^*(\theta\sigma^\mu\bar{\epsilon}), \tag{5.49}$$

where we have demanded that $\delta x \in \mathbb{R}$ and c is a constant that we would like to determine. From an analogous argument as that for $\mathscr{P}$, we can look at infinitesimal transformations to determine the SUSY differential operators:

$$\delta S = i[S, \epsilon Q + \bar{\epsilon}\overline{Q}] = i\left(\epsilon\mathscr{Q} + \bar{\epsilon}\overline{\mathscr{Q}}\right)S, \tag{5.50}$$

from which we find

$$\epsilon^\alpha\mathscr{Q}_\alpha = -i\epsilon^\alpha\frac{\partial}{\partial\theta^\alpha} - c\epsilon^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^\mu} \tag{5.51}$$

$$\epsilon_{\dot{\alpha}}\overline{\mathscr{Q}}^{\dot{\alpha}} = -i\epsilon_{\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} + c^*\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\epsilon^{\dot{\alpha}}\frac{\partial}{\partial x^\mu}. \tag{5.52}$$

We would like to ‘peel off’ the transformation parameters ϵ and $\bar{\epsilon}$. This is straightforward for the first equation since the ϵ appears with the same index height and on the left of the spinor structure for every term,

$$\mathscr{Q}_\alpha = -i\frac{\partial}{\partial\theta^\alpha} - c(\sigma^\mu\bar{\theta})_\alpha\frac{\partial}{\partial x^\mu}. \tag{5.53}$$

Technically we should say that equation (5.51) holds for any value of ϵ^α, thus equation (5.53) must hold. However, we have to do a bit of work to remove the $\bar{\epsilon}_{\dot{\alpha}}$ from equation (5.52) and then subsequently lower the index on $\mathscr{Q}^{\dot{\alpha}}$,

$$\epsilon_{\dot{\alpha}}\overline{\mathscr{Q}}^{\dot{\alpha}} = -i\epsilon_{\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} + c^*(\theta\sigma^\mu)_{\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\alpha}}\epsilon_{\dot{\alpha}}\frac{\partial}{\partial x^\mu} \tag{5.54}$$

$$= -i\epsilon_{\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - c^*\epsilon_{\dot{\alpha}}(\theta\sigma^\mu)_{\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\alpha}}\frac{\partial}{\partial x^\mu} \tag{5.55}$$

$$\overline{\mathscr{Q}}^{\dot{\alpha}} = -i\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - c^*(\theta\sigma^\mu)_{\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\alpha}}\frac{\partial}{\partial x^\mu}. \tag{5.56}$$

To lower the index we must remember that we pick up a minus sign on the spinor derivative, c.f. equation (5.22).

$$\overline{\mathscr{Q}}_{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - c^*(\theta\sigma^\mu)_{\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial x^\mu} \tag{5.57}$$

$$= i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + c^*(\theta\sigma^\mu)_{\dot{\gamma}}\frac{\partial}{\partial x^\mu}, \tag{5.58}$$

where we've used $\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\gamma}}_{\dot{\alpha}}$. In order to satisfy the SUSY anticommutation relation $\{\mathscr{Q}_\alpha, \overline{\mathscr{Q}}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} = \mathscr{P}_\mu$, one must have $\mathfrak{Re}\, c = 1$. We shall choose $c = 1$. In summary, the differential operators associated with our SUSY generators are given by,

$$\mathscr{P} = -i\frac{\partial}{\partial x^\mu} \tag{5.59}$$

$$\mathscr{Q}_\alpha = -i\frac{\partial}{\partial \theta^\alpha} - (\sigma^\mu\overline{\theta})_\alpha\frac{\partial}{\partial x^\mu} \tag{5.60}$$

$$\overline{\mathscr{Q}}_{\dot{\alpha}} = i\frac{\partial}{\partial \overline{\theta}^{\dot{\alpha}}} + (\theta\sigma^\mu)_{\dot{\alpha}}\frac{\partial}{\partial x^\mu} \tag{5.61}$$

5.3.3 Differential Operators as a Motion in Superspace

There is an alternate way of viewing these differential operators in terms of a motion on the coset space Poincaré/Lorentz[6]. We may exponentiate the SUSY algebra (a graded Lie algebra) using the translation parameter a and the SUSY Grassmann parameters ϵ, $\overline{\epsilon}$, yielding a Lie group element

$$G(x, \theta, \overline{\theta}) = e^{i(\pm x^\mu P_\mu + \theta Q + \overline{\theta Q})}, \tag{5.62}$$

where we have written the translation with a '$\pm$' to indicate some arbitrariness in the convention for how we define the translation operator (FLIP ***: I don't think it'll matter in the end. Also has to do with active vs passive transformations. Further, the choice of sign DEFINES what we mean by a forward translation. I SHOULD review the translation case?) Because the product of two Grassmann variables (e.g. θQ) is a commuting object, we may apply the Baker-Campbell-Hausdorff formula to products of group elements,

$$e^A\, e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}. \tag{5.63}$$

Thus we find,

$$G(0, \epsilon, \overline{\epsilon})\, G(x^\mu, \theta, \overline{\theta}) = e^{i(\epsilon Q + \overline{\epsilon Q}) + i(\pm x^\mu P_\mu + \theta Q + \overline{\theta Q}) - \frac{1}{2}[\epsilon Q + \overline{\epsilon Q}\,,\, \pm x^\mu P_\mu + \theta Q + \overline{\theta Q}]}. \tag{5.64}$$

[6] This is the approach used in standard texts like Wess & Bagger and Bailin & Love. While the general procedure is identical, note that these two references differ from us and from each other by minus signs and factors of i.

We'd like to work out the commutator using the SUSY algebra of equations (3.3) - (3.9).

$$[\epsilon Q + \overline{\epsilon Q}\,,\ \pm x^\mu P_\mu + \theta Q + \overline{\theta Q}] = [\epsilon Q, \overline{\theta Q}] + [\overline{\epsilon Q}, \theta Q]. \tag{5.65}$$

Since it can be illustrative to work out the arithmetic explicitly at least once in one's life, let's do the first commutator on the right-hand side. One just has to be careful with anticommuting the Grassmann numbers.

$$\begin{aligned}
[\epsilon Q, \overline{\theta Q}] &= \ \ \epsilon^\alpha Q_\alpha \overline{\theta}_{\dot\beta} \overline{Q}^{\dot\beta} - \overline{\theta}_{\dot\beta} \overline{Q}^{\dot\beta} \epsilon^\alpha Q_\alpha &(5.66)\\
&= -\epsilon^\alpha \overline{\theta}_{\dot\beta} Q_\alpha \overline{Q}^{\dot\beta} + \overline{\theta}_{\dot\beta} \epsilon^\alpha \overline{Q}^{\dot\beta} Q_\alpha &(5.67)\\
&= -\epsilon^\alpha \overline{\theta}_{\dot\beta} Q_\alpha \overline{Q}^{\dot\beta} - \epsilon^\alpha \overline{\theta}_{\dot\beta} \overline{Q}^{\dot\beta} Q_\alpha &(5.68)\\
&= -\epsilon^\alpha \overline{\theta}_{\dot\beta}\, [Q_\alpha, \overline{Q}^{\dot\beta}] &(5.69)\\
&= -\epsilon^\alpha \overline{\theta}_{\dot\beta}\, 2(\sigma^\mu)_\alpha{}^{\dot\beta} P_\mu &(5.70)\\
&= -2(\epsilon \sigma^\mu \overline{\theta}) P_\mu. &(5.71)
\end{aligned}$$

Note that the last line doesn't introduce an overall sign since $(\sigma^\mu)_\alpha{}^{\dot\beta} = (\sigma^\mu)_{\alpha\dot\gamma}\epsilon^{\dot\gamma\dot\beta}$ is a matrix of *commuting* numbers. Doing the same manipulation for the second commutator in equation (5.65), we may thus write equation (5.64) as

$$\begin{aligned}
G(0,\epsilon,\overline{\epsilon})\, G(x^\mu,\theta,\overline{\theta}) &= e^{i[\pm x^\mu P_\mu + (\epsilon+\theta)Q + (\overline{\epsilon}+\overline{\theta})\overline{Q}] - \frac{1}{2}[2(\epsilon\sigma^\mu\overline{\theta})P_\mu - 2(\theta\sigma^\mu\overline{\epsilon})P_\mu]} &(5.72)\\
&= e^{i[(\pm x^\mu - i\epsilon\sigma^\mu\overline{\theta} + i\theta\sigma^\mu\overline{\epsilon})P_\mu + (\epsilon+\theta)Q + (\overline{\epsilon}+\overline{\theta})\overline{Q}]} &(5.73)\\
&= G(x \mp i\epsilon\sigma^\mu\overline{\theta} \pm i\theta\sigma^\mu\overline{\epsilon}\,,\ (\epsilon+\theta)\,,\ (\overline{\epsilon}+\overline{\theta})). &(5.74)
\end{aligned}$$

Notice that the $\pm$ and $\mp$ signs have moved to the SUSY-generated Minkowski translations. This is simply a statement of our convention for a forward spacetime translation. Thus we may associate the left-multiplication of group elements as a motion in the parameter space (which is identified with superspace),

$$g(\epsilon,\overline{\epsilon}) : (x^\mu,\theta,\overline{\theta}) \to (x^\mu \mp i\epsilon\sigma^\mu\overline{\theta} \pm i\theta\sigma^\mu\overline{\epsilon}\,,\ (\theta+\epsilon)\,,\ (\overline{\theta}+\overline{\epsilon})). \tag{5.75}$$

We may thus write out a representation of differential operators for the SUSY generators following the 'template' of equation (5.44),

$$e^{ia^\mu \mathscr{P}_\mu + i\epsilon\mathscr{Q} + i\overline{\epsilon}\overline{\mathscr{Q}}} S(x,\theta,\overline{\theta}) = S(x^\mu \mp i\epsilon\sigma^\mu\overline{\theta} \pm i\theta\sigma^\mu\overline{\epsilon}\,,\ (\theta+\epsilon)\,,\ (\overline{\theta}+\overline{\epsilon})), \tag{5.76}$$

so that (keeping particular care with factors of i and signs):

$$\epsilon^\alpha \mathscr{Q}_\alpha = -i\epsilon^\alpha \frac{\partial}{\partial\theta^\alpha} \mp i\epsilon^\alpha (\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta} \mathscr{P}_\mu \tag{5.77}$$

$$= -i\epsilon^\alpha \frac{\partial}{\partial\theta^\alpha} \mp i\epsilon^\alpha (\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta} (\mp i\partial_\mu) \tag{5.78}$$

$$= -i\epsilon^\alpha \frac{\partial}{\partial\theta^\alpha} - \epsilon^\alpha (\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta} \partial_\mu. \tag{5.79}$$

Note that the dependence on our '$\pm$' convention has dropped out. We can then 'peel off' the ϵ^α to find the differential operator

$$\mathscr{Q}_\alpha = -i\frac{\partial}{\partial\theta^\alpha} - (\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu. \tag{5.80}$$

We may now do the same manipulation for the $\bar\epsilon\overline{\mathscr{Q}}$ term,

$$\bar\epsilon_{\dot\alpha}\overline{\mathscr{Q}}^{\dot\alpha} = -i\bar\epsilon_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}} \pm i\theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\bar\epsilon^{\dot\alpha}\mathscr{P}_\mu \tag{5.81}$$

$$= -i\bar\epsilon_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}} \pm i\theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\bar\epsilon^{\dot\alpha}(\mp i\partial_\mu) \tag{5.82}$$

$$= -i\bar\epsilon_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + \theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\bar\epsilon^{\dot\alpha}(\partial_\mu). \tag{5.83}$$

This requires a bit more care to peel off the $\bar\epsilon_{\dot\alpha}$. We will make use of the relations in equation (2.60) to swap the order of a spinor contraction

$$\bar\epsilon_{\dot\alpha}\overline{\mathscr{Q}}^{\dot\alpha} = -i\bar\epsilon_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + \theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\bar\epsilon^{\dot\alpha}(\partial_\mu). \tag{5.84}$$

These match equations (5.51) and (5.52) that we derived using the operator/field approach.

ACK... my signs are wrong all over the place. I should just work it all out myself and see what happens. GRARGHRGH. This is a total mess. See Aitchison, sec 6.1; Binetruy, app C. I think Aitchison 6.1 p. 94 is the most useful.

We have to pcik a representation for P.

- QUEVEDO: $e^{-iaP}\phi e^{iaP}$, $P = -i\partial$, $\exp(\theta Q + \overline{\theta Q} - xP)$
- AITCHISON: $e^{iaP}\phi e^{-iaP}$, $P = i\partial$, $\exp(\theta Q + \overline{\theta Q} + xP)$
- BAILIN LOVE: $e^{iaP}\phi e^{-iaP}$, $P =$? neg sign?,$\exp(\theta Q + \overline{\theta Q} - xP)$
- WESS BAGGER: $e^{-iaP}\phi e^{iaP}$, $P = -i\partial$, $\exp(\theta Q + \overline{\theta Q} - xP)$

- BINETRUY: $e^{iaP}\phi e^{-iaP}$, $P = i\partial$, $\exp(\theta Q + \overline{\theta Q} + xP)$

I should make a table of this in the back. Also, I should follow the argument of Aitchison.

Chapter 6

SUSY Breaking

"Supersymmetry is elegant, beautiful, and broken."
— Fernando Quevedo

...

6.1 ...

2-1 relationship

Chapter 7

XD Basics

"Insert quote."
— Quote [2]

7.1 Notation and conventions used in this document

XD Basics

7.2 Notes

... for "orbifold or interval" discussion, see Csaba/Jay/Patrick's notes.

Chapter 8

Philosophy of Extra Dimensions

"I need to change the title."
— Fill this in

Here we'd like to talk about XD as an effective field theory. Also perhaps motivate study of RS through Strong Coupling. See [18].

8.1 5D as EFT

5D Theory is nonrenormalizable, thus it only makes sense as an EFT. That's fine. The more religous among us can wave our hands and point to string theory at some high scale. However, there are things that we can calculate that are manifestly finite, these are real predictions.

8.2 RS and Holography

Now perhaps a more phenomenologically appealing reason why warped extra dimensions is interesting is as a playground for the AdS/CFT duality. This way we can use RS to study strong coupling.

Chapter 9

Propagators in Extra Dimensions

"Fill this in with a real quote."
— Fill this in

It is instructive to go over XD propagators. We'll introduce them in mixed position-/momentum-space. See [18].

9.1 Propagators on $\mathbb{R}^5$

9.1.1 Scalar Propagator

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_A\phi\partial^A\phi - \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\left(\partial_5\phi\right)^2 - \frac{1}{2}m^2\phi^2\end{aligned}$$

$$\left(-\Box + \left(\partial_5\right)^2 - m^2\right)\phi = 0 \tag{9.1}$$

Recall that the propagator is just the Green's function of this operator,

$$\left(-\Box + (\partial_5)^2 - m^2\right) S_F(x - x', y, y') = i\delta^{(d-1)}(x - x')\delta(y - y')$$

We already know the solution to this equation

$$S_F(x - x', y, y') = \int \frac{d^d p}{(2\pi)^2} \frac{i}{p^2 - (p^5)^2 - m^2 + i\epsilon} e^{-i(p\cdot(x-x') - p^5(y-y'))} \tag{9.2}$$

9.1.2 Fermion Propagator

This is nontrivial? $(i\Gamma^A \partial_A - m)\psi = 0$.

$$\begin{pmatrix} \partial_5 - m & i\sigma^\mu \partial_\mu \\ i\overline{\sigma}^\mu \partial_\mu & -\partial_5 - m \end{pmatrix} \psi = 0. \tag{9.3}$$

Using Dirac representation used in Peskin & Schroeder.

9.1.3 Gauge boson propagator

9.2 Propagators on $\mathbb{R}^4 \times S_1$

Let's now compactify the fifth dimension on S_1. Let us begin with a bit of a digression. There is a very interesting relation between statistical physics and quantum field theory that one can see just from looking at the partition function. (See Zee) The heart of the matter is that there is a relation between the inverse temperature β and the Planck constant $\hbar$. Here's the beautiful thing that comes from this relation: when one wants to do *finite temperature* field theory, one simply does a wick rotation to impose periodic boundary conditions on the Euclidean time. Of course in the Euclidean theory one can't really tell space and time apart, so we may borrow from the framework of finite temperature field theory to work with a compactified (i.e. periodic boundary conditions) *spatial* dimension. Good reference: Kasputin.

We will write the 4D momentum as p^μ. If p^μ_{5D} is the 5D momentum, then

$$\begin{aligned} p^2_{\text{5D}} &= m^2 \\ &= p^2 - \frac{n^2}{R^2}. \end{aligned}$$

Thus it will be useful to write

$$p^2 = m^2 + \frac{n^2}{R^2}. \tag{9.4}$$

9.2.1 Scalar Propagator

For the scalar propagator we can be a bit more general and work with $d-1$ infinite spacetime dimensions with on dimension compactified on S_1. We shall take a hint from finite temperature field theory and write our compactified-space propagator S^c_F as an infinite sum of uncompactified propagators from equation (9.2):

$$S^c_F(x-x', y, y') = \sum_{n=-\infty}^{\infty} S_F(x-x', y+2\pi Rn, y'). \tag{9.5}$$

This, by construction, satisfies the periodic boundary conditions imposed by the compactification. More intuitively, the propagator represents the summing of all paths between two points weighted by the exponential of the action. Thus in equation (9.5) we are pretending that the space is *not* compact but that instead of just summing over all paths between the initial point and the class of points that are identified by the compactification. This is shown heuristically in Figure 9.1... though here I've added the $2\pi Rn$ to the end point not the initial point. I should fix this. Really what we're doing is a sum over n winding modes around the compact dimension.

One can go ahead and compute the p^5 integral explicitly and writing $\chi = \sqrt{p^2 - m^2 + i\epsilon}$,

$$\begin{aligned} S^c_F(x-x', y, y') &= \sum_{n=-\infty}^{\infty} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \int \frac{dp^5}{2\pi} \frac{ie^{-i(p\cdot(x-x')-p^5(y+2\pi Rny'))}}{p^2-(p^5)^2-m^2+i\epsilon} \\ &= \sum_{n=-\infty}^{\infty} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{e^{i\chi|y-y'+2\pi Rn|}}{2\chi} e^{-ip\cdot(x-x')}. \end{aligned}$$

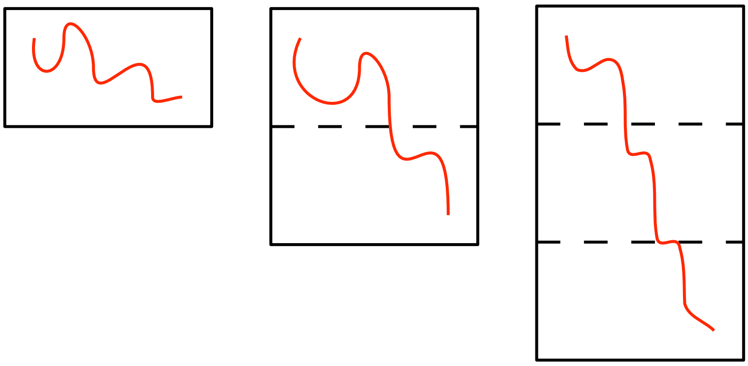


Figure 9.1: Examples of paths that are implicitly summed in equation (9.5).

And now performing the summation for $y, y' \in [0, 2\pi R]$,

$$= \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{i\cos[\chi(\pi R - |y-y'|)]}{2\chi \sin \chi\pi R} e^{-ip\cdot(x-x')}. \tag{9.6}$$

As a preview of things to come, the same derivation on a warped background will replace the trigonometric functions with Bessel functions. This will lead to nice asymptotic forms which will be useful for understanding convergence of loops.

There is one further feature that we can explore. For the heck of it, let's explicitly separate the $n = 0$ propagator from the rest of the sum,

$$\begin{aligned} S_F^c(x-x', y, y') &= S_F(x-x', y, y') + \sum_{n\neq 0} S_F(x-x', y+2\pi Rn, y') \\ &= S_F(x-x', y, y') + \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{e^{i\chi|y-y'|} + e^{-i\chi|y-y'|}}{2\chi(e^{-i\chi(2\pi R)} - 1)} e^{-ip\cdot(x-x')}. \end{aligned}$$

The second term is an analytic function that is finite for $y \to y'$, it is often written as $S_F^{\text{analy.}}$. We can now perform a Wick rotation on this term and look at the behavior for large Euclidean momentum. Remembering that $\chi = i\chi_E$ due to the $i\epsilon$ term, we find that for $\chi_E \gg R^{-1}$,

$$\frac{e^{i\chi|y-y'|} + e^{-i\chi|y-y'|}}{2\chi(e^{-i\chi(2\pi R)} - 1)} \to \frac{1}{2\chi_E e^{\chi_E(2\pi R - |y-y'|)}}. \tag{9.7}$$

Since $|y-y'| < 2\pi R$, we arrive at a very important result. $S_F^{\text{analy.}}$ is *exponentially damped* at high Euclidean momentum. This means that loop integrals containing *any* factors of $S_F^{\text{analy.}}$ are manifestly *finite*. Hence UV-divergences only occur in the term in which *every* loop propagator is $S_F(x-x', y, y')$. The counterterms of the compact and uncompactified theories are exactly the same. This makes sense since UV-divergences are short-distance effects that do not 'see' the compactification radius. In thermal language, a $T \neq 0$ theory has the exact same divergences as its corresponding $T = 0$ limit.

As a sanity check, taking $R \to \infty$ (uncompactifying the extra dimension) we find $S_F^{\text{analy.}} \to 0$ since the winding modes decouple.

9.2.2 Fermion Propagator

At this point, one might guess the mixed position/momentum-space propagator for fermions. Since any particle obeying the Dirac equation simultaneously obeys the Klein-Gordon equation, one can expect a factor resembling the Klein-Gordon propagator in equation (9.6). comparing to the relation of the 4D Dirac and Klein-Gordon propagators, one could eventually guess that the correct form of the fermion propagator,

$$S_F^c(x-x', y, y') = \int \frac{d^4p}{(2\pi)^4} \left(p_\mu\gamma^\mu + i\gamma^5\partial_5 + m\right) \frac{i\cos[\chi(\pi R - |y-y'|)]}{2\chi\sin\chi\pi R} e^{-ip\cdot(x-x')}. \tag{9.8}$$

We use the same convention as in the scalar case an write $\chi = \sqrt{p^2 - m^2 + i\epsilon}$, with p^μ the 4D momentum. To check that this is correct, one simply needs to plug back into the 5D Dirac equation, (9.3).

We can derive this equation a bit more formulaically by retracing our derivation of the 4D Dirac equation while adding some 5D space dependence. We first start by reconstructing the ['classical'] plane wave solutions to the Dirac equation, the usual $u(p)$ and $v(p)$ spinors of 4D QFT.

We start by writing out an explicit positive-energy 4D plane wave expansion, leaving the y-dependence explicit in the plane wave spinors,

$$\psi(x, y) = u(p, y)e^{ip\cdot x}. \tag{9.9}$$

The periodic boundary conditions of S_1 imply that these 4D plane waves must depend on the compactified dimension as a superposition of $\sin(ny/R)$ and $\cos(ny/R)$. For the case $n = 0$ we should recover the usual 4D plane wave spinor,

$$u^+_{s,0}(p,y) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\overline{\sigma}}\xi^s \end{pmatrix}. \tag{9.10}$$

Here ξ is a normalized Weyl 2-spinor such that $\xi^{s\dagger}\xi^r = \delta^{sr}$. The index s indicates the spinor index, the 0 refers to the mode number in the periodic direction, and the '+' superscript indicates that we're looking at the cos solutions of the periodic boundary conditions–we'll get to on this in a bit. More generally, we can write down the y dependence right into the plane waves:

$$u_{s,n}(p,y) = \begin{pmatrix} (A\cos(ny/R) + B\sin(ny/R))\sqrt{p\cdot\sigma}\xi^s \\ (C\cos(ny/R) + D\sin(ny/R))\sqrt{p\cdot\overline{\sigma}}\xi^s \end{pmatrix}. \tag{9.11}$$

This explicitly satisfies our boundary conditions in the y-direction. We fix the coefficients by requiring that $u(p,y)$ is in the kernel of the Dirac operator of equation (9.3), which we shall write succinctly as $\mathbb{D}$. It is sufficient to only look at only the upper 2 components.

$$\begin{aligned}(\mathbb{D}u)^{\text{up}}_{s,n} = &\left(-\frac{n}{r}A\sin\left(\frac{ny}{R}\right) + \frac{n}{r}B\cos\left(\frac{ny}{R}\right) - mA\cos\left(\frac{ny}{R}\right) - mB\sin\left(\frac{ny}{R}\right)\right)\sqrt{p\cdot\sigma}\xi \\ &+ \left(C\cos\left(\frac{ny}{R}\right) + D\sin\left(\frac{ny}{R}\right)\right)p\cdot\sigma\sqrt{p\cdot\overline{\sigma}}\xi. \end{aligned} \tag{9.12}$$

Recall that the '$\sqrt{p\cdot\sigma}$' is really just shorthand notation for the square root of the positive eigenvalue of the matrix, following the convention of Peskin & Schroeder's derivation of the 4D plane waves. Further, recall from equation (9.4) that $p\cdot\sigma\, p\cdot\overline{\sigma} = \overline{m}^2$, where

$$\overline{m}^2 = m^2 + \frac{n^2}{R^2}. \tag{9.13}$$

Collecting the coefficients of the sine and cosine and mandating that they each sum to zero (hence satisfying the Dirac condition), we find

$$-\frac{n}{R}A - mB + \overline{m}D = 0 \tag{9.14}$$

$$\frac{n}{R}B - mA + \overline{m}C = 0. \tag{9.15}$$

These relate the coefficients of the lower Weyl spinor (C,D) to those in the upper Weyl spinor (A,B). We are free to choose basis spinors with $A = 1, B = 0$ and $A = 0, B = 1$. We shall label the former with a '+' superscript and the latter with a '−' superscript.

Thus our positive-energy plane waves are

$$u^{+}_{s,n}(p,y) = \sqrt{2}\begin{pmatrix} \cos\left(\frac{ny}{R}\right) & \sqrt{p\cdot\sigma}\xi^s \\ \frac{m}{\overline{m}}\cos\left(\frac{ny}{R}\right) + \frac{n}{\overline{m}R}\sin\left(\frac{ny}{R}\right) & \sqrt{p\cdot\overline{\sigma}}\xi^s \end{pmatrix} \tag{9.16}$$

$$u^{-}_{s,n}(p,y) = \sqrt{2}\begin{pmatrix} \sin\left(\frac{ny}{R}\right) & \sqrt{p\cdot\sigma}\xi^s \\ \frac{m}{\overline{m}}\sin\left(\frac{ny}{R}\right) - \frac{n}{\overline{m}R}\cos\left(\frac{ny}{R}\right) & \sqrt{p\cdot\overline{\sigma}}\xi^s \end{pmatrix}. \tag{9.17}$$

We've added an overall factor of $\sqrt{2}$ to satisfy the normalization condition

$$\int d^4x \int_0^{2\pi R} dy\ \overline{u}^{\sigma}_{s,n}(p,y)u^{\sigma'}_{s',n'}(p',y')e^{-i(p-p')\cdot x} = 2m\delta^{ss'}\delta^{\sigma\sigma'}(2\pi R)\delta^{nn'}(2\pi)^4\delta^{(4)}(p-p'),$$

as well as the spin-sums

$$\sum_s u^{+}_{s,0}(p,y)\,\overline{u}^{+}_{s,0}(p,y') = (\not{p}+m) \tag{9.18}$$

$$\sum_s\sum_{\sigma=\pm}\left(u^{\sigma}_{s,0}(p,y)\,\overline{u}^{\sigma}_{s,0}(p,y')\right) = 2(\not{p}+i\gamma^5\partial_5+m)\cos\left(\frac{n(y-y')}{R}\right). \tag{9.19}$$

One can do the same rigmarole for the negative-energy solutions of the Dirac equation, finding the plane waves $v(p,y)$ in terms of Weyl spinors η^s. They differ by minus signs that are easy to trace.

$$v^{+}_{s,n}(p,y) = \sqrt{2}\begin{pmatrix} \cos\left(\frac{ny}{R}\right) & \sqrt{p\cdot\sigma}\eta^s \\ -\frac{m}{\overline{m}}\cos\left(\frac{ny}{R}\right) - \frac{n}{\overline{m}R}\sin\left(\frac{ny}{R}\right) & \sqrt{p\cdot\overline{\sigma}}\eta^s \end{pmatrix} \tag{9.20}$$

$$v^{-}_{s,n}(p,y) = \sqrt{2}\begin{pmatrix} \sin\left(\frac{ny}{R}\right) & \sqrt{p\cdot\sigma}\eta^s \\ -\frac{m}{\overline{m}}\sin\left(\frac{ny}{R}\right) + \frac{n}{\overline{m}R}\cos\left(\frac{ny}{R}\right) & \sqrt{p\cdot\overline{\sigma}}\eta^s \end{pmatrix}. \tag{9.21}$$

The normalization and spin sums also differ by the minus signs that one would expect:

$$\int d^4x \int_0^{2\pi R} dy\ \overline{v}^{\sigma}_{s,n}(p,y)v^{\sigma'}_{s',n'}(p',y')e^{-i(p-p')\cdot x} = -2m\delta^{ss'}\delta^{\sigma\sigma'}(2\pi R)\delta^{nn'}(2\pi)^4\delta^{(4)}(p-p')$$

$$\sum_s v^{+}_{s,0}(p,y)\,\overline{v}^{+}_{s,0}(p,y') = (\not{p}-m)$$

$$\sum_s\sum_{\sigma=\pm}\left(v^{\sigma}_{s,0}(p,y)\,\overline{v}^{\sigma}_{s,0}(p,y')\right) = 2(\not{p}-i\gamma^5\partial_5-m)\cos\left(\frac{n(y-y')}{R}\right).$$

Phew. That's a lot of algebra floating around. Let's move on and now quantize these fermion fields. Unfortunately our creation and annihilation operators now end up being covered in a gaudy display of indices:

$$\left\{a^{\sigma\, n}_{p\, s},\ a^{\sigma'\, n'\, \dagger}_{p'\, s'}\right\} = \left\{b^{\sigma\, n}_{p\, s},\ b^{\sigma'\, n'\, \dagger}_{p'\, s'}\right\} = (2\pi)^3\delta^{(3)}(p-p')(2\pi R)\delta^{nn'}\delta^{ss'}\delta^{\sigma\sigma'}. \tag{9.22}$$

As a reminder, $\sigma = \pm$ indicates whether we take cosine- or sine-like behavior in the upper Weyl spinor of our plane wave Dirac spinors, p is the four-momentum, n is the mode number in the y direction, and $s = 1, 2$ is the index of the Weyl spinors ξ, η. We can construct one-particle states of our Fock space in the usual way,

$$|p, n, s, \sigma\rangle = \sqrt{2E_{p,n}}\, a^{\sigma\, n\, \dagger}_{p\, s}|0\rangle, \tag{9.23}$$

this is normalized according to

$$\langle p, n, s, \sigma|p', n', s', \sigma'\rangle = 2E_{p,n}(2\pi)^3\delta^{(3)}(p-p')(2\pi R)\delta^{nn'}\delta^{ss'}\delta^{\sigma\sigma'}. \tag{9.24}$$

Continuing with the usual procedure, we now write down the decomposition of the quantum fields,

$$\psi(x, y) = \frac{1}{2\pi R}\sum_{n=0}^{\infty}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{p,n}}}\sum_{s,\sigma}\left(a^{\sigma,n}_{p,s}\, u^{\sigma}_{s,n}(p, y)e^{-ip\cdot x} + b^{\sigma,n\,\dagger}_{p,s}\, v^{\sigma}_{s,n}(p, y)e^{ip\cdot x}\right) \tag{9.25}$$

$$\overline{\psi}(x, y) = \frac{1}{2\pi R}\sum_{n=0}^{\infty}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{p,n}}}\sum_{s,\sigma}\left(b^{\sigma,n}_{p,s}\, \overline{v}^{\sigma}_{s,n}(p, y)e^{-ip\cdot x} + a^{\sigma,n\,\dagger}_{p,s}\, \overline{u}^{\sigma}_{s,n}(p, y)e^{ip\cdot x}\right). \tag{9.26}$$

Note that $\int\frac{dp^5}{(2\pi)} \to \frac{1}{2\pi R}\sum_{n=0}^{\infty}$ corresponding to the Fourier transform of a compact dimension. Why does the sum go from $n = 0$ rather than $n = -\infty$? Recall that n counts sin and cos modes, hence the Fourier series only sums over positive n. We can use these fields to calculate the 2-point Green's functions,

$$\begin{aligned}\langle 0|\psi_a(x,y)\overline{\psi}_b(x',y')|0\rangle &= \frac{1}{2\pi R}\sum_{n=0}^{\infty}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{p,n}} \\ &\quad\times\sum_{s,\sigma}u^{\sigma}_{s,n}(p,y)\overline{u}^{\sigma}_{s,n}(p,y')e^{-ip\cdot(x-x')} \qquad (9.27)\\ &= \frac{1}{\pi R}\sum_{n=0}^{\infty}\left(i\partial\!\!\!/ + i\gamma^5\partial_5 + m\right)_{ab}\left[\cos\left(\frac{n(y-y')}{R}\right)-\frac{\delta_{n0}}{2}\right] \\ &\quad\times\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{p,n}}e^{-ip\cdot(x-x')} \qquad (9.28)\\ \langle 0|\overline{\psi}_a(x,y)\psi_b(x',y')|0\rangle &= \frac{1}{2\pi R}\sum_{n=0}^{\infty}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{p,n}} \\ &\quad\times\sum_{s,\sigma}v^{\sigma}_{s,n}(p,y)\overline{v}^{\sigma}_{s,n}(p,y')e^{-ip\cdot(x-x')} \qquad (9.29)\\ &= -\frac{1}{\pi R}\sum_{n=0}^{\infty}\left(i\partial\!\!\!/ + i\gamma^5\partial_5 + m\right)_{ab}\left[\cos\left(\frac{n(y-y')}{R}\right)-\frac{\delta_{n0}}{2}\right] \\ &\quad\times\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{p,n}}e^{-ip\cdot(x-x')}. \qquad (9.30)\end{aligned}$$

By now one can start to see the form of the propagator. Let's proceed with taking the time-ordered product,

$$\begin{aligned}\langle 0|\mathrm{T}\left[\psi(x,y)\overline{\psi}(x',y')\right]|0\rangle =& \frac{1}{\pi R}\left(i\partial\!\!\!/ + i\gamma^5\partial_5 + m\right)\sum_{n=0}^{\infty}\left[\cos\left(\frac{n(y-y')}{R}\right)-\frac{\delta_{n0}}{2}\right] \\ &\times\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+\frac{n^2}{R^2}+i\epsilon}e^{-ip\cdot(x-y)}. \qquad (9.31)\end{aligned}$$

Ordinary mortals would be stuck at this point, but there is a nice mathematical relation that we may invoke. For $x \in [2\pi m, 2\pi(m+1)]$ and $a \notin \mathbb{Z}$,

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^2-a^2} = \frac{1}{2a^2} - \frac{\pi\cos\left(a(2m+1)\pi - ax\right)}{2a\sin(a\pi)}. \qquad (9.32)$$

This brings us to our desired result, equation (9.8). One can check explicitly that this form is the Green's function of the 5D Dirac operator,

$$\left(\partial\!\!\!/ + i\gamma^5\partial_5 - m\right)S_F(x-x',y,y') = i\delta^{(4)}(x-x')\delta(y-y'). \qquad (9.33)$$

9.2.3 Gauge Boson Propagator

The main nontrivial thing is that you can choose a gauge such that the A_5 decouples.

In $\mathbb{R}^4 \times S_1$, because the 5th dimension is compact the theory only has 4 dimensional Lorentz invariance. (What? That can't be true... locally it looks like $\mathbb{R}^5$.) The point is that the gauge fixing term in the Lagrangian can be written as

$$\mathcal{L}_{g.f.} = -\frac{1}{2}\left(\xi_4 \partial_\mu A^\mu + \xi_5 \partial_5 A^5\right)^2 \tag{9.34}$$

Appendix A

Literature Guide

"There are more conventions than there are authors."
— Adrian Signer

...

A.1 General Textbooks

... Aitchison is a relatively new book that is extremely pedagogical. Maybe a little chatty for those who are already familiar with the topic, but very useful for looking for a discussion of basic topics. ... Wess and Bagger is the classic with an approach closes to these notes. Rather technical for reading by itself ... Binetruy has nice appendices, but strange ordering of topics, Majorana spinors. ... Freund is ... short ... Weinberg is only comprehensible by Weinberg. But it's quite good. These notes were (indireclty) heavily influence by Weinberg. Stupid notation and stupid conventions.

... Drees... sparticles? ... Tata ... Weak scale SUSY? ... Terning ... Modern SUSY is an advanced book.

... Buchbinder and Kuzenko ... formalism of symmetries

A.2 Canonical Reviews

... Stephen Martin. Rather complimentary to our approach, doesn't use superfields but instead works directly with component fields to understand physical significance of susy. Lots of collider stuff. Sign convention may be flipped using tex.

... Argyres. Well written, many versions

A.3 Specialized Reviews

... strassler

A.4 SUSY Breaking

Appendix B

Notation and Conventions

"Insert quote."
— Quote [2]

B.1 Notation and conventions used in this document

We use the West Coast 'mostly-minus' metric that is standard for particle physicists,

$$\eta = \text{diag}(+,-,-,-). \tag{B.1}$$

When indices are not important, we shall refer to vector and tensor quantities by writing them in boldface. Thus we might write $\mathbf{M}$ to refer to the tensor $M^{\mu\nu}$.

SUSY: Greek lowercase letters denote the usual indices in Minkowski space, $\mu \in \{0,1,2,3\}$. Roman lowercase letters around i denote 3D Euclidean indices, $i,j,k \in 1,2,3$.

XD: hm.

Lie algebra is written as $\mathcal{L}(SO(3))$ rather than in gothic.

Epsilon tensor: $\epsilon^{12} = -\epsilon_{12} = -\epsilon^{\dot{1}\dot{2}} = \epsilon_{\dot{1}\dot{2}} = 1$ Unindexed spinor are in what representations.

B.2 Blah

B.3 Comparison with other sources

Things to check: definition of the epsilon tensor, order of indices for generators, minus signs...

Source	Metric	ϵ^{12}	ϵ_{0123}	SUSY generators
These notes	$(+,-,-,-)$	$+$	$-$	Weyl
Wess & Bagger	$(+,-,-,-)$	$+$	$-$	Weyl
Bailin & Love	$(+,-,-,-)$	$-$		Weyl
Binetruy		$+$		Majorana
Terning	$(+,-,-,-)$	$+$	$-$	Weyl
Weinberg	$(-,+,+,+)$			Majorana?
Martin				Weyl
Aitchison				Weyl
Argyres 1996				Weyl
Argyres 2001				Majorana

See P.453 of Binetruy

Source	σ^0	σ^1	σ^2	σ^3	γ^μ
These notes	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & \sigma^\mu\\ \overline{\sigma}^\mu & 0\end{pmatrix}$
Wess & Bagger	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & \sigma^\mu\\ \overline{\sigma}^\mu & 0\end{pmatrix}$
Bailin & Love					
Binetruy					
Terning	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & \sigma^\mu\\ \overline{\sigma}^\mu & 0\end{pmatrix}$
Weinberg					
Martin					
Aitchison					
Argyres 1996					
Argyres 2001					

Conventions for superfields:

BAILIN AND LOVE: something weird about sign for momentum. See (1.166, 167) vs (2.5)

$\hat{\phi} \to U^\dagger(P)\phi U(P)$

$\phi \to \mathcal{U}(\mathscr{P})\phi$

Source	$U(P)$	$\mathcal{U}(\mathscr{P})$	$\mathscr{P}$	$\ln(G(x,\theta,\overline{\theta}))$
These notes				
Wess & Bagger				$i(-x^\mu P_\mu \theta Q + \overline{\theta Q})$
Bailin & Love	$e^{-ix^\mu P_\mu}$	$e^{-ix^\mu \mathscr{P}_\mu}$	$i\partial_\mu$	$i(\theta Q + \overline{\theta Q} - x^\mu P_\mu)$
Binetruy				
Terning				
Weinberg				
Martin				
Aitchison				
Argyres 1996				
Argyres 2001				

Source	$\mathscr{Q}_\alpha$	$\overline{\mathscr{Q}}_{\dot\alpha}$	$\mathscr{D}_\alpha$	$\overline{\mathscr{D}}_{\dot\alpha}$
These notes				
Wess & Bagger	$\partial_\alpha - i(\sigma^\mu\overline{\theta})_\alpha\partial_\mu$		$\partial_\alpha + i(\sigma^\mu\overline{\theta})_\alpha\partial_\mu$	
Bailin & Love	$-i\partial_\alpha - (\sigma^\mu\overline{\theta})_\alpha\partial_\mu$	$i\partial_{\dot\alpha} + (\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$	$\partial_\alpha + (\sigma^\mu\overline{\theta})_\alpha\partial_\mu$	$\partial_{\dot\alpha} - (\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$
Binetruy				
Terning				
Weinberg				
Martin				
Aitchison				
Argyres 1996	$\partial_\alpha - i(\sigma^\mu\overline{\theta})_\alpha\partial_\mu$	$-\overline{\partial}_{\dot\alpha} + i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$	$\partial_\alpha + i(\sigma^\mu\overline{\theta})_\alpha\partial_\mu$	$\overline{\partial}_{\dot\alpha} - i(\theta\sigma^\mu)_\alpha\partial_\mu$
Argyres 2001				

B.4 Convention-independent expressions

Appendix C

Useful Identities

"Insert quote."
— Quote [2]

C.1 Pauli Matrices

...

C.2 Gamma Matrices

C.3 Fierz Identities

...

C.4 Miscellaneous

$$\int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^\alpha}{(k^2 - A^2)^\beta} = \frac{(-)^{\alpha+\beta} i}{(4\pi)^{d/2}} \left(\frac{1}{A^2}\right)^{\beta-\alpha-\frac{d}{2}} \frac{\Gamma(\alpha + d/2)\Gamma(\beta - (\alpha + d/2))}{\Gamma(d/2)\Gamma(\alpha)} \tag{C.1}$$

Appendix D

Representations of the Poincaré Group

"Insert quote."
— Quote [2]

Here we present a more in-depth presentation of the representations of the Poincaré group, based on the lectures by Jan Gutowski (reference). For more information, the reader is urged to see the relevant appendix of Wess and Bagger or Weinberg Volume I.

D.1 $SL(2\mathbb{C})$

2-1 relationship

D.2 Projective representations

Review Weinberg's argument in detail here.

D.3 Further Reading

Choquet-Bruhat, Nakahara/Frankel, Moshe Carmeli, ...

Appendix E

Review of Gauge Theories

"Dimensions change as you change dimensions."
— Hugh Osborn, Part III Advanced Quantum Field Theory 2007

This is a short review of relevant topics in gauge theory. This is neither meant to be comprehensive nor succinct, it's a subset of topics which the author felt was relevant and/or interesting. Most of this treatment comes from [19] as well as the standard Quantum Field Theory literature.

E.1 Lie Algebras

> ***Caveat emptor.*** The following synopsis is meant to review the general idea of Lie algebras and will *not* attempt to be mathematically rigorous. A more systematic and mathematically honest treatment can be found in [10], from which this presentation is condensed.

A **Lie group**, G, is a smooth, differentiable manifold which is also has a group structure. In particular, the group multiplication of two elements of the manifold is a smooth map and the inverse operator under this multiplication is also a smooth map. From differential geometry we know that at each point, p, on the manifold we may define a set of tangent vectors that span the $n = \dim G$ dimensional **tangent space**, $T_p(G)$.

These tangent vectors map real functions ($f : U \in G \to \mathbb{R}$) to $\mathbb{R}$. Suppose $\gamma(t) : \mathbb{R} \to U \in G$ is a curve such that $\gamma(0) = p$. Then the tangent vector $\dot{\gamma}_p$ maps $f \to \left[\frac{d}{dt}(f \circ \gamma(t))\right]_{t=0}$.

The tangent space of a group is called its **Lie algebra**, which we shall denote $\mathcal{L}(G)$. The Lie algebra of a group is equipped with a **Lie bracket** $[\,\cdot\,,\,\cdot\,] : \mathcal{L}(G) \times \mathcal{L}(G) \to \mathcal{L}(G)$.

Lie algebra notation. When we discuss gauge groups, we really are interested in Lie algebras.

For $X, Y \in \mathcal{L}(G)$

E.2 Comparison with other sources

Appendix F

Review of Renormalization and Effective Field Theory

"Q: What's purple and changes with scale?
A: The renormalization grape."
— Bastardization of a popular mathematics.

This is a summary of relevant ideas of the renormalization group and effective field theory. Our prediction for new physics is based on the modern Wilsonian perspective of renormalization.

There exist a few exceptionally intuitive expositions on the renormalization group. These include the treatment based on dimensional analysis by Stevenson [20] and the pedagogical approach to the Wilsonian paradigm by Hollwood [21]. In addition, many modern quantum field theory texts (such as [9] and [22]) do a very good job explaining renormalization.

F.1 Intuitive picture

F.2 Understanding UV divergences

Log log log.

F.3 Naturalness

Colophon

These notes were made in $\LaTeX\,2_\varepsilon$ using KDE Integrated $\LaTeX$ Environment [23] and TextMate [24] the "hepthesis" class [25].

Bibliography

[1] F. Quevedo, Supersymmetry and extra dimensions, lecture notes taken by Philip Tanedo, 2007, CASM 'Part III' course.

[2] G. Kane and M. Shifman, *The Supersymmetric World: The Beginnings of the Theory* (World Scientific, 2000).

[3] I. L. Buchbinder and S. M. Kuzenko, Bristol, UK: IOP (1998) 656 p.

[4] E. D. Bolker, The American Mathematical Monthly **80**, 977 (1973).

[5] H. K. Dreiner, H. E. Haber, and S. P. Martin, (2008), 0812.1594.

[6] H. J. W. Muller-Kirsten and A. Wiedemann, Print-86-0955 (KAISERSLAUTERN).

[7] J. Polchinski, Cambridge, UK: Univ. Pr. (1998) 531 p.

[8] S. Weinberg, Cambridge, UK: Univ. Pr. (2000) 419 p.

[9] S. Weinberg, *The Quantum Theory of Fields, Vol I-III* (Cambridge University Press, 2005).

[10] J. Gutowski, Symmetries and particle physics, 2007, Michaelmas term Part III course, Department of Applied Mathematics and Theoretical Physics, Cambridge University.

[11] S. Weinberg, Cambridge, UK: Univ. Pr. (1995) 609 p.

[12] A. Salam and J. A. Strathdee, Phys. Rev. **D11**, 1521 (1975).

[13] A. Salam and J. A. Strathdee, Nucl. Phys. **B76**, 477 (1974).

[14] B. S. DeWitt, *Supermanifolds* , Cambridge, UK: Univ. Pr. (1992) 407 p. (Cambridge monographs on mathematical physics). (2nd ed.),.

[15] F. Gieres, Lect. Notes Phys. **302**, 1 (1988).

[16] M. Dine, *Supersymmetry and string theory: Beyond the standard model* , Cambridge, UK: Cambridge Univ. Pr. (2007) 515 p.

[17] P. Binetruy, *Supersymmetry: Theory, experiment and cosmology* , Oxford, UK: Oxford Univ. Pr. (2006) 520 p.

[18] M. Puchwein and Z. Kunszt, Annals Phys. **311**, 288 (2004), hep-th/0309069.

[19] H. Osborn and S. Gielen, Advanced quantum field theory, 2007, Lent term Part III course, Department of Applied Mathematics and Theoretical Physics, Cambridge University.

[20] P. M. Stevenson, Ann. Phys. **132**, 383 (1981).

[21] T. Hollowood, Cut-offs and continuum limits: a wilsonian approach to field theory, 1998, Lecture Notes.

[22] M. Srednicki, *Quantum Field Theory* (Cambridge University Press, 2007).

[23] B. P. Jeroen Wijnhout, Holger Danielsson, Kde integrated LaTeX environment, 2006.

[24] Micromates, Textmate: The missing editor, 2008.

[25] A. Buckley, The hepthesis LaTeXbook class, 2007.

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