# Notes on Seibergology

## EM-like duality SUSY and applications

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### Abstract

This is a set of unfinished LaTeX'ed notes on Seiberg duality and related topics in phenomenology. It subsumes an older set of notes on metastable supersymmetry breaking. Corrections are welcome, just don't expect me to get around to it any time soon.

# Contents

# 1 Introduction

These are personal notes originally based on the spring 2009 lectures by Csaba Csáki at Cornell University. They have been augmented with material following the author's interests in the field and draw *heavily* from some of the very excellent 'Seibergology' review literature:

- Intriligator and Seiberg's lectures on SUSY breaking [1] and electromagnetic duality [2]
- Strassler's lecture notes on SUSY gauge theory [3] and the duality cascade [4].
- Terning's book on supersymmetry [5] and the accompanying slides which are available online.

Additionally, there are several elements taken from various lectures, review articles, talks, current literature, and discussions with colleagues. I have tried to give credit and links to further literature where appropriate. To be clear, this document includes no new results and the pedagogical approach is an amalgamation from other sources. If unlisted, it is fair to assume that material and the style of presentation came from one of the aforementioned references. All I did was interpolate between several sources to weave a narrative which makes sense to me. Comments, constructive criticism, and corrections are especially welcome.

## 1.1 Pre-requisites

The reader is expected to already be familiar with supersymmetric gauge theories at the level of an introductory graduate-level course, e.g. [6]. For more foundational reading on supersymmetric gauge theories there is now a plethora of review articles and recorded lectures available. As a rule of thumb any set of lectures after 1996 should review Seiberg duality while any lectures after 2006 should at least mention metastable vacua. Recent reviews which mention the metastable SUSY-breaking program include Dine's 2008 Cargese lectures [7] (see also his more recent review [8]) and Shirman's 2008 TASI lectures [9]. One may find further references in those papers. The textbooks by Terning [5] and Dine [10] also mention useful material in modest depth. The multimedia-inclined are encouraged to view Brian Wecht and Nathan Seiberg's lectures at the Isaac Newton Institute's 2007 "Gauge Fields and Strings" workshop[1], Seiberg's lectures at PiTP 2010[2], or Csaba Csáki's lectures on supersymmetry[3]

Further references to appropriate pedagogical or otherwise important literature will be mentioned as appropriate in this document.

## 1.2 A selected non-technical history

[**To do:** This section to be written.]

[1]http://www.newton.ac.uk/programmes/SIS/sisw02p.html
[2]http://video.ias.edu/pitp-2010
[3]Available to the Cornell LEPP theory group.

#### 1.2.1 Dynamical SUSY breaking

#### 1.2.2 $\mathcal{N} = 1$ electromagnetic duality and beyond

#### 1.2.3 Metastable SUSY breaking

# $\mathcal{N} = 1$ Duality in SQCD

## 2 Nonperturbative SUSY QCD

The basic tools we will need come from supersymmetric QCD. We shall now review the key aspects of supersymmetric $SU(N)$ gauge theory with $F$ flavors, in particular its nonperturbative for various values of $F$ and $N$. Our review will be loosely based on Csáki's lectures on Beyond the Standard Model physics[4], which is in turn based on Terning's textbook [5]. Additional review articles and lectures include Intriligator and Seiberg's lectures on SUSY gauge theory and electromagnetic (i.e. Seiberg) duality [2], Wecht's lectures at the Newton Institute's *Gauge Fields and Strings* program[5], Argyres' Cornell lectures on supersymmetry[6], and Peskins TASI lectures on duality in super Yang-Mills theories [11]. Those unfamiliar with this topic are encouraged to read some of these references for a proper pedagogical introduction to the subject.

### 2.1 Effective Actions

We must make an important distinction between two types of effective action. While we don't usually care about the exact sense in which an effective action is 'effective,' both the 1PI and Wilsonian effective actions play a key role in SUSY and it is important to be clear which sort of effective action one is working with. The importance of the distinction between these two was first elucidated by Shifman and Vainshtein when they explained the apparent contradiction between the nonrenormalization theorem for the superpotential and their exact-to-all orders (NSVZ) beta function [12]. Some discussion can also be found in Intriligator and Seiberg's SUSY electromagnetic duality lectures [2] and Burgess' introduction to effective field theory [13].

The **1PI action**, $\Gamma$, comes from integrating out all quantum effects such that the *classical* equations of motion from the 1PI action $\Gamma$ yield the *complete* equations of motion including quantum effects. This object is the generator of 1PI diagrams, i.e. the so-called 'skeleton diagrams' that are tree-level but encode multi-loop processes. The momentum-independent part of the 1PI action is the rather-important-for-this-paper **Coleman-Weinberg effective potential**. The 1PI action (and the Coleman-Weinberg potential) by definition take into account all loop effects so that any actual calculation of these objects must be done as a loop expansion. More formally, the 1PI effective action is the Legendre transform of the classical action with respect to sources and background fields. The NSVZ 'exact to all orders' beta function for the real gauge coupling $g$ is obtained from the 1PI action.

[4] Spring 2009, Cornell University.

[5] Notes and recordings: `http://www.newton.ac.uk/programmes/SIS/sisw02p.html`.

[6] `http://www.physics.uc.edu/~argyres/661/index.html`

The **Wilsonian action**, on the other hand, comes from integrating out heavy fields and the high-momentum modes of light fields. This is the action that one obtains from the [Wilsonian] renormalization group flow to lower energies. Unlike the 1PI action, the Wilsonian action must still be treated quantum mechanically, i.e. one still has to perform the path integral and it is inherently a 'theory with a cutoff.' The one-loop exact ('Seiberg') beta function for the holomorphic gauge coupling $\tau$ is obtained with from the Wilsonian action.

Because *all* quantum excitations are integrated out, 1PI action contains contributions from massless particles running in loops ($\sim \log k$) and can thus have infrared divergences. A tangible example of this can be seen in the chiral Lagangian for pions. These "IR ambiguities" can lead to "holomorphic anomalies." In other words these divergences tell us that our theory is missing something important. The Wilsonian action does not have any problems with massless particles since it only integrates out heavy modes.

## 2.2 Gauge theory facts

> This section is based on Gutowski's lecture notes, which uses anti-Hermitian generators. I've tried to change these int Hermitian generations, I may have missed a few places. [**Even better:** instead I should use Hugh Osborn's notes... they're slightly better on gauge theories.]

The fundamental covariant derivative $D_\mu$ is $D_\mu = \partial_\mu + ieA_\mu$, so that $D_\mu\phi$ transforms in the same way as the fundamental field $\phi$. This imposes that $ieA'_\mu = gieA_\mu g^{-1} - \partial_\mu g g^{-1}$. More generically, the covariant derivative associated with a field $\varphi$ in representation $\mathbf{r}$ of the gauge group is

$$D_\mu\varphi = \partial_\mu\varphi + d(ieA_\mu)\varphi, \tag{2.1}$$

where $d(ieA_\mu)$ is the associated representation of the Lie algebra on $\mathbf{r}$. In other words, if $\varphi \to \mathcal{D}(g)\varphi$ where $\mathcal{D}(g)$ is the transformation $g$ in the $\mathbf{r}$ representation of the gauge group, then $D_\mu\varphi \to \mathcal{D}(g)D_\mu\varphi$. This requires two lemmas to prove,

$$d(gvg^{-1})\mathcal{D}(g) = \mathcal{D}(g)d(v) \tag{2.2}$$

$$\frac{d\mathcal{D}(g)}{dt}\mathcal{D}g^{-1} = d\left(\frac{dg}{dt}g^{-1}\right). \tag{2.3}$$

These can be proved by writing $g = e^{th}$ where $t$ is a parameter which we set to $t = 1$ after taking appropriate derivatives. See Gutowski or Hugh Osborn's notes for details. The result for the adjoint covariant derivative is

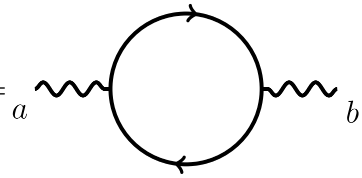
$$D_\mu\psi = \partial_\mu\psi + (\mathrm{ad}\ ieA_\mu)\psi = \partial_\mu\psi + [ieA_\mu, \psi]. \tag{2.4}$$

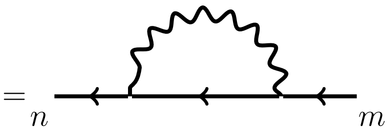
For an SU($N$) gauge theory with representation $\mathbf{r}$ generators $T^a_\mathbf{r}$ the (quadratic) Casimir $C_r(\mathbf{r})$ and the Dynkin index $T(\mathbf{r})$ are given by [5]

$$C_2(\mathbf{r})\delta^m_n = (T^a_\mathbf{r})^m_\ell (T^a_\mathbf{r})^\ell_n \tag{2.5}$$

$$T(\mathbf{r})\delta^{ab} = (T^a_\mathbf{r})^m_n (T^b_\mathbf{r})^n_m. \tag{2.6}$$

These can be understood diagrammaticaly.

$T(\mathbf{r})\delta^{ab} =$ 


$C(\mathbf{r})\delta^m_n =$ 


In fact, using this diagrammatic interpretation, we can close the external lines of each diagram to obtain an equivalent two-loop diagram. This leads to the relation $d(\mathbf{r})C_2(\mathbf{r}) = d(\mathbf{Ad})T(\mathbf{r})$. We use the standard normalization that

$$d(\Box) = N \qquad\qquad d(\mathbf{Ad}) = N^2 - 1 \tag{2.7}$$

$$T(\Box) = \frac{1}{2} \qquad\qquad T(\mathbf{Ad}) = N \tag{2.8}$$

$$C_2(\Box) = \frac{N^2-1}{2N} \qquad\qquad C_2(\mathbf{Ad}) = N. \tag{2.9}$$

**Comments on gauge redundancy**. Gauge (local) symmetries are different from global symmetries: they are redundancies in the way that a system is described. For example, for a U(1) gauge theory the photon has only two physical polarizations but is described by a four-component field. The longitudinal polarization is removed due to the photon being massless, but the additional degree of freedom that must be removed is precisely the gauge redundancy—we are free to add to the vector potential (photon) any gauge transformation since it is projected out in any physical quantity. Note, somewhat subtly, that there is also a global component of a gauge transformation which gives the current by which the gauge fields couple to matter fields. Finally, this picture of gauge redundancy is much more subtle when this 'symmetry' is broken in the Higgs phase. It is not technically the case that the gauge symmetry 'breaks' to a smaller subgroup as in the case of a global symmetry—though the end result is the same. What technically happens is that the parameterization of the order parameter (Higgs) field introduces an additional redundancy (precisely the subgroup that one 'breaks' to) while giving masses to the heavy gauge fields. For a good discussion of this last point, see chapter 8 of the QFT textbook by Banks [14].

## 2.3 Reminder: basic SUSY gauge theory facts

Let us quickly remind ourselves of some aspects of a supersymmetric gauge theory. A good refresher is Section 4 of the latest version ($v > 6$) of Stephen Martin's SUSY Primer [15], which covers superspace techniques.

Recall that under a gauge transformation, a chiral superfield $\Phi$ transforms as $e^{\Lambda^a T^a}\Phi$, where $\Lambda$ is a chiral superfield parameterizing the gauge transformation. The canonical Kähler potential $\Phi^\dagger\Phi$ is clearly not gauge invariant and doesn't include the correct gauge covariant derivative terms which couple the matter fields to the vector fields. Using Terning's conventions [5], the Kähler potential and vector superfield transform as,

$$K = \Phi^\dagger e^{gT^aV^a}\Phi \qquad\qquad V^a \to V^a + \Lambda^a + \Lambda^{a\dagger} + \mathcal{O}(V^a\Lambda^a). \tag{2.10}$$

More generally, the gauge-invariant Kähler potential takes the form $K = K(\Phi^\dagger, e^{gTV}\Phi)$ and the transformation of the vector superfield obeys

$$e^{T^a V^a} \to e^{T^a \Lambda^{a\dagger}} e^{T^a V^a} e^{T^a \Lambda^a}. \tag{2.11}$$

In writing the gauge parameter as a chiral superfield we see explicitly that supersymmetry also enlarges that gauge symmetry. It is conventional to work in the **Wess-Zumino gauge** in which the vector superfield is restricted to the vector, gaugino, and auxiliary $D$ term. This *super*-gauge choice breaks supersymmetry (by projecting to a subspace of the supermultiplet) but preserves the usual gauge redundancy of a non-supersymmetric quantum field theory.

**Gauge transformations of a vector superfield**. It is worth remarking that even through the entire vector supermultiplet is in the adjoint representation of the gauge group, the definition of a gauge transformation has also been supersymmetrized. That is to say that the gauge transformation parameter is itself a superfield with different components. Because of this, the different components of a vector multiplet transform differently under a gauge transformation. In Wess-Zumino gauge this can be seen from the different gauge transformation properties of the vector versus the gaugino. While the gaugino transforms precisely as one would expect for a *matter* field in the adjoint representation, the vector plays a special role as the connection of the gauge redundancy and thus transforms with an additional derivative term. (One could have also argued this based on Lorentz indices.)

When writing down a supersymmetric Lagrangian we may make use of $F$ terms and $D$ terms since these are invariant under SUSY transformations modulo total derivatives which vanish in the action. Of course, the superfields which furnish the $F$ and $D$ terms are generally not the same as the matter and vector content of the theory. Instead, one must form from these the appropriate chiral and vector superfields which are gauge invariant and then take the $F$ and $D$ terms (respectively) of these products as Lagrangian terms.

Note that one may alternately write even the $D$ terms as $F$ terms by using the identity $D^\alpha D_\alpha(\theta\theta) = -4$, where $D_\alpha$ is the SUSY covariant derivative. The conjugate identity also holds, $\bar{D}\bar{D}(\bar{\theta}\bar{\theta})$, where we have suppressed the spinor indices of $\bar{D}$. Thus the $D$-term of a vector superfield $V$ can be written as

$$V|_D = \int d^4\theta\, V = -\frac{1}{4}\int d^4\theta\, V\, \bar{D}\bar{D}(\bar{\theta}\bar{\theta}) = -\frac{1}{4}\int d^4\theta(\bar{\theta}\bar{\theta})\bar{D}\bar{D}V = -\frac{1}{4}\left(\bar{D}^2 V\right)_F, \tag{2.12}$$

where we dropped total derivatives upon integration by parts. This is the origin of the funny form of the gauge field strength superfield,

$$\mathbb{W}_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V, \tag{2.13}$$

from which we can see that this is the same as a Kähler potential term for $D_\alpha V$, which better resembles the non-supersymmetric gauge field strength $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$.

## 2.4 Moduli space

We will see that in SYM theories one typically finds **flat directions** or **moduli** in the field space. These are directions in the scalar fields with vanishing potential. When supersymmetry is broken these tree-level flat directions are often lifted through quantum corrections, i.e. by the Coleman Weinberg potential. In that case these directions are called **pseudomoduli**. We can now study how these flat directions arise in super QCD. At the bare minimum this theory will have a $D$-term potential since it is a gauge theory. It needn't necessarily have any superpotential, so we will ignore the superpotential contribution for now[7].

Recall that the $D$ terms are the auxiliary fields of the vector superfield. Their value is fixed in terms of the *matter* fields by virtue of the on-shell equation of motion which is algebraic. To see this, recall that the $D$ term pops up both in the canonical gauge-invariant Kähler potential for the matter fields $\Phi^\dagger e^{gTV}\Phi$ and in the field strength superpotential term $\frac{1}{4}\mathbb{W}^2$. The Kähler potential gives a piece which is linear in $D$, $\sim \phi^\dagger g T^a D^a \phi$, while the superpotential term gives a piece which is quadratic in $D$. As a reminder for the latter, recall that $\mathbb{W} \sim \lambda + \theta D + \cdots$. Thus the equation of motion sets $D^a \sim \phi^\dagger g T^a \phi$. This is important for determining the moduli space of the theory.

### 2.4.1 Example: $U(1)$

As a toy example, let's start with $U(1)$ SQED. The squark potential then takes the form

$$V = (Q^\dagger Q - \widetilde{Q}^\dagger Q)^2. \tag{2.14}$$

The moduli space is given by the values of $\langle Q\rangle$ and $\langle\widetilde{Q}\rangle$ such that $V$ is minimized. Modding out by the gauge redundancy the moduli space is parameterized by a complex parameter $a$,

$$\langle Q\rangle = \langle\widetilde{Q}\rangle = a. \tag{2.15}$$

Note that the different values of $a$ parameterize *inequivalent* vacua of the theory. Compare this to the the vacuum manifold of the usual Higgs mechanism where each point on the vacuum manifold is physically equivalent to any other since those points only differ by a transformation by the unbroken gauge generator.

For $a \neq 0$ the gauge group is broken by the **super Higgs mechanism**. In the usual Higgs mechanism a massless gauge field obtained a mass by 'eating' a scalar field which took the place of the longitudinal mode. Here we promote the fields to superfields. The gauge superfield acquires a mass $|a|$ by eating an entire chiral superfield. It is easy to check that the usual Higgs mechanism is subsumed in the super Higgs mechanism; consider the squark kinetic term,

$$\mathcal{L}_{Q,\text{kin}} \sim \int D_\mu Q^\dagger D^\mu Q + D_\mu \widetilde{Q}^\dagger D^\mu \widetilde{Q}, \tag{2.16}$$

upon giving the squarks vevs as in (2.15), this manifestly gives a mass term $|a|^2$ to the photon. The $U(1)$ gauge invariance is broken and the photon acquires a mass that depends on the particular point in the moduli space in which our theory happens to land.

[7] In general the superpotential is highly constrained by the global symmetries of the theory.

There is an excellent discussion of this theory (and its cousins) in Strassler's unorthodox review [3]. We will highlight just one small part of the story. The $D$-term condition that minimizes (2.14) tells us that

$$|Q|^2 - |\widetilde{Q}|^2 = 0, \tag{2.17}$$

which, in words, says that gauge invariance (the $D$ term conditions) imposes that the vevs of the two squark fields should have the same magnitudes. In other words, instead of minimizing the $D$-term potential we could as well have *complexified* the gauge redundancy. In other words, (2.17) modded out by the usual $U(1)$ gauge invariance is completely equivalent to modding out by

$$Q \to \alpha Q \tag{2.18}$$

$$\widetilde{Q} \to \alpha^* \widetilde{Q} \tag{2.19}$$

for $\alpha \in \mathbb{C}$. It is natural to parameterize our moduli space in terms of the vev of a gauge invariant object, $M \equiv Q\widetilde{Q}$, so that $\langle M \rangle = a^2$ tells us everything about where we live on the moduli space. (We can call $M$ the modulus field.) The classical Kahler potential can also be written in terms of $M$,

$$K_{\text{cl}} = Q^\dagger e^V Q + \widetilde{Q}^\dagger e^{-V} \widetilde{Q} = 2\sqrt{M^\dagger M}. \tag{2.20}$$

The 'meson' $N$ is the effective gauge-invariant low-energy degree of freedom so that $2\sqrt{M^\dagger M}$ can be understood to be the effective Kähler potential at low energies. The Kähler potential has a singularity at $X = 0$. This is actually a 'singularity' and not just a 'zero' since the Lagrangian comes from taking derivatives of the Kähler potential. Singularities are a signal of our theories trying to tell us something. In this case the theory is telling us that there are new degrees of freedom that should be in the effective action. We know exactly what these are: at $\langle X \rangle = 0$ the gauge symmetry is unbroken and there are massless gauge fields that cannot be excluded from any 'low energy' action.

**Reminder about the Kähler potential.** As an aside, remember that the scalar potential is given by

$$V(\phi_i) = W^\dagger_{\bar{j}} (K^{-1})^{i\bar{j}} W_i. \tag{2.21}$$

For a non-canonical Kähler potential this gives us a non-trivial Kähler metric, $K_{i\bar{j}} \neq \delta_{i\bar{j}}$. When this is true it is possible that $W_i = 0$ is *no longer sufficient* to determine that SUSY is unbroken, even in a theory of only chiral superfields. Consider, for example, an exercise from Strassler's lectures [3]. Given a theory of a single chiral superfield $W = y\Phi^3$, we can define a new chiral superfield $\Sigma = \Phi^3$ so that $dW/d\Sigma \neq 0$ even when $\Sigma = 0$. For an excellent introductory analysis of supersymmetry with a general Kähler potential, see the lectures by Argyres [16] and Bilal [17]. Such theories are often called SUSY nonlinear sigma models (NL$\Sigma$M) because the Kähler potential forces the scalars to live on a complex manifold for which there is a very geometric interpretation for the quantities that appear in the Lagrangian. (An analogous and much simpler thing occurs in spontaneous symmetry breaking phenomena in 'vanilla' QFT, but in those cases the manifold's geometry is usually trivial.)

### 2.4.2 Case $F < N$

We assume that we have an $SU(N)$ theory with $F < N$ flavors of 'quarks' $\phi_{im}$ in the fundamental and 'antiquarks' $\overline{\phi}^{im}$ in the anti-fundamental, where $i = 1, \cdots, F$ and $m = 1, \cdots, N$. The $D$-term for this theory are

$$
\begin{aligned}
D^a &= \sum_i \phi_i^\dagger T^a \phi_i + \overline{\phi}_i^\dagger \overline{T}^a \overline{\phi}_i \\
&= \left[ \sum_i \left(\phi^\dagger\right)^{in} \phi_{im} - \sum_i \overline{\phi}^{in} \left(\overline{\phi}^\dagger\right)_{im} \right] (T^a)_n{}^m .
\end{aligned}
$$

where we understand that the $\phi$s really mean $\langle \phi \rangle$. We can define the $N \times N$ matrices $D^n{}_m$ and $\overline{D}^n{}_m$,

$$
\begin{aligned}
D^n{}_m &= \left(\phi^\dagger\right)^{in} \phi_{im} \\
\overline{D}^n{}_m &= \overline{\phi}^{in} \left(\overline{\phi}^\dagger\right)_{im} .
\end{aligned}
$$

The condition that our $D$-term scalar potential vanishes (the '$D$-flatness condition') then imposes $D^a = 0$. Since the generators $T^a$ are traceless, a solutions is

$$
D^n{}_m - \overline{D}^n{}_m = \alpha \mathbb{1}
$$

for some overall constant $\alpha$. We may now use an $SU(N)$ gauge transformation to diagonalize the $D$ and $\overline{D}$ matrices. In the case $F < N$. Then from their definition we see that the $D$ and $\overline{D}$ matrices can have at most $F$ nonzero eigenvalues. Thus they must take the form

$$
D = \mathrm{diag}(v_1^2, v_2^2, \cdots, v_F^2, \underbrace{0, \cdots, 0}_{(N-F)}).
$$

Imposing $D - \overline{D} = \alpha \mathbb{1}$ then imposes that $\overline{D}$ must also be a diagonal matrix. By the structure of the zero and non-zero entries, we establish that the $D$-flatness condition can only be satisfied for $\alpha = 0$. From this we may write the solutions for our quark fields,

$$
\langle \phi \rangle = \langle \overline{\phi}^\dagger \rangle = \left( \begin{array}{ccc} v_1 & & \\ & \ddots & \\ & & v_F \\ \hline 0 & \cdots & 0 \end{array} \right) . \tag{2.22}
$$

This spontaneously breaks $SU(N) \to SU(N-F)$. We observe the **super Higgs mechanism** at work: we started with $(2F) \times N$ chiral superfields and found a vev where we have a number of broken generators

$$
(N^2 - 1) - \left((N-F)^2 - 1\right) = 2NF - F^2,
$$

each of which 'eats' a chiral superfield. The number of $D$-flat directions is then the number of chiral superfields minus the number of broken generators,

$$(2NF) - (2NF - F^2) = F^2.$$

In the usual Higgs mechanism a massless vector eats a massless Goldstone boson. The exact same effect occurs here, but due to supersymmetry the entire superfields must be included. Conceptually the actual 'coupling' of the two superfields occurs between the massless vector component and the Goldstone scalar, so one can think of the super Higgs mechanism as the joining of two superfields due to the mixing of one of each of their components due to the regular Higgs mechanism. After this feast, the remaining $F^2$ massless degrees of freedom are parameterized by an $F \times F$ meson field,

$$M_i^{\ j} = \overline{\phi}^{jn} \phi_{ni}. \tag{2.23}$$

There is actually a more general theorem by Luty and Taylor [18] regarding this:

**Theorem 2.1** (Luty-Taylor)**.** *The classical moduli space of degenerate vacua can always be parameterized by independent, holomorphic, gauge-invariant polynomials.*

*Proof.* A heuristic proof is provided in Intriligator and Seiberg's lecture notes on Seiberg duality [2]. Setting the [$D$-term] potential to zero and modding out by the gauge group is equivalent to modding out by the complexified gauge group. The space of chiral superfields modulo the complexified gauge group can be parameterized by the gauge invaraint polynomials modulo any classical relations. Then, Intriligator and Seiberg claim, this theorem follows from geometrical invariant theory [19]. For a proper proof the reader is directed to the original paper by Luty and Taylor [18]. □

### 2.4.3 Case $F \geq N$

Before moving on let's quickly cover the case $F \geq N$. As before the $D$-flatness condition is still $D - \overline{D} = \rho \mathbb{1}$, where $\rho$ is some constant. We can again use the $SU(N)$ gauge degree of freedom to diagonalize the $D = (\phi^\dagger)^i \phi_i$ and $\overline{D}$ matrices, though now they are of full rank and we may use the $D$-flatness condition to write $\overline{D}$ in terms of the eigenvalues of $D$ and the constant $\rho$,

$$D = \begin{pmatrix} |v_1|^2 & & \\ & \ddots & \\ & & |v_N|^2 \end{pmatrix} \qquad \overline{D} = \begin{pmatrix} |v_1|^2 - \rho & & \\ & \ddots & \\ & & |v_N|^2 - \rho \end{pmatrix}. \tag{2.24}$$

This implies that we may write the $\langle \phi \rangle$ and $\langle \overline{\phi} \rangle$ matrices as

$$\langle \phi \rangle = \left( \begin{array}{ccc:c} v_1 & & & \\ & \ddots & & 0 \\ & & v_n & \end{array} \right) \qquad \langle \overline{\phi} \rangle = \left( \begin{array}{ccc} v_1 & & \\ & \ddots & \\ & & v_N \\ \hdashline & 0 & \end{array} \right). \tag{2.25}$$

Now we see that $SU(N)$ is completely broken at a generic point on the moduli space. This means that we have $(N^2-1)$ broken generators and thus $[2NF-(N^2-1)]$ light $D$-flat directions in field space. Again we parameterize these degrees of freedom by 'gauge-invariant polynomials',

$$M_i{}^j = \overline{\phi}^{jn}\phi_{ni} \tag{2.26}$$
$$B_{i_1\cdots i_N} = \phi_{n_1 i_1}\cdots\phi_{n_N i_N}\epsilon^{n_1\cdots n_N} \tag{2.27}$$
$$\overline{B}_{i_1\cdots i_N} = \overline{\phi}^{n_1 i_1}\cdots\overline{\phi}^{n_N i_N}\epsilon_{n_1\cdots n_N}. \tag{2.28}$$

But wait! We find that we have too many degrees of freedom. That's okay. We've forgotten to impose the classical constraints to which these fields are subject,

$$B_{i_1\cdots i_N}\overline{B}^{j_1\cdots j_N} = M_{[i_1}{}^{j_1}\cdots M^{j_N}{}_{i_N]} \sim \det M \tag{2.29}$$

## 2.5 The holomorphic gauge coupling

Recall that the action for a vector superfield is conventionally written as

$$\mathscr{L} = \frac{1}{4}\int d^2\theta\, \mathbb{W}^{a\alpha}\mathbb{W}^a_\alpha + \text{h.c.} \tag{2.30}$$

In this case, the gauge coupling $g$ shows up in the kinetic term for the chiral superfields

$$\mathscr{L}_{\text{kin}} = \int d^4\theta\, \phi^\dagger e^{gV^aT^a}\phi. \tag{2.31}$$

We can redefine $\mathbb{W}$ by absorbing the coupling into the vector superfield,

$$\widetilde{V}^a = gV^a, \tag{2.32}$$

where we are no longer canonically normalized, but we are in some sense using a natural normalization[8]. Then the vector Lagrangian takes the form

$$\mathscr{L} = \frac{1}{4g^2}\int d^2\theta\, \mathbb{W}^{a\alpha}\mathbb{W}^a_\alpha + \text{h.c.} \tag{2.33}$$

We know that there are also non-perturbative effects that contribute to this Lagrangian, i.e. the CP-violating $\Theta_{\text{YM}}$ term. We can include this effect by defining a **holomorphic gauge coupling**[9],

$$\tau \equiv \frac{4\pi i}{g^2} + \frac{\Theta_{\text{YM}}}{2\pi} \tag{2.34}$$

[8]This can be understood, for example, by considering the renormalization of the gauge coupling in ordinary (non-supersymmetric) field theory. The only diagrams that contribute to this renormalization come from loop contributions to the gauge field propagator. This tells us that $g$ is 'really' something associated to the vector field, not necessarily the coupling of the vector to fermions.

[9]As noted in Appendix A, there seem to be many 'standard' normalizations for $\tau$ which differ by factors of, e.g., $2\pi$. I audibly groan every time I read a paper with a different normalization.

Our vector superfield Lagrangian finally takes the form

$$\mathscr{L} = \frac{1}{16\pi i}\int d^2\theta\, \tau \mathbb{W}^{a\alpha}\mathbb{W}^a_\alpha + \text{h.c.} \tag{2.35}$$

The canonically normalized Lagrangian is trivially obtained by multiplying through by $g^2$. Since $\tau$ only appears under the $d^2\theta$ of the superpotential, it is manifestly a holomorphic parameter. Note that some people will write this as the imaginary part of the first term, accounting for the appropriate factors of $i$; this seemingly odd notation is just the analog of saying $2\text{Re}X = X + \text{h.c.}$ Recall the RG equations for the perturbative coupling,

$$\mu\frac{dg}{d\mu} = -\frac{b}{16\pi^2} \tag{2.36}$$

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2}. \tag{2.37}$$

By the way, from (2.36) we should *already* know what the value of $b$ is at any given scale:

$$b = 3N_{\text{eff}} - F_{\text{eff}}, \tag{2.38}$$

where we have been very careful to write that this is the *effective* number of colors $N_{\text{eff}}$ and the *effective* number of flavors $F_{\text{eff}}$. This is important since in the following sections we'll be exploring the moduli space of SQCD with $N$ colors and $F$ flavors, but as we get away from the origin of the moduli space the *effective* number of colors and flavors changes.

Applying (2.36) to $\tau$, we may write

$$\tau_{\text{1-loop}} = \frac{1}{2\pi i}\left[b\log\left(\frac{|\Lambda|}{\mu} + i\Theta_{\text{YM}}\right)\right] \tag{2.39}$$

$$= \frac{b}{2\pi i}\log\left(\frac{\Lambda}{\mu}\right), \tag{2.40}$$

where have defined the **holomorphic dynamical scale**

$$\Lambda = |\Lambda|e^{i\Theta_{\text{YM}/b}}. \tag{2.41}$$

The real quantity $|\Lambda|$ plays the role of $\Lambda_{\text{QCD}}$ from non-supersymmetric chromodynamics, but the holomorphic quantity $\Lambda$ is what will be very important for us. We can also invert the expression to write

$$\Lambda = \mu e^{2\pi i b/\tau}. \tag{2.42}$$

Now we claim that this does not receive any further corrections within perturbation theory, i.e. that (2.40) is the full perturbative expression.

**Theorem 2.2.** *The holomorphic coupling is only perturbatively renormalized at one loop. It does, however, receive non-perturbative corrections from instanton effects.*

*Proof.* We've written the one-loop renormalization of $g$ in Eq. (2.40). We now have to show that this only gets corrections from instantons. The key will be to consider the $\Theta_{\text{YM}}$ dependence. We know that $\Theta_{\text{YM}}$ is a term which multiplies an $F\widetilde{F}$ in the Lagrangian,

$$F\widetilde{F} = 4\epsilon^{\mu\nu\rho\sigma}\partial_\mu \text{Tr}\left(A_\nu \partial_\rho A_\rho + \frac{2}{3}A_\nu A_\rho A_\sigma\right). \tag{2.43}$$

This is a total derivative and has no effect in perturbation theory (as expected from a non-perturbative instanton effect); in perturbation theory $\Theta_{\text{YM}}$ is just a constant because it is a total derivative. However, this term contributes to a topological winding number, $n$,

$$\frac{\Theta_{\text{YM}}}{32\pi^2}\int d^4x\, F\widetilde{F} = n\Theta_{\text{YM}}, \tag{2.44}$$

c.f. the usual index theorem. In the path integral $\int dA \exp(iS) \sim \int dA \exp(in\Theta_{\text{YM}})$. Thus we see that the $\Theta_{\text{YM}}$ must be periodic in $2\pi$, i.e. $\Theta_{\text{YM}} \to \Theta_{\text{YM}} + 2\pi$ must be a symmetry of the theory. Under this transformation the dynamical scale goes as

$$\Lambda \to e^{2\pi i/b}\Lambda. \tag{2.45}$$

This, in turn, affects the effective superpotential $W_{\text{eff}} = \tau/(16\pi i)\mathbb{W}^2$ through the dependence of the holomorphic coupling on $\Lambda$,

$$\tau = \frac{b}{2\pi i}\log\left(\frac{\Lambda}{\mu}\right) + f(\Lambda,\mu), \tag{2.46}$$

where the first term is the one-loop result that we derived and the second term represents an arbitrary function that includes any higher-loop corrections. Remember from (2.34) that we may write $\tau$ in terms of $\Theta_{\text{YM}}$. Under the transformation of $\Lambda$ in (2.45), we see from (2.41) and (2.34) that $\tau \to \tau + 1$. But we can also see that the expected shift is already saturated by the first term on the right-hand size of (2.46). Since the first term already saturates the correct behavior, the second term must be invariant under the transformation. We can then write out the second term as

$$f(\Lambda,\mu) = \sum_{n=1}^{\infty} a_n \left(\frac{\Lambda}{\mu}\right)^{bn}, \tag{2.47}$$

where the form is set by demanding weak coupling as $\Lambda \to 0$ (we want the perturbative result in this limit). Terms of this form, however, just represent instanton effects. Recall the instanton action,

$$S_{\text{inst}} = \frac{8\pi^2}{g^2} \qquad \Rightarrow \qquad e^{S_{\text{inst}}} \sim e^{2\pi i\tau} = \left(\frac{\Lambda}{\mu}\right)^b. \tag{2.48}$$

Thus instanton effects in SUSY gauge theories will always appear with a prefactor of $(\Lambda/\mu)^b$. Thus we have the result that $\tau$ is only [perturbatively] renormalized at one-loop order. $\square$

One can also determine the instanton corrections. For example, Seiberg and Witten famously found exact expressions for the $a_n$ coefficients in $\mathcal{N}=2$ SYM. For review see, e.g., [20, 21, 22].

At this point you might want to brush up on your instantons. In a pinch, one can look over the relevant chapter in Terning [5]. A well-written and more pedagogical treatment of instantons can be found in the author's A-exam[10]. Comprehensive guides to calculations in supersymmetry can be found in the notes by Shifman and Vainshtein (the S and V in NSVZ) [23] and a separate set by Bianchi, Kovacs, and Rossi [24]. More general expositions can be found in Dine [10], Coleman's 'The uses of instantons' in *Aspects of Symmetry* [25], Vandoren and van Nieuwenhuizen's lectures [26], Manton and Sutcliffe [27], Rajaraman [28], and a lecture from Michael Peskin's 2005 course on quantum field theory at Stanford University, [11].

## 2.6 The NSVZ $\beta$-function

The first piece of 'Seibergology' that students of supersymmetry learn is the clever use of holomorphy (and asymptotics) to prove the non-renormalization of the superpotential. We also know that the (holomorphic) gauge coupling is only renormalized at one-loop order. We know, however, that this isn't the whole story. Even though the *holomorphic* couplings in superpotential are not renormalized, there's nothing protecting the Kähler potential from running. The renormalization of the Kähler potential then changes the canonical normalization of fields and hence changes the *physical* couplings for interactions in the superpotential. With this in mind, now want the $\beta$ function for the *canonically normalized* gauge coupling. NSVZ is an *all*-loop expression for the $\beta$ function for $\tau$. There will be two sources: the canonical normalization of chiral superfields and that of vector superfields.

Let's start with one of our favorite toy models of chiral superfields, the Wess Zumino model,

$$W = \frac{1}{2}\hat{m}\Phi^2 + \frac{1}{3}\hat{\lambda}\Phi^3. \tag{2.49}$$

As we have written holomorphic quantities with a hat to distinguish them from physical quantities $m$ and $\lambda$. At tree level $m = \hat{m}$ and $\lambda = \hat{\lambda}$, but at loop level we have to account for the renormalization of the kinetic term $K = Z\Phi^\dagger\Phi$, leading to

$$\mathcal{L}_{\text{kin}} = Z\left|\partial\phi\right|^2 + Z\bar{\psi}i\bar{\partial\!\!\!/}\psi, \tag{2.50}$$

where $Z$ depends on the parameters of the theory (including the renormalization point $\mu$) via

$$Z = 1 + c\frac{\hat{\lambda}\hat{\lambda}^*}{16\pi^2}\log\left(\frac{\hat{\Lambda}^2}{\mu^2}\right). \tag{2.51}$$

It should be clear that this is true. This represents the wavefunction renormalization from loop corrections to the two point scalar function. Consider the loop of internal fermions with Yukawa couplings to the external scalars (the loop with an internal scalar and auxilliary field is related by SUSY). The prefactor of $|\hat{\lambda}|^2/16\pi^2$ is easy to read off. The logarithm should also be expected: we

[10] http://www.lepp.cornell.edu/~pt267/files/documents/A_instanton.pdf

[11] http://www.slac.stanford.edu/~mpeskin/Physics332/instantons.pdf

know that the contribution to the scalar *mass* term can be quadratically divergent in a general QFT. The divergence of the *kinetic* term is two powers less and is thus only logarithmically divergent.

To canonically normalize we have to rescale our field

$$\Phi \to \Phi' = Z^{1/2}\Phi \tag{2.52}$$

and define physical parameters

$$m = \frac{\hat{m}}{Z} \qquad\qquad \lambda = \frac{\hat{\lambda}}{Z^{3/2}}. \tag{2.53}$$

The important quantity describing the running of these physical parameters is the familiar **anomalous dimension**,

$$\gamma = -\frac{\partial \log Z}{\partial \log \mu}. \tag{2.54}$$

**Why is it called an anomalous dimension?** In a general (not necessarily supersymmetric) theory, quantum (loop) effects generate corrections to the kinetic terms that are *dimensionful*,

$$\mathcal{L} = z|\partial\phi|^2 + \cdots \tag{2.55}$$

with $[z] \neq 0$. Like any dimensionful parameter, we define a dimensionless parameter in terms of the natural scale of the theory (the renormalization scale $\mu$),

$$Z \equiv \frac{z}{\mu^\gamma}, \tag{2.56}$$

where $\gamma$ is defined here to be whatever power is required to make $Z$ dimensionless. The Lagrangian can then be written as

$$\mathcal{L} = Z\mu^\gamma|\partial\phi|^2 = |\partial\phi'|^2 + \cdots \tag{2.57}$$

after re-absorbing a dimensionless factor of $Z^{1/2}$ and a dimensionful factor of $\mu^{\gamma/2}$ into the definition of the field $\phi'$. This means that the field $\phi'$ has now picked up an 'anomalous dimension',

$$[\phi'] = 1 + \frac{1}{2}\gamma, \tag{2.58}$$

where the 1 represents the tree-level dimension. The true scaling dimension of $\phi$ has changed due to quantum effects. The formula for $\gamma$ can be understood by further that noting we could have written a $\beta$ function for the dimensionless coupling $Z$, giving

$$\beta_Z = -\gamma Z \Rightarrow \frac{\partial \ln Z}{\partial \ln \mu} = -\gamma. \tag{2.59}$$

These anomalous dimensions turn out to be very important for (super-)conformal theories. Even in the neighborhood of a conformal theory, one can make use of 'deep theorems' in QFT to give constraints on the anomalous dimension. See, for example, [29]. (See also Strassler's lectures for an appropriately unorthodox persepctive [3].) The meaning of the anomalous dimension, by the way, really comes from the anomaly of dilatations. See Coleman's Erice lectures for an excellent discussion [25].

In this chiral theory, the anomalous dimensions encode everything there is to know about how the renormalization of the physical couplings. As explained in Strassler's lectures [3], the *exact* $\beta$ function for the Yukawa coupling of the Wess-Zumino model is

$$\beta_y = \frac{3}{2} y\gamma(y), \tag{2.60}$$

where in practice the anomalous dimension must be calculated to some finite order in perturbation theory. Before moving on to the vector superfields, let us point out that this rescaling of the chiral superfields may affect the anomaly due to the fermion's rescaling. The anomaly contributes to the $F\widetilde{F}$ term, so we would expect such a modification to feed into the physical gauge coupling renormalization.

We now perform the analogous analysis for the gauge fields. In the holomorphic basis (annotated by a subscript 'h') the gauge kinetic term takes the form

$$\mathcal{L}_{\rm h} = \frac{1}{4}\int d^2\theta \, \frac{1}{g_h^2} \mathbb{W}^{a\alpha}(V_{\rm h})\mathbb{W}^a{}_\alpha(V_{\rm h}) + \text{h.c.} \tag{2.61}$$

The factor of $1/g_{\rm h}^2$ is really shorthand for the holomorphic coupling as $\tau/4\pi i$. To pass to the canonical basis one must perform a rescaling of the variables to absorb the coupling into the gauge field,

$$\mathcal{V}_{\rm h} = g_{\rm c} V_{\rm c}, \tag{2.62}$$

so that the Lagrangian for canonically-normalized fields takes the form

$$\mathcal{L}_{\rm c} = \frac{1}{4}\int d^2\theta \left(\frac{1}{g_{\rm c}^2} - i\frac{\Theta_{\rm YM}}{8\pi^2}\right) \mathbb{W}^{a\alpha}(g_{\rm c}V_{\rm c})\mathbb{W}^a{}_\alpha(g_{\rm c}V_{\rm g}) + \text{h.c.} \tag{2.63}$$

Again, we are rescaling the *whole* vector superfield, including the gaugino. For the gaugino picking up a phase is like a chiral transformation so that this can give an anomaly. Recall that anomalies come from the change in the path integral measure due to a transformation $\psi \to e^{i\alpha}\psi$. The formula for a fermion in representation $\mathbf{r}$ is

$$D[g_c V_c] = D[V_c] \exp\left(-\frac{i}{4}\int d^4x \int d^2\theta \, \frac{T(\mathbf{r})}{8\pi^2} \alpha \mathbb{W}^\alpha(g_{\rm c}V_{\rm c})\mathbb{W}_\alpha(g_{\rm c}V_{\rm c}) + \text{h.c.}\right). \tag{2.64}$$

Note that $\mathbb{W}^\alpha\mathbb{W}_\alpha$ contains our favorite $F\widetilde{F}$ term. In the present case of interest the transformation is $V_h = g_{\rm c}V_{\rm c}$ so that the analog of the $e^{i\alpha}$ transformation is $g_{\rm c}$, or $\alpha = -i\log g_{\rm c}$. The additional term in the path integral measure is called the **Konishi anomaly**. This can be understood as an IR effect associated with massless particles; while the Wilsonian effective action is holomorphic in the RG scale $\mu$, the 1PI effective action becomes singular because of the anomaly.

Applying this formula to pure Yang-Mills we obtain

$$D[V_h] \to D[V_h] \exp\left(-\frac{i}{4}\int d^4x \int d^2\theta \, \frac{1}{g_{\rm h}^2} \mathbb{W}^\alpha(V_{\rm h})\mathbb{W}_\alpha(V_{\rm h}) + \text{h.c.}\right) \tag{2.65}$$

$$= D[V_c] \exp\left(-\frac{i}{4}\int d^4x \int d^2\theta \left(\frac{1}{g_{\rm h}^2} - \frac{2T(\text{Ad})}{8\pi}\log g_{\rm c}\right) \mathbb{W}^\alpha(g_{\rm c}V_c)\mathbb{W}_\alpha(g_{\rm c}V_c) + \text{h.c.}\right). \tag{2.66}$$

In other words, the partition function is

$$Z = \int D[V_h] e^{\frac{i}{4}\int d^2\theta d^4x \frac{1}{g_h^2}\mathbb{W}(V_h)\mathbb{W}(V_h)+\text{h.c.}} \tag{2.67}$$

$$= \int D[V_c] e^{\frac{i}{4}\int d^2\theta d^4x \left(\frac{1}{g_h^2} - \frac{2T(\text{Ad})}{8\pi} \log g_c\right)\mathbb{W}(g_c V_c)\mathbb{W}(g_c V_c)+\text{h.c.}}. \tag{2.68}$$

This tells us that the canonically normalized gauge coupling is

$$\frac{1}{g_{\text{c}}^2} = \text{Re}\left(\frac{1}{g_{\text{h}}^2} - \frac{2T(\text{Ad})}{8\pi^2}\log g_{\text{c}}\right), \tag{2.69}$$

where this expression includes the anomaly. This is the ‘real’ relation between the holomorphic and canonically normalized gauge coupling.

Now let’s see what happens when we include matter fields. In the pure Yang-Mills case there was a contribution to the Konishi anomaly coming from the gaugino zero modes. For matter fields we should also expect a contribution from the matter fermions (quarks). The point is that integrating out a sliver of momentum space for a species of matter field will generate a non-holomorphic wavefunction renormalization factor $Z$. Canonically normalizing with respect to this wavefunction renormalization shifts the path integral measure so that the contribution to the anomaly takes the form $\ln Z\ \mathbb{W}\mathbb{W}$.

The chiral superfield rescaling is $Q' = Z^{1/2}Q$. The path integral measure for the $Q$ and $\widetilde{Q}$ fields in SQCD is

$$D[Q]D[\widetilde{Q}] = D[Q']D[\widetilde{Q}'] \exp\left(-\frac{i}{4}\int d^2\theta d^4x \frac{T(\text{Ad})}{8\pi^2}\log Z^{1/2}\mathbb{W}\mathbb{W} + \text{h.c.}\right). \tag{2.70}$$

Thus the full expression for the canonically normalized gauge coupling comes from including the anomalies in the $D[V_c]D[Q']D[\widetilde{Q}']$ measure,

$$\frac{1}{g_{\text{c}}^2} = \text{Re}\left(\frac{1}{g_{\text{h}}^2} - \frac{2T(\text{Ad})}{8\pi^2}\log g_c - \sum_j \frac{T(r_j)}{8\pi^2}\log Z_j\right). \tag{2.71}$$

Finally we arrive at an expression for the NSVZ $\beta$ function. For further discussion see the original literature [30, 31, 12] or the follow-up works by Arkani-Hamed and Murayama [32, 33]. The $\beta$ function for the canonically normalized gauge coupling is

$$\frac{d}{d\log\mu}\left(\frac{1}{g_{\text{c}}^2}\right) = \frac{-b}{16\pi^2} = \frac{d}{d\log\mu}\left(\frac{1}{g_h^2}\right) - \frac{2T(\text{Ad})}{8\pi^2}\frac{d\log g_{\text{c}}}{d\log\mu} - \sum_j \frac{T(r_j)}{8\pi^2}\frac{d\log Z_j^{1/2}}{d\log\mu}. \tag{2.72}$$

The first term here is just the flow of the one-loop exact coupling that we wrote in (2.37),

$$\frac{1}{d\log\mu}\left(\frac{1}{g_{\text{h}}^2}\right) = \frac{-b}{16\pi^2} = \frac{-1}{16\pi^2}\left(3T(\text{Ad}) - \sum_j T(r_j)\right). \tag{2.73}$$

The second term in (2.72) is simply

$$-\frac{2T(\mathrm{Ad})}{8\pi^2}\frac{d\log g_{\mathrm{c}}}{d\log\mu} = -\frac{g_{\mathrm{c}}^2}{2}\frac{d}{d\log\mu}\left(\frac{1}{g_{\mathrm{c}}^2}\right), \tag{2.74}$$

which we can move to the left-hand side of the equation. Finally, the last term is simply written in terms of the anomalous dimensions of the matter fields since

$$\frac{d\log Z_j^{1/2}}{d\log\mu} \equiv \gamma_j. \tag{2.75}$$

We thus end up with

$$\frac{d}{d\log\mu}\left(\frac{1}{g_{\mathrm{c}}^2}\right)\left(1-\frac{T(\mathrm{Ad})g_{\mathrm{c}}^2}{8\pi^2}\right) = -\frac{1}{16\pi^2}\left(3T(\mathrm{Ad})-\sum_j T(r_j)\right)-\sum_j\frac{T(r_j)}{16\pi^2}\gamma_j, \tag{2.76}$$

from which we derive the NSVZ $\beta$ function for the running of the canonically normalized gauge coupling at all-loop order,

$$\frac{d}{d\log\mu}\left(\frac{1}{g_{\mathrm{c}}^2}\right) = -\frac{1}{16\pi^2}\frac{3T(\mathrm{Ad})-\sum_j T(r_j)(1-\gamma_j)}{1-\frac{T(\mathrm{Ad})g_{\mathrm{c}}^2}{8\pi^2}}. \tag{2.77}$$

## 2.7 The Konishi Anomaly

This section comes from Jesse Thaler's TASI 2012 lectures. Suppose you have a gauge theory with U(1)-charged fermions. The [chiral] anomaly is generated by chiral rotations:

$$\psi \to e^{+i\alpha}\psi \tag{2.78}$$

$$\bar{\psi}^c \to e^{+i\alpha}\psi^c \tag{2.79}$$

$$\mathcal{L}(\psi,\psi^c) \to \mathcal{L}(e^{+i\alpha}\psi, e^{+i\alpha}\psi^c) + \frac{alpha}{64\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \tag{2.80}$$

Now let us promote the chiral rotation parameter $\alpha$ to a chiral superfield. This means we must have

$$\mathcal{L}(\psi,\psi^c) \to \mathcal{L}(e^{+i\alpha}\psi, e^{+i\alpha}\psi^c) + \frac{1}{16\pi^2}\int d^2\theta\,\alpha\mathbb{W}^\alpha\mathbb{W}_\alpha + \text{h.c.} \tag{2.81}$$

Consider the following Lagrangian:

$$\mathcal{L} = \int d^4\theta\,\left(Q^\dagger e^{2V}Q + \widetilde{Q}^\dagger e^{-2V}\widetilde{Q}\right)\left(1+\frac{X}{\Lambda}+\frac{X^\dagger}{\Lambda}+\cdots\right) \tag{2.82}$$

$$+\int d^2\theta m Q\widetilde{Q} + \text{h.c.} + \int d^2\theta\frac{1}{4g^2}\mathbb{W}^\alpha\mathbb{W}_\alpha + \text{h.c.} \tag{2.83}$$

If we now integrate out $Q\widetilde{Q}$ at the scale $m$, is there a loop-level coupling $X\mathbb{W}\mathbb{W}$? Holomorphy suggests suggests that this wouldn't happen since we only get the combination $X + X^\dagger$. However, if we do a field redefinition

$$Q \to Qe^{-X/\Lambda} \tag{2.84}$$

$$\widetilde{Q} \to \widetilde{Q}e^{-X/\Lambda}, \tag{2.85}$$

then we get rid of the linear $X + X^\dagger$ terms in the Kähler potential, at the cost of changing the superpotential by a term

$$\Delta W = me^{-2X/\Lambda}Q\widetilde{Q}. \tag{2.86}$$

Note, however, that the Konishi anomaly also gives us a term

$$\frac{1}{4g^2(\mu)} \to \frac{1}{4g^2(\mu)} - \frac{1}{16\pi^2}\frac{X}{\Lambda}. \tag{2.87}$$

Thus, at the scale $\mu$,

$$\frac{1}{4g^2(\mu, me^{-2X/\Lambda})} - \frac{1}{16\pi^2}\frac{X}{\Lambda} = \frac{1}{4g^2(\mu, m)} - \frac{b_0 - b_1}{32\pi^2}\frac{2X}{\Lambda} + \mathcal{O}(X^2) - \frac{1}{16\pi^2}\frac{X}{\Lambda}. \tag{2.88}$$

Since $b_0 - b_1 = -1$ for integrating out $Q\widetilde{Q}$, this is just $g^{-2}(\mu, m)/4 + \mathcal{O}(X^2)$. And indeed, the coupling to $X + X^\dagger$ vanishes as expected from holomorphy. Note that if $m = 0$ then there is no cancellation, and this falls under the purview of **anomaly mediation**.

## 2.8 Symmetries of SQCD

We will make extensive use of the symmetries of SQCD. Here we summarize the main results.

The charge table is:

| | $SU(N)$ | $SU(F)_L$ | $SU(F)_R$ | $U(1)_A$ | $U(1)_B$ | $U(1)_{R'}$ |
|---|---|---|---|---|---|---|
| $Q$ | $\square$ | $\square$ | 1 | 1 | 1 | 0 |
| $\widetilde{Q}$ | $\overline{\square}$ | 1 | $\square$ | 1 | $-1$ | 0 |
| $\lambda$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\psi_Q$ | $\square$ | $\square$ | 1 | 1 | 1 | $-1$ |
| $\psi_{\widetilde{Q}}$ | $\overline{\square}$ | 1 | $\square$ | 1 | $-1$ | $-1$ |
| $\Theta_{\text{YM}}$ | 1 | 1 | 1 | $2F$ | 0 | $2N - 2F$ |

Here we've written the $SU(N)$ gauge symmetry along with the non-Abelian $SU(F)^2$ flavor symmetry. In addition to these, we have three $U(1)$ symmetries. Naïvely from QCD we only expect baryon number $U(1)_B$ and the anomalous axial symmetry $U(1)_A$, but we have to remember that there are other fermions in the theory, namely the gaugino $\lambda$. The gaugino is a Weyl spinor and so may have a $U(1)$ charge, but since the other components in its supermultiplet are real, they cannot carry this $U(1)$ charge. This means that the gaugino's $U(1)$ charge must be an $R$-symmetry, which we call $R'$ for now.

We've also written the charges of the quark and anti-quark fermions: these are the same as their bosons with the $R$-charge decremented by one since the superspace coordinate $\theta$ soaks up one unit. As a sanity check, the gaugino having an $R$-charge (rather than an ordinary global $U(1)$) is consistent with the usual SQCD matter vertices: gauge-quark$^2$ and gaugino-quark-squark.

The last line of the table is the $\Theta_{\text{YM}}$ angle in Yang-Mills theory. Below we will be more sophisticated and package this into the holomorphic scale $\Lambda$. For now we can afford to be prosaic. Under a rotation $\alpha$ of the anomalous axial symmetry, $\Theta_{\text{YM}}$ is shifted by $2\alpha F$ coming from the $F$ quarks and the $F$ anti-quarks running in the triangle diagram. This shift is precisely what is meant when we say a symmetry is anomalous. We note, however, that $\Theta_{\text{YM}}$ also shifts under a rotation of the $R'$ symmetry: it shifts by $2N$ coming from the gauginos (in the adjoint representation) and by $-2F$ from the quarks.

Of course, for a simple gauge group there can only be one anomalous U(1). We may replace $R'$ take a linear combination of $R'$ and $A$ such that it is anomaly-free. Looking at the $\Theta_{\text{YM}}$ charges we can see that this combination is:

$$R = R' + \frac{F-N}{F} A. \tag{2.89}$$

With this choice $\Theta_{\text{YM}}$ is invariant and we have the standard assignment of $R$-charge to the squarks:

| | $U(1)_A$ | $U(1)_B$ | $U(1)_R$ |
|---|---|---|---|
| $Q$ | 1 | 1 | $\frac{F-N}{F}$ |
| $\widetilde{Q}$ | 1 | $-1$ | $\frac{F-N}{F}$ |
| $\lambda$ | 1 | 1 | 1 |
| $\psi_Q$ | 1 | 1 | $-\frac{N}{F}$ |
| $\psi_{\widetilde{Q}}$ | 1 | $-1$ | $-\frac{N}{F}$ |
| $\Theta_{\text{YM}}$ | $2F$ | 0 | 0 |

Here we've only written the U(1) charges. We note with foresight that one may choose many different $U(1)_R$ charge assignments, but there is a single unique anomaly-free $\text{U}(1)_R$ which is special since it is the symmetry that lives inside the superconformal algebra.

# 3 $F < N$: the ADS superpotential

We now review the famous result by Affleck, Dine, and Seiberg in the 1980s that instantons generate the so-called ADS superpotential [34, 35]. Along the way we'll learn how to use the moduli space to go to regions in parameter space where we can make definitive statements that carry over to the nonperturbative regime. In the following section we'll make use of the tools that we've developed to go over the $F \geq N$ case and finally address Seiberg duality.

## 3.1 Holomorphic scale as a spurion

The trick that we will employ is to promote the instanton power of the holomorphic scale $\Lambda^b$ to a spurion for anomalous symmetries. In particular, anomalies break global symmetries through instanton effects which manifest themselves via the **'t Hooft operator**[12],

$$\mathcal{O}_{\text{'t Hooft}} = \Lambda^b \prod_i {\psi_i}^{2T_i}, \tag{3.1}$$

where $T_i = T(\square) = 1/2$ for the fundamental representation. For a one-instanton background and under a chiral rotation, i.e. a rotation that acts independently on each chiral fermion $\psi_i$,

$$\psi_i \to e^{i\alpha q_i}\psi_i \tag{3.2}$$

$$\Theta_{\text{YM}} \to \Theta_{\text{YM}} - \alpha \sum_r n_r \cdot 2T(r) \tag{3.3}$$

$$\Lambda^b \to \Lambda^b e^{-i\sum_r n_r(2T(r))}, \tag{3.4}$$

where $n_r$ is the number of fermions in the representation $r$. If we recall that $\Lambda = |\Lambda| \exp(i\Theta_{\text{YM}}/b)$, we note that we can assign a fake (i.e. spurious) charge to $\Lambda$ so that the 't Hooft operator preserves the chiral symmetry,

$$q_\Lambda = -\sum_r 2n_r T(r). \tag{3.5}$$

For more on the NSVZ $\beta$ function and the Konishi anomaly, see the notes by Xi Yin[13].

## 3.2 The ADS Superpotential

Our goal is to write down the effective superpotential. We know that this is given by gauge-invariant polynomials. In fact, the symmetries of the theory allow us to further constrain the superpotential. Let's explicitly write out the representations of the relevant fields under all of these symmetries using a funny table of boxes,

| | $SU(N)$ | $SU(F)$ | $SU(F)$ | $U(1)_1$ | $U(1)_2$ | $U(1)_R$ |
|---|---|---|---|---|---|---|
| $Q$ | $\square$ | $\square$ | 1 | 1 | 0 | 0 |
| $\widetilde{Q}$ | $\overline{\square}$ | 1 | $\square$ | 0 | 1 | 0 |
| $\Lambda^b$ | 1 | 1 | 1 | $F$ | $F$ | $-2F+2N$ |

The $\Lambda^b$ charge under $U(1)_1$ and $U(1)_2$ are given by the prescription above to absorb anomalous charges. For example, because of (3.3), we can see that the $U(1)_1$ charge of $\Lambda^b$ must be

$$q_1[\Lambda^b] = -\sum_r 2T_r q_r = -2\left(\frac{1}{2}\right)(q_1[Q] + q_1[\widetilde{Q}])F = -2\left(\frac{1}{2}\right)F. \tag{3.6}$$

For the $U(1)_R$ let us remember that the bosons and fermion within a supermultiplet contain different $R$-charges,

$$R[\text{fermion}] = R[\text{boson}] - 1, \tag{3.7}$$

[12] I thank Leo van Nierop for teaching me how to correctly spell an pronounced Gerard 't Hooft's name. Ironically I still cannot properly pronounce "van Nierop."

[13] `http://www.people.fas.harvard.edu/~xiyin/beta.pdf`

so that $R[\psi_Q] = R[\psi_{\widetilde{Q}}] = -1$. Remembering that the Dynkin index for fermions is still 1/2, the quarks combine to contribute $-2F$ to the $R$-charge of the spurion, $R[\Lambda^b] = -2F$. We must remember, however, that there are other fermions in the theory coming from the gauge supermultiplet. Since $R[W] = 2$, we know that $R[\mathbb{W}^a\mathbb{W}_a] = 2$, and so the gaugino has $R$-charge $R[\lambda^a] = 1$. The gaugino Dynkin index is just $T(\text{adj}) = 1$, so this gives a contribution of $2N$ to $R[\Lambda^b]$. Thus we find $R[\Lambda^b] = 2(N - F)$. Note that all of the $U(1)$ symmetries defined here are anomalous, though two linear combinations are anomaly-free. In particular, we could have written a non-anomalous $U(1)_B$ and a new $U(1)_R$ along with an anomalous $U(1)_A$. We don't have to worry about this for now, but for reference the revised table looks like

| | $SU(N)$ | $SU(F)$ | $SU(F)$ | $U(1)_A$ | $U(1)_B$ | $U(1)_R$ |
|---|---|---|---|---|---|---|
| $Q$ | $\square$ | $\square$ | 1 | 1 | 1 | $\frac{F-N}{F}$ |
| $\widetilde{Q}$ | $\overline{\square}$ | 1 | $\square$ | 1 | -1 | $\frac{F-N}{F}$ |
| $\Lambda^b$ | 1 | 1 | 1 | $2F$ | 0 | 0 |

Finally, since the holmorphic scale is the only quantity carrying $R$-charge, we know that the superpotential must go as

$$W \sim \Lambda^{\frac{3N-F}{N-F}}. \tag{3.8}$$

Invariance under the $U(1)$ symmetries forces additional factors,

$$W \sim \left(\frac{\Lambda^{3N-F}}{Q^F\overline{Q}^F}\right)^{\frac{1}{N-F}}. \tag{3.9}$$

Further imposing flavor invariance and writing the superpotential in terms of gauge invariant polynomials (which parameterize the moduli space), we get the **ADS superpotential**,

$$W_{\text{ADS}} = C_{N,F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}}, \tag{3.10}$$

where we've written $M$ to be the gauge-invariant meson field and $C_{N,F}$ is a coefficient that we have to determine. We'll now do this for the particular case $F = N - 1$ and then we'll show that there are neat tricks we can do to derive more general combinations $(F, N)$.

**More generally:** We could have also written a term

$$b \ln \Lambda \mathbb{W}^a\mathbb{W}_a \tag{3.11}$$

which would be invariant under the above symmetries due to the transformation of the path integral under the anomaly. The gauge-invariant chiral superfields that we have available to us are for constructing a superpotential are $\mathbb{W}^\alpha\mathbb{W}_\alpha$, $\Lambda^b$, and $\det M$. The term above corresponds to a Wess-Zumino term. More generally, we could have written

$$W = \Lambda^{bn}(\mathbb{W}\mathbb{W})^m(\det M)^p \tag{3.12}$$

from which $U(1)_A$ and $U(1)_R$ symmetries impose

$$2 = 2m + 2p(F - N) \Rightarrow n = -p = \frac{1-m}{N-F}. \tag{3.13}$$

Requiring that our superpotential makes sense in the weak coupling limit $\Lambda \to 0$ (this boils down to requiring a *Wilsonian* effective theory) forces power of $\Lambda$ in (3.12) to be non-negative. We know from (2.38) that $b = 3N - F > 0$, so that we require $n \geq 0$. This, in turn, implies $p \leq 0$ andhence $m \leq 1$. On the other hand, $\mathbb{W}^a\mathbb{W}_a$ contains derivatives terms and so locality requires that it comes with a non-negative as well, $m \geq 0$ and $m \in \mathbb{Z}$. (The low-energy Wilsonian effective action must have a sensible derivative expansion.) Thus we are left with $m = 0, 1$. The case $m = 1$ gives (3.11), while $m = 0$ is precisely (3.10).

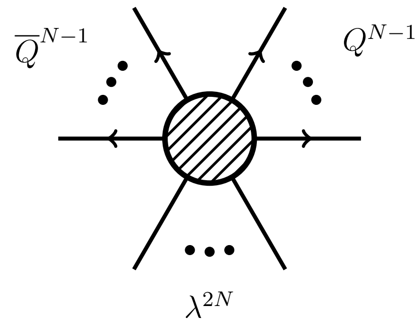
**Holomorphic?** This superpotential might make you a bit unhappy—it's not *holomorphic*! Isn't one of the mantras of SUSY that our superpotential must be holomorphic? The ADS superpotential appears to have a pole; certainly having a negative power of a superfield is not holomorphic, i.e. analytic—infinitely differentiable—over the entire complex plane. The [pedantic] point is that when physicists refer to the 'holomorphy' of the superpotential, what we really *technically* mean is meromorphy, i.e. holomorphy up to isolated singularities. (I won't bother with a technical definition.) *Practically*, what we mean is that the superpotential depends only on the superfield and not its complex conjugate. For most of our favorite pedagogical toy SUSY superpotentials, the symmetries of the theory require that the superfields appear in positive powers. However, this needn't be true—as evidenced by the ADS superpotential.

What is the physical meaning of such a pole in the superpotential? Well, the divergence implies that the potential is very large near that region of field space and the universe will not want to settle nearby. Further, the divergence is a signal that this is a regime where the theory breaks down, as we shall see below.

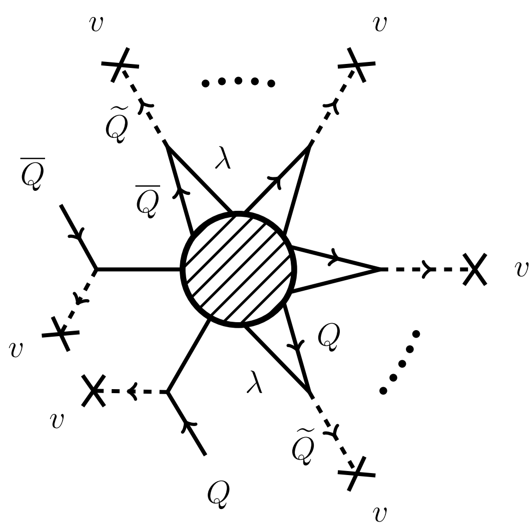
## 3.3 ADS: $F = N - 1$, instantons

For $F = N - 1$, $W \sim \Lambda^{3N-F} = \Lambda^b$, so that the ADS superpotential smells like an instanton effect. This case is rather special since we know that instanton effects requires an integer power of $\Lambda$; this is manifestly guaranteed by setting $F = N - 1$. In this case the $SU(N)$ gauge symmetry is completely Higgsed to $SU(N - (N - 1)) = SU(1) =$ nothing. We can see this since each vev that we turn on breaks a flavor symmetry; turning on one vev breaks $SU(N) \to SU(N - 1)$, turning on two vevs breaks $SU(N) \to SU(N - 2)$, and so forth. We have $F = N - 1$ flavors, so a vev $\langle M \rangle$ breaks $SU(N) \to SU(1)$. (We have assumed that we are away from the $\det M = 0$ point of the moduli space; we do not yet fully understand such a theory [5].)

Does this buy us anything? It sounds bad, this puts us in an asymptotically free ($\beta > 0$), strongly interacting region. However, we can go to a region in moduli space where $\langle M \rangle$ is very large. In particular, we can go to a theory where the gauge group breaks *before* the theory becomes strongly coupled so that our instanton calculations are reliable in this weakly interacting regime. Before we jump ahead of ourselves, though, let's convince ourselves that these really are instanton effects. The 't Hooft operator can be drawn as a vertex with an external leg for each zero mode fermion: the quarks, anti-quarks, and the gauginos.




This doesn't quite look like our superpotential. However, we can go along the flat directions to points in the moduli space where the squarks have very large vevs, $v$. Now recall that we have the coupling between squarks and gauginos, $\lambda Q\widetilde{Q}^*$ and $\lambda\overline{Q}\widetilde{\overline{Q}}^*$. We can use these couplings to connect the $\lambda$ and $Q, \overline{Q}$ legs of the 't Hooft operator. We have two gaugino legs left over, which we may convert into quarks as shown in the diagram[14].




This rather complicated diagram gives us a contribution to the 'quark' mass (where we're being lax about $v$ versus $v^*$)

$$v^{2N}Q\overline{Q}\Lambda^{2N+1}. \tag{3.14}$$

[14]As of the time of this writing, this is the sexiest TikZ diagram that I have ever drawn using purely hand-typed commands. It is important for two key techniques when drawing Feynman diagrams: (a) using `clip` and `foreach` to draw a shaded blob, and (b) rotating and translating 'x's to have them uniform at any angle.

To get the right term for the ADS superpotential we need to suppress by the length scale of the instanton. In the presence of the squark vev, this length scale is

$$\rho^2 \sim \frac{b}{16\pi^2|v|^2},$$

and so we can write our instanton-background Lagrangian as

$$\begin{aligned}
\mathscr{L} &\sim v^{2N} Q\overline{Q}\Lambda^{2N+1}(\rho)^{2N} & (3.15)\\
&= v^{-2N} Q\overline{Q}\Lambda^{2N+1}. & (3.16)
\end{aligned}$$

This is just the fermion mass term that we get from the ADS superpotential.

$$W_{\text{ADS}} = \frac{\Lambda^{2N+1}}{\det M} \quad \to \quad \sim \Lambda^{2N+1}\frac{Q\overline{Q}}{\left(\widetilde{Q}\widetilde{\overline{Q}}\right)^N} \sim \Lambda^{2N+1}\frac{Q\overline{Q}}{v^{2N}}. \tag{3.17}$$

Thus we see that the ADS superpotential for $F = N - 1$ is really just a one-instanton term. Grown-ups can do the exact instanton calculation [36]. I don't know how they do it, and for the moment I don't really care. The magical result however, is that the coefficient $C_{NF}$ for $F = N-1$ is... drum-roll...

$$C_{N,N-1} = 1. \tag{3.18}$$

Now we understand what we need for the particular case $F = N-1$. That's useful for very specific models, but we are more ambitious. In Section 3.4 and 3.5 we will describe two general tools for taking a given theory of SQCD with $F$ flavors and $N$ colors and deforming it to a theory with a different $F'$ and $N'$. The principle will be to go out along the moduli space of the original theory where and either give a squark a vev or otherwise add a mass term to the superpotential so that the low energy theory below these introduced scales is described by a different pair $F'$ and $N'$. We can determine the ADS coefficient by matching the two theories. The procedure of Higgsing or adding a mass term to a quark is known as **deformations** of the original theory and will hold for general $F$ and $N$, even when $F \geq N$.

## 3.4 Deforming SQCD: Higgsing a squark

Our first trick will be to assign a large vev to one squark flavor,

$$\langle q_F \rangle = \langle \overline{q}_F \rangle = v. \tag{3.19}$$

We thus have two scales in the theory that we'd like to relate via the Wilsonian renormalization group. The original theory has an $SU(N)$ gauge group with $F$ flavors, while the low-energy Higgsed theory has $SU(N) \to SU(N-1)$ and one flavor eaten, i.e. $SU(N-1)$ with $(F-1)$ flavors. Thus this Higgsing has taken us from $(N, F)$ to $(N-1, F-1)$. By matching these two theories, we can find a way to relate the coefficients $C_{N,F}$ and $C_{N-1,F-1}$.

We now match of the low energy (with a subscript $L$) and UV couplings at the scale $v$. Using (2.40),

$$\frac{8\pi}{g_L^2(v)} = \frac{8\pi}{g^2(v)} \quad \Rightarrow \quad b_L \log\left(\frac{\Lambda_L}{v}\right) = b \log\left(\frac{\Lambda}{\mu}\right) \quad \Rightarrow \quad \left(\frac{\Lambda_L}{v}\right)^{b_L} = \left(\frac{\Lambda}{v}\right)^{b}. \tag{3.20}$$

The value of the $\beta$-function coefficients are well known in SUSY QCD,

$$b = 3N - F \quad \Rightarrow \quad b_L = 3(N-1) - (F-1), \tag{3.21}$$

from which we obtain the so-called **scale-matching** conditions,

$$\left(\Lambda_{N,F}\right)^{3N-F} = v^2 \left(\Lambda_{N-1,F-1}\right)^{3N-F-2}. \tag{3.22}$$

> **Scale matching**. This will be one of our most powerful tools to explore the moduli space of SUSY gauge theories. The 'big picture' is that we deform the theory in the UV—in this case by Higgsing a quark, but in the next section my integrating out a quark—and then check the effect on the low-energy theory which now has a different number of colors and/or flavors and that does not care about the particular tweaks we performed at a high scale. This language should sound very familiar: it is nothing more than the usual story of **effective field theory**.

We can represent the $(F-1)^2$ light [scalar] degrees of freedom as an $(F-1)\times(F-1)$ matrix $M_L$. This can be related to the analogous $F \times F$ matrix in the original (UV) theory via

$$\det M = v^2 \det M_L. \tag{3.23}$$

Going back and plugging Eqs. (3.20 - 3.23) into the ADS superpotential in Eq. (3.10), we get

$$\begin{aligned} C_{N,F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{1/N-F} &= C_{N,F}\left(\frac{\cancel{v^2}\Lambda_{N-1,F-1}^{3N-F-2}}{\cancel{v^2}\det M_L}\right)^{1/N-F} \\ &\equiv C_{N-1,F-1}\left(\frac{\Lambda_{N-1,F-1}^{3N-F-2}}{\det M_L}\right)^{1/N-F}, \end{aligned} \tag{3.24}$$

where in the last line we've reminded ourselves of the form of the ADS potential with $N-1$ colors and $F-1$ flavors. They take precisely the same form. Coincidence? No, the Higgsed theory is *exactly* the same as the $(N-1, F-1)$ theory at low energies since in this limit the effects of the Higgsed flavors decouples. (This is the lesson of Wilsonian renormalization.) Thus what we've discovered is that

$$C_{N-1,F-1} = C_{N,F}. \tag{3.25}$$

In particular, this means that $C$ only depends on $(N-F)$, i.e. $C_{N,F} = C_{N-F}$. Thus thanks to our $N = F-1$ solution, we now have a set of solutions for $(N-F) = -1$. It turns out there's still one more trick we can play.

The astute reader will wonder how we came to find such a simple relation in Eq. (3.25). What ever happened to the usual complications, namely threshold effects? Usually when we integrate out a field, we get some remnant of the matching in the solutions to the RG equations. The matching we've written without any threshold effects implicitly reflects a choice of the $\overline{DR}$ subtraction scheme [37]. In other words, the threshold effects are absorbed into the particular definition of the cutoff scale.

## 3.5 Deforming SQCD: mass perturbations

The general principle is clear now: how do we can perturb the UV limit of a super QCD and work out the consequences for the low energy theory. In that limit the UV perturbations are negligible effects so that the IR theory characterized by $(N', F')$ is 'really' the $(N', F')$ super QCD theory. We can match the $C$ coefficients of the two theories to obtain a relation between $C_{N,F}$ and $C_{N',F'}$.

The next perturbation we have at our disposal is to give mass $m$ to a flavor without Higgsing the group,

$$\Delta W_{\text{mass}} = mQ_F\overline{Q}_F = mM_{FF}. \tag{3.26}$$

This allows us to integrate out that flavor in the low energy theory, $(N, F) \to (N, F-1)$. We can go ahead and play our scale matching game (really just matching in effective field theory),

$$\left(\frac{\Lambda}{m}\right)^b = \left(\frac{\Lambda_L}{m}\right)^{b_L} \quad \Rightarrow \quad \left(\frac{\Lambda_{N,F}}{m}\right)^{3N-F} = \left(\frac{\Lambda_{N,F-1}}{m}\right)^{3N-(F-1)} \tag{3.27}$$

so that we finally obtain

$$\Lambda_{N,F-1}^{3N-F+1} = m\Lambda_{N,F}^{3N-F}. \tag{3.28}$$

Now we'd like to solve the equation of motion in the presence of the mass term. We start with the mass-perturbed superpotential

$$W_{\text{ADS}} + \Delta W_{\text{mass}} = C_{N,F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}} + mM_{FF}. \tag{3.29}$$

The relevant functional derivative of $W = W_{\text{ADS}} + \Delta W_{\text{mass}}$ is

$$\frac{\partial W}{\partial M_i^{\ j}} = 0 = \frac{C_{N,F}}{N-F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}-1}\frac{(-\Lambda^{3N-F})}{(\det M)^2}\left(\frac{\partial \det M}{\partial M_i^{\ j}}\right) + m\delta_i^F\delta_F^j, \tag{3.30}$$

where $(\partial \det M/\partial M_i^{\ j}) = (M^{-1})^i_{\ j}\det M$ can be written in terms of cofactors (subdeterminants) by using the matrix identity

$$\left(M^{-1}\right)^i_{\ j} = \frac{\operatorname{cof}(M_j^{\ i})}{\det M}. \tag{3.31}$$

The equations of motion for ${M_i}^F$ (similarly for ${M_F}^i$) and ${M_F}^F$ are

$$\frac{\partial W}{\partial {M_i}^F} = 0 = \frac{-C_{N,F}}{N-F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}} \frac{\mathrm{cof}({M_F}^i)}{\det M} \tag{3.32}$$

$$\frac{\partial W}{\partial {M_F}^F} = 0 = \frac{-C_{N,F}}{N-F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}} \frac{\mathrm{cof}({M_F}^F)}{\det M} + m. \tag{3.33}$$

The first of these equations tells us that $\mathrm{cof}({M_F}^i) = 0$, or ${(M^{-1})^F}_i = 0$. This, in turn, tells us that $M$ must take a block diagonal form,

$$M = \begin{pmatrix} \widetilde{M} & 0 \\ 0 & {M_F}^F \end{pmatrix}. \tag{3.34}$$

Combining this with (3.33), we find

$$\frac{C_{N,F}}{N-F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}} = m{M_F}^F. \tag{3.35}$$

Now plugging this back into our superpotential $W$, we obtain

$$W = C_{N,F}\left(1 + \frac{1}{N-F}\right)\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}}. \tag{3.36}$$

As before, the real trick is to write this in terms of low-energy variables,

$$\det M = {M_F}^F \det\widetilde{M} \tag{3.37}$$

$$m\Lambda^{3N-F} = \left(\Lambda_{N,(F-1)}\right)^{3N-F+1}, \tag{3.38}$$

from which we ultimately obtain a recursion relation,

$$W_{\mathrm{ADS}}^{(N,F-1)} = (N-F+1)\left(\frac{C_{N,F}}{N-F}\right)^{\frac{N-F}{N-F+1}}\left(\frac{(\Lambda_{N,F-1})^{3N-F+1}}{\det\widetilde{M}}\right)^{\frac{1}{N-F+1}}. \tag{3.39}$$

Note that this is indeed in the correct form that we proposed in (3.10). In fact, comparing to (3.10), we can deduce the relation

$$C_{N,F-1} = (N-F+1)\left(\frac{C_{N,F}}{N-F}\right)^{\frac{N-F}{N-F+1}}. \tag{3.40}$$

## 3.6 The coefficient of the ADS superpotential

Now we're doing quite well. From Higgsing through the moduli space we found a nice relation (3.25) between the coefficients $C_{N,F}$ and $C_{N-1,F-1}$. Further, mass perturbations gave us (3.40), which relates the coefficients $C_{N,F}$ and $C_{N,F-1}$. Using these perturbations to explore the moduli space, we can relate the ADS superpotential for any $N, F$ with that of any other $N', F'$. All we need to decisively write down the explicit form is the value for $C_{N,F}$ at any particular value. This is precisely what the *we-won't-derive-it-here* instanton calculation in (3.18) gave us.

More explicitly, (3.25) required that $C_{N,F}$ is only a function of $(N-F)$,

$$C_{N,F} = f(N-F). \tag{3.41}$$

Further, the instanton calculation (3.18) told us the particular value at $F = N-1$,

$$C_{N,N-1} = f(1) = 1. \tag{3.42}$$

Finally, the mass perturbation (3.40) gives a recursion relation

$$f(k+1) = (k+1)\left(\frac{f(k)}{k}\right)^{k/(k+1)}, \tag{3.43}$$

which can better be written

$$\left(\frac{f(k+1)}{k+1}\right)^{k+1} = \left(\frac{f(k)}{k}\right)^{k}, \tag{3.44}$$

which has the solution $f(k) = k$.

This gives us the explicit form of the ADS coefficient for any $N$ and $F$,

$$C_{N,F} = N - F.. \tag{3.45}$$

The ADS superpotential (3.10) is thus

$$W_{\text{ADS}} = (N-F)\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}}. \tag{3.46}$$

We'll get back to the ADS superpotential one last time in Section 4.4, where we'll meet an even slicker derivation.

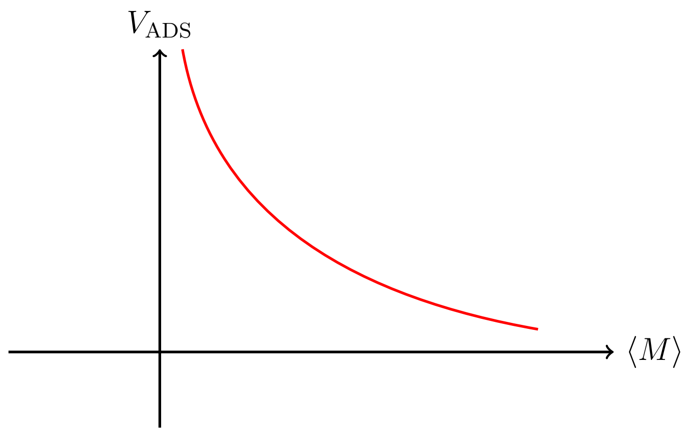
## 3.7 Run, run, runaway

*Run run run run runnin' / Here I go I'm wantin' you*
*Run run run run runnin' / Here I go I'm needing you*
*Run run run run runnin' / Here I go I'm loving you*
*Run run run run runnin' / Run run run run runaway*
–Jefferson Starship

What kind of scalar potential does $W_{\text{ADS}}$ generate? In other words, given (3.46), what do we know about the moduli space of our theory? Since $\det M \sim M^F$, the scalar potential looks heuristically like

$$V_{\text{ADS}} = \left|\frac{\partial W_{\text{ADS}}}{\partial M}\right|^2 \sim |M|^{\frac{-2N}{N-F}}. \tag{3.47}$$

Since we're presently interested in the regime where $F < N$, we find a rather unsettling potential of the form




Whoa there. This is like a slide that never ends. It has a minimum infinitely far away. In fact, you can convince yourself that the minimum at $\langle M\rangle = \infty$ is supersymmetric since $V_{\text{ADS}}(\infty) = 0$. This is indeed what we would expect from a Witten index analysis of SQCD.

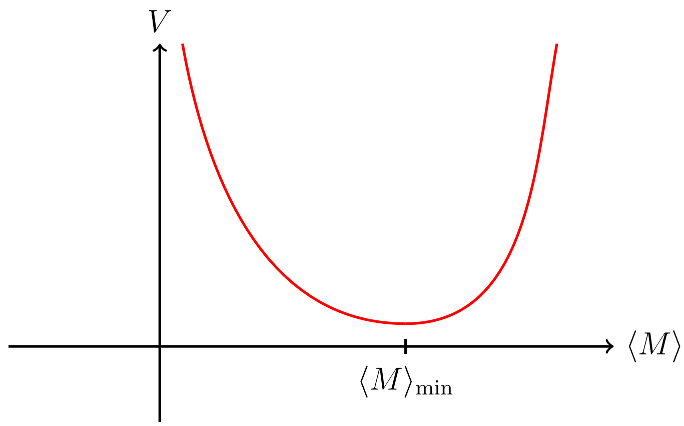
But effectively the potential has *no ground state*, we call this a **run away** potential because the vacuum just runs, runs, runs away to infinity. Let us make a few remarks about this [16]

- Is it possible for quantum effects to bring the $\langle M\rangle = \infty$ point to some finite value? No; the regime of very large $\langle M\rangle$ is one where we trust perturbation theory.
- Is it possible for quantum effects to generate a minimum for *small* $\langle M\rangle$? For example, perhaps the inverse Kähler metric $g^{a\bar{a}}$ has some weird behavior. However, the scalar potential $V \sim g^{a\bar{a}}\partial_a W \partial_{\bar{a}} W$ only has zeros when $\partial_a W = 0$ and we now know that this oly occurs at $\langle M\rangle = \infty$. Note that modifying the Kähler potential can generate *metastable* (i.e. only local) minima, this is a key insight for the industry of metastable SUSY breaking that we will explore later in this document.
- Finally, one last possibility is that for some finite $\langle M\rangle$ the Kähler metric becomes singular so that the theory tells us that there are new massless degrees of freedom. This is what happens for the gauge multiplet when $\det M = 0$. One could say that we do not yet understand this sort of theory without vevs [5].

Let us emphasize that this is what happens when we write down a pure SQCD theory with no additional tree-level superpotential. In this case the tree-level theory has many classical flat directions in the moduli space. The ADS superpotential is dynamically generated and produces a potential which pushes the moduli to infinity where a SUSY-restoring vacuum is waiting. In this sense the ADS superpotential is the avatar of the SUSY-preserving minima predicted by the Witten index.

**Lift your flat directions!** This leads us to a very important lesson for SUSY model-builders. Usually the goal of a nice SUSY theory is to find a clever way to break supersymmetry, i.e. to write a model where we live in a nice SUSY-breaking vacuum. One must always make sure that this nice SUSY-breaking vacuum has no flat directions in the potential, in other words, one should "lift" these flat directions. If you don't, then the dynamically-generated ADS superpotential will likely push your SUSY-breaking vacuum to a SUSY-preserving minimum at infinite vev. This is a surprisingly common pitfall that causes papers to get withdrawn.

Of course, if we add tree-level masses to our theory, then there should be no problems. These mass terms generate a quadratic potential that, for large field values, will pull back towards the origin. Thus we would expect the potential to be modified to the following heuristic form,




in which we can see that a minimum is generated for finite $\langle M \rangle$. Let us thus see what happens when we give mass terms to *all* quark flavors and integrate them out. This will take us to a theory of SQCD without any matter, i.e. pure super Yang-Mills theory. Fortunately, we already have to tools to navigate the moduli space, so this should be a piece of cake.

In Section 3.5 We learned how to integrate out flavors one at a time by adding mass perturbations. It's easy to generalize to the case where we integrate out all flavors:

$$W = (N-F)\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}} + m^i{}_j M^j{}_i. \tag{3.48}$$

The equation of motion $(\partial W/\partial M_i^j) = 0$ completely determines the meson matrix,

$$0 = -(N-F)\frac{1}{N-F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}-1}\frac{\Lambda^{3N-F}}{(\det M)^2}(M^{-1})^i{}_j(\det M) + m^i{}_j, \tag{3.49}$$

where we've used $\partial_{M_i^j}\det M = (M^{-1})^i_j \det M$. Cleaning this mess up we obtain

$$M_i{}^j = (m^{-1})_i^{\ j}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}}. \tag{3.50}$$

Now to simplify this further we'd like to get rid of the $\det M$ on the right-hand side. We can do this by taking determinants of both sides, remembering that the tedious expression on the right is just an overall number multiplying a matrix element. We find

$$\det M = \frac{1}{\det m}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{F}{N-F}} \tag{3.51}$$

$$(\det M)^{\frac{N}{N-F}} = \frac{1}{\det M}\Lambda^{F\frac{3N-F}{N-F}}. \tag{3.52}$$

Plugging this back into (3.50) we obtain the meson matrix as promised,

$$\langle M_i{}^j\rangle_{\text{min}} = (m^{-1})_i^j \left(\det m\,\Lambda^{3N-F}\right)^{1/N}. \tag{3.53}$$

This formula, while derived for $F < N$, will be true in the IR limit of theories with $F > N$ massive flavors since one can always integrate out flavors to get to the $F < N$ and $F = 0$ limits.

We still have to do scale matching at the mass thresholds. We may do these either step by step or all at once. The condition $g_{\text{high-}E} = g_{\text{low-}E}$ gives us

$$\left(\frac{\Lambda_{\text{HE}}}{m}\right)^{b_{\text{HE}}} = \left(\frac{\Lambda_{\text{LE}}}{m}\right)^{b_{\text{LE}}}. \tag{3.54}$$

Removing one flavor gives us

$$\Lambda^{3N-F}m = \widetilde{\Lambda}^{3N-F+1}, \tag{3.55}$$

so that integrating out all flavors gives

$$\Lambda^{3N-F}\det m = \widetilde{\Lambda}^{3N}, \tag{3.56}$$

where the strong coupling scale $\widetilde{\Lambda}$ on the right-hand side what one obtains for pure SYM with no flavors ($F = 0$). This tells us that the ADS superpotential for SYM is

$$W_{\text{ADS}}^{F=0} = N\widetilde{\Lambda}^3_{F=0}. \tag{3.57}$$

To be a bit pedantic, it is common to write $W_{\text{ADS}}^{F=0} \sim (\Lambda^{3N})^{1/N}$, where the $N^{\text{th}}$ root is related to the vacua of the theory, as we will see shortly. Is it weird that $W = N\widetilde{\Lambda}^3$, which looks like a constant? No, $\widetilde{\Lambda}$ is *not* a constant since it is a function of $\det M$. The main results (3.53) and (3.57) were first derived in [38].

## 3.8 Gaugino condensation

It is now prudent to wonder what kind of physics might generate the ADS superpotential (3.46). We saw in Section 3.3 that instantons can generate the $F = N - 1$ ADS superpotential. What about the case $F < N - 1$? (The case $F > N - 1$ will be the focus of the next section.) From our previous analysis, we know that this manifestly *cannot* be an instanton effect. What generates $W_{\text{ADS}}$ in this case?

For $F = N-1$ we saw that the gauge group is completely broken (Higgsed) by the vevs. For $F < N-1$, there simply aren't enough flavors to break the $\mathrm{SU}(N)$ gauge group, there is always some unbroken $\mathrm{SU}(N-F)$ subgroup. This SYM theory (with $F^2$ singlets) is asymptotically free and becomes strongly coupled. We will now show that it is this strong coupling which generate $W_{\mathrm{ADS}}$. In particular, there will be a leftover coupling between the mesons to the pure Yang-Mills theory that will cause the gauginos to condense.

Let us start with pure $\mathrm{SU}(N)$ SYM, i.e. $F = 0$. Here there is no anomaly-free $\mathrm{U}(1)_R$ symmetry, $\lambda \to e^{i\alpha}\lambda$. This is because the gaugino—the only fermion in the theory—would have $R$-charge $R[\lambda] = 1$. Triangle diagrams with the $R$-current are anomalous. A nice way to see this is to think about the 't Hooft operator, which goes like $\mathcal{O}_{\mathrm{tH}} \sim \lambda^{2N}$ and hence breaks $R$-symmetry. Note, however, that the $R$-symmetry is not completely broken. In fact it is broken down to a discrete subgroup, $\mathbb{Z}_{2N}$ given by

$$\lambda \to e^{\frac{2\pi i}{2N}}\lambda. \tag{3.58}$$

Another way to see this is to note that

$$\Theta_{\mathrm{YM}} \to \Theta_{\mathrm{YM}} - \alpha \sum_r n_r 2T(r) = \Theta_{\mathrm{YM}} - 2N\alpha \tag{3.59}$$

under an $R$ transformation, using $T(\mathrm{Ad}) = N$. When the transformation angle is $\alpha = k\pi/N$ this is just a shift by an integer multiple of $2\pi$.

Now let's remember that this $\mathrm{SU}(N)$ theory also has $F$ flavors of $Q$ and $\widetilde{Q}$ matter. Unlike the case of pure super Yang-Mills, this SQCD theory *does* have an anomaly-free $R$-symmetry coming from the combined rotation of the matter superfields and gauginos. (Of course, this $R$-symmetry needn't be the 'canonical' $R$-symmetry.) This seems to be a contradiction: on the one hand we know that $\mathrm{SU}(N)$ with $F < N-1$ massive flavors goes to a pure SYM theory, but the former theory has an anomaly-free $R$-symmetry while the latter theory does not.

We claim that this means that there must be some coupling between the mesons and gauginos (living in $\mathbb{W}_\alpha\mathbb{W}^\alpha$) that compensates for the anomaly and restores the anomaly-free $R$-symmetry.

We can do the scale matching for the high and low holomorphic scales (3.20),

$$\left(\frac{\Lambda}{v}\right)^{3N-F} = \left(\frac{\widetilde{\Lambda}}{v}\right)^{3(N-F)}, \tag{3.60}$$

where $v$ is meant to be the mass scale of the mesons, $\det M \sim v^{2F}$. We have written $\Lambda$ for the high-scale theory ($\mathrm{SU}(N)$, $F$ flavors) and $\widetilde{\Lambda}$ for the low-scale theory ($\mathrm{SU}(N-F)$ SYM). Thus the holomorphic scale depends on the meson vev and hence induces a dependence of the holomorphic coupling on the meson vev. This is just the avatar of our purported coupling between $M$ and $\mathbb{W}_\alpha\mathbb{W}^\alpha$. Cleaning up the scale matching above,

$$\widetilde{\Lambda}^{3(N-F)} = \frac{1}{v^{2F}}\Lambda^{3N-F} = \frac{1}{\det M}\Lambda^{3N-F}, \tag{3.61}$$

where in the last step we have restored the $\det M$ dependence of the scale matching. We should read this equation as the dependence of $\widetilde{\Lambda}$ on $\det M$ for a *fixed* UV holomorphic scale $\Lambda$.

The low-energy gauge coupling can be read off of the kinetic term which, in terms of the low-energy holmorphic coupling $\widetilde{\tau}$, is

$$\frac{1}{16\pi i}\int d^2\theta\,\widetilde{\tau}\,\mathbb{W}_\alpha\mathbb{W}^\alpha + \text{h.c.} \tag{3.62}$$

Explicitly, $\widetilde{\tau}$ is given by

$$\widetilde{\tau} = \frac{\widetilde{b}}{2\pi i}\log\frac{\widetilde{\Lambda}}{\mu}, \tag{3.63}$$

where $\widetilde{b} = 3(N-F)$ since the unbroken gauge group is $\mathrm{SU}(N-F)$. We should think of $\widetilde{\tau}$ as a function of $\det M$ through its dependence on $\widetilde{\Lambda}$, $\widetilde{\tau} = \widetilde{\tau}(\det M)$. We can read off $\log\widetilde{\Lambda}$ from (3.61),

$$3(N-F)\log\widetilde{\lambda} = -\log\det M + \cdots, \tag{3.64}$$

where we've neglected terms that are independent of the meson vevs. Plugging this into the kinetic term we get an expression of the form

$$\mathcal{L} \supset \frac{1}{32\pi^2}\int d^2\theta(\log\det M)\mathbb{W}_\alpha\mathbb{W}^\alpha + \text{h.c.} \tag{3.65}$$

Now we can already qualitatively see how this is going to give us an anomaly-free $R$-symmetry. The $R$ transformation induces a shift in the $F\widetilde{F}$ term inside the $\mathbb{W}_\alpha\mathbb{W}^\alpha$ term. This is the signal that $R$-symmetry is anomalous in the pure SYM theory. The $R$ transformation also induces a phase in $M$, which becomes a shift when we take $\log\det M$. This has the correct form to cancel the transformation of the $\Theta_{\text{YM}}$ term. This is indeed what happens. Note that the logarithm plays an important role in converting the phase to a shift. In fancy parlance, (3.65) is called a **Wess-Zumino term**. It is a term which can be understood as being generated in a low energy theory to protect the anomaly structure of the UV theory. For a delightful exposition, see [39] (check: I might be confusing the WZ term of Skyrme significance with the WZW term).

We can do this more explicitly in components. The relevant terms in the Lagrangian are

$$\mathcal{L} = \frac{1}{32\pi^2}\text{Tr}(F_M M^{-1})\lambda^a\lambda^a + \text{Arg det}\,M\,F\widetilde{F} + F_M^2 + \cdots. \tag{3.66}$$

The Arg det $M$ term is precisely the phase that restores $R$ symmetry while the other terms are required by SUSY. Taking the equations of motion we obtain

$$F_M = \frac{1}{32\pi^2}M^{-1}\langle\lambda^a\lambda^a\rangle, \tag{3.67}$$

where on the right-hand side we've restored the angle brackets that we typically leave implicit when discussing the moduli space. Completely independently of this, however, we can write down $F_M$ coming from the full high-scale ADS superpotential to which we must match,

$$F_M^{\text{ADS}} = \frac{\partial W_{\text{ADS}}}{\partial M} = \frac{N-F}{N-F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}-1}\frac{-1}{(\det M)^2}\Lambda^{3N-F}M^{-1}\det M. \tag{3.68}$$

This should look very familiar from our analogous calculation earlier in (3.49). The whole point is that $F_M^{\text{ADS}}$ must match with $F_M$ in (3.67). Setting them equal and cleaning things up a little, we finally obtain

$$\frac{1}{32\pi^2}\langle\lambda^a\lambda^a\rangle = -\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}}. \tag{3.69}$$

The right-hand side is just $-\widetilde{\Lambda}^3$ in terms of the low energy holomorphic scale defined in (3.61). What this means is that the theory has a gaugino condensate,

$$\langle\lambda^a\lambda^a\rangle = -32\pi^2\widetilde{\Lambda}^3, \tag{3.70}$$

this condensate generates the ADS superpotential and explains the $\widetilde{\Lambda}^3$ superpotential in the pure SYM case (3.57).

What we have *not* explained is what dynamics actually forms the gaugino condensate. This is a hard question. It is possible that the condensate is formed through instantons, but unlike the 'easy' instanton calculation that we omitted for the $F = N - 1$ case, this instanton calculation is in the strong coupling regime and is much more difficult.

The miracle is that we have been able to deduce gaugino condensation, which is a purely strongly coupled phenomenon. It seems like we've gotten away with information that we had no business deriving. How did we manage this voo-doo? We used holomorphy to connection regions of strong and weak coupling in the moduli space.

Finally, let's mention that the gaugino condensate spontaneously breaks the $\mathbb{Z}_{2N}$ symmetry that we originally found in the pure SYM theory:

$$\langle\lambda\lambda\rangle : \ \mathbb{Z}_{2N} \to \mathbb{Z}_2. \tag{3.71}$$

This is because under an $R$-transformation, $\Lambda^{3N} \to e^{2iN\alpha}\Lambda^{3N}$, so that $\langle\lambda\lambda\rangle \to e^{2i\alpha}\langle\lambda\lambda\rangle$. We end up with degenerate but *distinct* vacua in SYM essentially coming from the $N^{\text{th}}$ root that we had to take. This is, of course, no surprise from a Witten index analysis.

## 3.9 Integrating in

We now turn to a related topic that provides a slightly more general technique to understand gaugino condensation. This technique is called '**integrating in**' and, as the name implies, it is in some sense the opposite of 'integrating out.' Recall that we usually integrate out a heavy field to write down a low-energy theory without that field. Now we would like to ask when is this process invertible? In other words, we will take a theory and include new operators to account for additional heavy modes. We can then, in certain cases, interpolate to the case when the additional modes are massless.

### 3.9.1 The 1PI effective action

Let's start by reviewing something completely different that will turn out to be handy: the Legendre transform in field theory[15]. Consider the case of a single scalar field, $\phi(x)$ with some

[15] As a helpful reminder, the geometric meaning of the Legendre transform is explained in chapter 7.3 of Ryder's textbook. For more review, see Ramond chapter 3.3 [40] or Peskin chapter 11.3 [41].

action $S[\phi]$. Recall that the partition function (vacuum-to-vacuum amplitude) for a field theory is given by

$$Z[J] = \int d[\phi] e^{iS+J\phi} \equiv \exp(iWJ), \tag{3.72}$$

where $W[J]$ is the generating functional of connected diagrams and $J$ is a source for the quantum field. From this we may calculate Green's functions such as the expectation value for the field $\phi(x)$ itself in the background of a source $J(x)$,

$$\langle\phi(x)\rangle_J = \frac{\delta W[J]}{\delta J(x)} \equiv \varphi(x). \tag{3.73}$$

Here we see that $\varphi(x)$ is the classical value of the field. The 1PI effective action $\Gamma[\varphi]$ written in terms of the classical field $\varphi$ is given by the Legendre transform

$$W[J] = \Gamma[\varphi] + \int d^d x J\varphi, \tag{3.74}$$

where $\varphi$ is, by definition, the solution of

$$\frac{\delta}{\delta\varphi(x)}\Gamma[\varphi] = -J(x) \tag{3.75}$$

in the presence of the source $J(x)$. This is our main point: the effective action whose tree-level Green's functions represents a resummation of quantum effects is given by a Legendre transform.

In general, the actual analytic form of $\Gamma[\varphi]$ must be calculated in a loop expansion in $\hbar$. Recall that for $\varphi(x) = \varphi_0$ one obtains the effective potential which determines the quantum vacuum structure (moduli space) of a theory. It is crucial now to remember that $\Gamma$ is the 1PI effective action and should absolutely not be confused with the Wilsonian effective action, see Section 2.1. In no sense have we integrated out any heavy modes. The 1PI effective action is effective in the sense that if we did not know about quantum effects, $\Gamma[\varphi]$ would be the action that we would write down to describe results from experiments of the theory $S[\phi]$.

Has this jogged your memory? Good. Let's get back to the art of integrating in. This was first presented by Intriligator in [42] and is mentioned in pedagogical contexts in [2, 16, 43]. Before jumping into the details, the main idea is this: the process of integrating out is 'invertible' when the low-energy superpotential is a Legendre transform of the high-energy superpotential. In this case one can take a known low-energy superpotential and invert the Legendre transform to obtain information about the vevs of the high-energy degree of freedom and hence information about the phase of the gauge theory. Let's do this systematically. We'll start by integrating *out* heavy fields to go to a low-energy effective theory, pointing out relevant features along the way.

### 3.9.2 Integrating out

Remembering that we are still working with the 1PI effective action, let us be a bit sly and refer to the act of 'integrating out' to obtain an effective theory. This notion is something that we are most familiar with in the Wilsonian picture where we very literally integrate out shells of UV momenta.

However, now we are working with an action which, in principle, encapsulates all quantum effects in its tree level couplings. The Legendre transform has replaced the quantum degree of freedom $\mathcal{O}$ by its classical value, $\langle\mathcal{O}\rangle$. The notation here is deliberate: the classical value of a field is, of course, simply its vacuum expectation value. In other words, the 1PI effective action obtained from performing a Legendre transform with respect to a particuar operator replaces that operator by its vev. Conceptually this is precisely what we mean when we integrate out a field, we remove the dynamical degree of freedom associated with a massive field and leave behind the vev.

First let us note that we can translate all of our results regarding the 1PI effective *action* to the superpotential. This is because adding a source term to a chiral superfield $\mathcal{O}$ contributes a term $\int d^2\theta J\mathcal{O}$ to the Lagrangian, which is equivalent to adding $J\mathcal{O}$ to the superpotential. If we are interested in $\langle\mathcal{O}\rangle$, the classical value of $\mathcal{O}$, then we would take functional derivatives with respect to $F_J$,

$$\langle O\rangle_J = \frac{\delta\mathcal{L}[J]}{\delta F_J} = \frac{\delta W[J]}{\delta J}. \tag{3.76}$$

Now we need to clarify some notation. In Section 3.9.1 we used the standard QFT notation where $S$ is the classical action, $Z$ is the partition function, $W$ is the generating functional of connected diagrams, and $\Gamma$ is the 1PI effecive action. From this point on we will work with superpotentials so that $W$ will *always* (unless otherwise stated) refer to a superpotential. We will refer to 1PI the effective superpotential with explicit subscripts. A useful analogy is thus

$$S[\phi]\ :\ \Gamma[\varphi] \quad :: \quad W[\mathcal{O}]\ :\ W_{\text{eff}}[\langle\mathcal{O}\rangle]. \tag{3.77}$$

Explicitly, given a superpotential $W$, we may write down a 1PI effective superpotential

$$W_{\text{eff}}(\langle\mathcal{O}\rangle, \Lambda^{b_0}) = W(J, \Lambda^{b_0}) + J\langle\mathcal{O}\rangle. \tag{3.78}$$

Let's now follow Intriligator's presentation in [42] (see also [44] for some context). This requires a bit of cumbersome notation, so we will write things out as explicitly as possible. Let us start with two SQCD theories: the **upstairs theory** with superpotential $W_u$ and holomorphic scale $\Lambda_u$, and the **downstairs theory** with superpotential $W_d$ and holomorphic scale $\Lambda_d$. Both theories describe chiral superfields $\phi_i$ and $\widetilde{\phi}_i$ which form gauge invariant polynomials $X_r$. The upstairs theory additionally includes a massive chiral superfield $\hat{\phi}$ which forms a meson $M = \hat{\phi}\widetilde{\hat{\phi}}$. There are also gauge invariant polynomials $Z_a$ which contain both $\phi$ and $\hat{\phi}$.

We know that as we integrate the heavy quark $\hat{\phi}$ (and the corresponding $\widetilde{\hat{\phi}}$) out of the upstairs theory, the low energy look something like the downstairs theory up to the vev of the heavy fields which manifest themselves as vevs of the $Z_a$ fields. Given a mass scale $m$ for $\hat{\phi}$, we know that the two holomorphic scales are related by (3.20),

$$\Lambda_d^{b_d} = m\Lambda_u^{b_u}, \tag{3.79}$$

where $b_d$ and $b_u$ are the $\beta$-function coefficients. Recall $b = 3N - F$ so that the power of $m$ is correct by dimensional analysis.

Let us explicitly write out the field dependence of the superpotential, $W_u = W_u(X, Z, \Lambda_u^{b_u})$, where we suppress the indices of $X$, $Z$, and $J$. Now let's set up the Legendre transform. Define the 'full' superpotential to be $W_u$ plus the usual source term,

$$W_{\rm f}(X, Z, \Lambda^{b_u}, J) = W_u(X, Z, \Lambda_u^{b_u}) + \sum JZ. \tag{3.80}$$

Note that the $\sum JZ$ implicitly includes a 'source' for the meson $M$, which we write as $mM$, the notation is meant to be evocative since such a source term is exactly a mass term for $\hat{\phi}$. Using the equation of motion in the background of a source $J$,

$$\left.\frac{\partial W_f}{\partial Z}\right|_J = 0, \tag{3.81}$$

we may write down the 1PI effective superpotential by taking the Legendre transform with respect to the $Z$ fields,

$$\begin{aligned} W_{\rm 1PI}(X, \Lambda_u^{b_u}, J) =& W_{\rm f}(X, \langle Z\rangle, \Lambda^{b_u}, J) && (3.82)\\ =& W_d(X, \Lambda^{b_d}) + W_I(X, \Lambda_u^{b_u}, J), && (3.83)\end{aligned}$$

where we've written out the field dependence explicitly. In the second line we've recovered our old friend, the downstairs theory. The leftover interactions coming from the $\langle Z\rangle$ vev is encapsulated in $W_I(X, \Lambda_u^{b_u}, J)$, which is RG irrelevant and vanishes for $J = 0$. We can see the irrelevance since $W_I \to 0$ as $m \to \infty$, as can be seen from the equation of motion (3.81). The final decomposition of $W_{\rm 1PI}$ into $W_d$ and $W_I$ depends on the fact that $W_f$ is linear in $J$ which can be taken as an assumption. (See [44] for some remarks on this.)

Remember that we have not thrown away any information to get to this low energy effective superpotential. Due to the linearity in the source(s) $J$, all we have done is performed a Legendre tranform. The great thing is that such a transform can be inverted.

### 3.9.3 Integrating back in

Let's explicitly reconstruct $W_u$ from $W_{\rm 1PI}$. First construct a new superpotential

$$W_n(X, Y, \Lambda_u^{b_u}, J) = W_{\rm 1PI}(X, \Lambda_u^{b_u}, J) - \sum JY, \tag{3.84}$$

where $Y_a$ is a set of new gauge invariants which do not affect $W_{\rm 1PI}$. The magic trick will be to transmogrify $Y_a$ back into the $Z_a$ fields which we integrated out. We make the bold claim that this can be done simply by integrating out the source(s) $J$. Playing the same game, we use the equation of motion $(\partial W/\partial J)_Y = 0$. Because $W$ is linear in $J$, the source is just an auxiiary superfield. Now observe that

$$W_n(X, Y, \Lambda_u^{b_u}, J) = W_u(X, \langle Z\rangle, \Lambda_u^{b_u}) + \sum J(\langle Z\rangle - Y), \tag{3.85}$$

so that $J$ is just a Lagrange multiplier that enforces $Y = \langle Z\rangle$ which, in turn, sets

$$W_n(X, Y, \Lambda_u^{b_u}, J) = W_u(X, Y, \Lambda_u^{b_u}). \tag{3.86}$$

So what we've found is that we can take $W_n(X, Y, \Lambda_u^{b_u}, J)$, which only depends on $W_d$ and $W_I$, integrate out the source $J$, and end up with the *high* scale theory, $W_u$. How's that for pulling a rabbit out of a hat?

To some extent all we've been doing is slight of hand using Legendre transforms. The real power comes from specific examples when the downstairs theory is much simpler than the upstairs theory. Integrating in fields then allows us a handle on the upstairs theory without having to meddle with it directly. One remark is worth making: because the upstairs theory includes a heavy field, it does not necessarily make sense to talk about it as a *dynamical* degree of freedom. Instead, it gives us information about the vaccum structure of that theory. The key point is that we're already armed with a very powerful tool, the exact ADS superpotential, (3.46), which gives us a natural handle on the low-energy theory. We'll put this to good use to rederive a now-familiar result.

### 3.9.4 Gaugino condensation (from another perspective)

Let's consider the simple case where the gauge singlets $Z$ are bilinears in $\hat{\phi}$ and $\widetilde{\hat{\phi}}$. This is particularly nice because the source terms in the superpotential $W_f$ are simply mass terms, $mM$, and the cumbersome term vanishes $W_I = 0$. You are referred to Intriligator's paper [42] for the not-so-simple case.

Fortunately, the simple case is enough to give an alternate demonstration of gaugino condensation. Our upstairs theory and downstairs theories are pure $SU(N)$ pure Yang-Mills, where the downstairs theory does not explicitly contain the glueball field. The $Z$ fields are objects like mesons $M$ and the source terms are simply $JZ = \mathrm{Tr}(mM)$, where the trace is over $\mathrm{SU}(F)_\mathrm{L} \times \mathrm{SU}(N)_\mathrm{R}$ flavor indices. In our case, $M$ is just the glueball superfield

$$S = -\mathbb{W}_\alpha \mathbb{W}^\alpha. \tag{3.87}$$

Let us work with $S$ as the gauge invariant polynomial that we'd like to integrate out, $Z = S$. The source for $S$ is the holomorphic coupling $\tau$, which we will simply write as $J = b \log \Lambda$, where $\Lambda$ is the SYM holomorphic scale and we're ignoring some overall constants. The gaugino condensate $\langle S \rangle \sim \langle \lambda^a \lambda^a \rangle$ is simply the classical value of $S$ so that

$$\langle S \rangle = \left. \frac{\partial W_{1\mathrm{PI}}}{\partial \log \Lambda^b} \right|_{\log \Lambda^b}. \tag{3.88}$$

Because our gauge invariant polynomial is quadratic, we know that $W_{1\mathrm{PI}}(X, \Lambda^b, \log \Lambda^b) = W_d(X, \Lambda^b)$ since $W_I$ vanishes. Note that we're a little redundant in the superpotential arguments (for example, there are no $X$ fields), but we do this to maintain the connection to our general discussion above where it was very important to keep track of the arguments of each functional. In fact, we know that the low-energy effective superpotential is exactly the SYM ADS superpotential (3.57), $W \sim \Lambda^3$. As promised, this rather simple (though it took a bit of tooth-pulling to derive), and we shall see that it is indeed much simpler than the $W_u$ that we'll eventually derive, justifying this entire procedure.

Let's spell things out now. From the expression above for $\langle S \rangle$ and remembering that $b = 3N$, we have

$$\langle S \rangle = \frac{N}{3N} \frac{\partial \Lambda^3}{\partial \log \Lambda} = \Lambda^3. \tag{3.89}$$

This is an expression for the vev $\langle S\rangle$, but the inverse Legendre transform restores this to $S$. In the general discussion above, this is just the $Y = \langle Z\rangle$ step from integrating out $J$. Thus we may trade $\Lambda^3 \to S$ in $W_n$ to obtain

$$\begin{aligned} W_n(X, S, \Lambda^b) &= W_{\text{ADS}}^{F=0} - N \log \Lambda^3 \, S && (3.90) \\ &= NS - NS \log S. && (3.91) \end{aligned}$$

We obtain the full 'upstairs' theory by also including the gauge kinetic term $W = b \log \Lambda \, S$,

$$W_{\text{VY}} \equiv W_u = S \left[ \log \left( \frac{\Lambda^3 N}{S^{3N}} \right) + N \right]. \tag{3.92}$$

This theory is the Veneziano-Yankielowicz superpotential, originally derived by anomaly considerations in [45, 46]. It is indeed uglier than the ADS superpotential, justifying the 'slick' integrating in of $S$. It may sound a bit fishy that $S$ gets pushed out of and into the theory with impunity, but it is important to remember that $S$ is a massive field whose dynamical degrees of freedom don't really belong in either theory. The whole point of reconstructing $W_{\text{VY}}$ is that it encodes the vacuum structure of the glueball field,

$$\frac{\partial W_{\text{VY}}}{\partial S} = 0 \quad \Rightarrow \quad \langle S\rangle = e^{2\pi i k/N} \Lambda^3, \tag{3.93}$$

as we saw at the end of Section 3.8.

Just for fun, we can integrate in one more time. This time let's take the VY theory to be our downstairs theory $W_d = W_{\text{VY}}$, and $0 < F < N$ SQCD be our upstairs theory. We would like to integrate in some matter fields which manifest themselves as gauge-singlet mesons $M$ with mass terms of the form $\operatorname{Tr} mM$. This time is a straightforward application of the steps outlined above. We simply write

$$W_n = W_{\text{VY}} - \operatorname{Tr} mM, \tag{3.94}$$

where we assume that in $W_{\text{VY}}$ the downstairs holomorphic scale $\Lambda_d = \Lambda$ is written in terms of the upstairs scale $\Lambda_u$ via

$$\Lambda_d^{3N} = \det m \, \Lambda_u^{3N-F}. \tag{3.95}$$

We then integrate out the source $m$ using its equation of motion $\partial W_n / \partial m = 0$, which gives $\langle m\rangle = SM^{-1}$. Finally, we obtain

$$W_u = S \left[ \log \left( \frac{\Lambda_u^{3N-F}}{S^{3N-F} \det Q\widetilde{Q}} \right) + N - F \right], \tag{3.96}$$

where we've written in the heavy squarks $Q$ and $\widetilde{Q}$. From here one should properly integrate out $S$ once again since it is always massive.

### 3.10 Relatied topics

There are some related topics which one may pursue at this point. A detailed discussion is left to more suitable references and, perhaps, future revisions of this document.

- Generalizing gaugino condensation for order parameters of dimension one. See Dine and Mason Section 4.3 [8] and the references therein.
- Integrating in for $W_I \neq 0$. See Intriligator's original paper, [42]. This is also mentioned in [43].
- Gaugino condensation can also be derived in a slick way using $\mathcal{N} = 2$ techniques due to Seiberg and Witten. We postpone a discussion of Seiberg-Witten theory to later in this document.
- Gaugino condensation is discussed in chapter 8.3 of Terning [5] using methods similar to those in Section 3.9.4 above, though in somewhat more plain language.

## 4 $F = N$ and $F = N + 1$: Some special cases

Now we get to the interesting regime. As before, we have to start by developing some tools.

### 4.1 't Hooft Anomaly Matching

The primary tool is the **'t Hooft anomaly matching condition**, which 't Hooft considered so trivial that the first reference to the technique is in his summer school lecture notes [47]. A more rigorous proof was presented by Banks, Frishman, Schwimmer, and Yankielowicz [48]. Anomaly matching is a way to check the spectrum of a strongly coupled theory in a regime where perturbative methods have no business being valid.

Suppose we have some asymptotically free gauge group $G_g$ in a theory which also has a flavor symmetry group $G_F$. By definition the gauge symmetry is just a redundancy in the description of a physical system, so it must be anomaly free. For global symmetries, however, all bets are off. There's nothing inconsistent about a global symmetry being anomalous. Another way of saying this is that the triangle diagrams associated with three global currents (we shall call this the $G_F^3$ anomaly) don't have any gauge bosons attached. 't Hooft developed a clever way to use these anomalies to teach us about the effective degrees of freedom at low energies.

He key step is to *pretend* that $G_F$ is **weakly gauged**, meaning the coupling constant $g_F$ is arbitrarily weak and perturbative in the range of energies of interest. Eventually we'll take the limit $g_F \to 0$ and turn off this gauging. Now the $G_F$ anomalies are a problem, albeit a pretend problem. In order to keep the theory consistent, we need to cancel the pretend $G_F^3$ anomalies. We do this by adding **spectator fields** to the theory which are $G_g$ singlets and only cary $G_F$ quantum numbers so that the anomaly coefficients for $G_F$ cancel. To be more precise, we only trust these calculations in the perturbative regime of the $G_g$ gauge theory (we define $g_F$ to be perturbative effectively everywhere). Since $G_g$ is asymptotically free, the precise statement is that the anomaly coefficients for $G_F$ cancel in the UV:

$$A^{\mathrm{UV}}(F) + A^{\mathrm{UV}}(\mathrm{spec}) = 0. \tag{4.1}$$

Now that we've introduced spectator fields to cancel the $G_F^3$ anomalies in the UV, the 'magic' is that the anomalies must *still* cancel in the IR theory,

$$A^{\mathrm{IR}}(F) + A^{\mathrm{IR}}(\mathrm{spec}) = 0. \tag{4.2}$$

This is because the IR regime where $G_g$ is strongly coupled looks the same as far as the $G_F$ weakly gauged sector is concerned. In fact, we have $A^{\mathrm{IR}}(\mathrm{spec}) = A^{\mathrm{UV}}(\mathrm{spec})$. Alternatively, this is just the consistency of the gauge theory. Of course we know that with respect to the 'real' gauge group $G_g$, the UV and IR theories are very different with totally different degrees of freedom. While the UV theory describes quarks, the IR theory describes confined states. Thus the calculable triangle diagrams contributing to $A^{\mathrm{UV}}(F)$ are totally different than those contributing $A^{\mathrm{UV}}(F)$, which may be much more difficult or impossible to calculate.

Now there are two possibilities as we go to the IR theory.

1. The strong dynamics spontaneously breaks the flavor symmetry so that $G_F$ is broken. This is what happens, for example, in QCD when the chiral condensate breaks $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \to \mathrm{SU}(3)_D$. When this is the case, the anomaly matching condition doesn't tell us anything useful.

2. $G_F$ is left unbroken by the strong dynamics. In this case we can do something.

When $G_F$ is unbroken, we can combine (4.1) and (4.2), remembering that $A^{\mathrm{IR}}(\mathrm{spec}) = A^{\mathrm{UV}}(\mathrm{spec})$,

$$A^{\mathrm{IR}}(F) = A^{\mathrm{UV}}(F). \tag{4.3}$$

In other words, the anomalies for the global $G_F$ symmetries must match in the UV and IR. At this point we can forget about the spectator fields and the weak gauging; they've served their purposes valiantly and we now have everything we need.

In fact, there's a handy corollary to this result. We know that anomalies come from zero-mode fermions, so if we calculate that the global anomaly is nonzero in the UV, then we can say that there must be massless fermions in the IR spectrum.

## 4.2 Moduli Space

We should also discuss the moduli space of the $F > N$ theory. Fortunately, we've already done this in Section 2.4.3. Please refer to that section for a quick refresher.

## 4.3 $N = F$

We can now take our first steps beyond the ADS superpotential. We will find an anomaly-matching spectrum, but some of the symmetries will be spontaneously broken. Our UV and IR fields are:

| | SU($N$) | SU($F$)$_L$ | SU($F$)$_R$ | U(1)$_B$ | U(1)$_A$ | U(1)$_R$ |
|---|---|---|---|---|---|---|
| $Q$ | $\square$ | $\square$ | 1 | 1 | 1 | 0 |
| $\widetilde{Q}$ | $\overline{\square}$ | 1 | $\overline{\square}$ | -1 | 1 | 0 |
| $Q\widetilde{Q} = M$ | 1 | $\square$ | $\overline{\square}$ | 0 | 2 | 0 |
| $Q^N = B$ | 1 | 1 | 1 | $N$ | $N$ | 0 |
| $\widetilde{Q}^N = \widetilde{B}$ | 1 | 1 | 1 | $-N$ | $N$ | 0 |
| $m$ | 1 | $\overline{\square}$ | $\square$ | 0 | $-2$ | 2 |
| $\Lambda^{2N}$ | 1 | 1 | 1 | 0 | $2F$ | 0 |

In the last two lines we've included the mass parameter from $W \subset mQ\widetilde{Q}$ and the holomorphic scale which should each be treated as spurions. The holomorphic scale carries the quantum numbers of the anomalous symmetries. We've written out the charges under the theories' U(1)s: baryon number, the anomalous abelian symmetry, and the anomaly-free $R$ symmetry. Note that we cannot use the anomalous symmetry for anomaly matching since the spectators would have to be charged under $G_g$.

We have $2N^2$ fields subject to $(N^2 - 1)$ $D$-flatness conditions, this leaves us with $N^2 + 1$ moduli. Looking at our gauge invariant polynomials, we have $N^2$ mesons $M$, and one of each type of baryon, $B$ and $\widetilde{B}$. Thus we have $N^2 + 2$ fields to fit into an $N^2 + 1$ moduli space. This just means that there is a **classical constraint** on the $(M, B, \widetilde{B})$ space to project it to the moduli space, as we learned in Section 2.4.3:

$$\det M = B\widetilde{B}. \tag{4.4}$$

This constraint describes a complex manifold which is singular at the origin $M = B = \widetilde{B} = 0$. This is a conical singularity associated with the fact that $M$, $B$, and $\widetilde{B}$ are complex so that at the origin their phases are undefined.

We know that physics doesn't like singularities. In this case, we'll see that the strong dynamics will want to smooth out this singularity by modifying the constraint. Let's look at what happens at the origin. Let's recall (3.53), the formula the vev $\langle M_i{}^j \rangle$ in the presence of a mass term,

$$\langle M_i{}^j \rangle_{\text{min}} = (m^{-1})_i^j \left(\det m \, \Lambda^{3N-F}\right)^{1/N}. \tag{4.5}$$

We originally derived this for $F < N$, but we remarked that it was general. This is why we pointed that out. Taking the determinant of both sides,

$$\det M = \frac{1}{\det m} \left( (\det m)^N \right)^{1/N} \Lambda^{2N} = \Lambda^{2N}. \tag{4.6}$$

The mass matrix $m$ drops out (a special feature of $F = N$), which means we can take the $m \to 0$ limit where there is no mass term in which case we would still have $\det M = \Lambda^{2N}$. There is one more remnant of the mass term: for $m \neq 0$ we have $B = \widetilde{B} = 0$ since all fields with baryon number can be integrated out. When we take the $m \to 0$ limit, we must also address the $B, \widetilde{B} \neq 0$ case. Recall that classically $\det M = B\widetilde{B} = 0$, so (4.6) is telling us that there really must be a quantum modification to the classical constraint. Is the $\Lambda^{2N}$ factor the whole correction?

We can use the symmetries of the theory to explicitly write out the most general quantum modification to the classical constraint:

$$\det M - B\widetilde{B} = \Lambda^{2N}\left(1 + (\Lambda^{2N})^a (B\widetilde{B})^b (\det M)^c\right), \tag{4.7}$$

where dimensional analysis requires $a + b + c = 0$. We also know that in the limit $B\widetilde{B} \to 0$ we must recover (4.6). This means that $b > 0$. Further, we don't want any divergences in the weak coupling limit ($\Lambda \to 0$) so that $a > 0$. We thus have

$$\det M - B\widetilde{B} = \Lambda^{2N}\left(1 + C_{ab}\frac{(\Lambda^{2N})^a (B\widetilde{B})^b}{(\det M)^{a+b}}\right). \tag{4.8}$$

Now let's slide along the baryonic branch of moduli space, $B\widetilde{B} \gg \Lambda^{2N}$. In this regime the constraint takes the form

$$\det M \sim \left(B\widetilde{B}\right)^{\frac{b-1}{a+b}}. \tag{4.9}$$

This can be seen by dividing both sides of (4.8) by $B\widetilde{B}$ and dropping terms which are $\ll 1$. The regime $B\widetilde{B} \gg \Lambda^{2N}$ corresponds to breaking the gauge group before the theory is strongly coupled so that this should match the classical result. This tells us that $C_{ab} = 0$, so that the full **quantum modified constraint** on the moduli space is

$$\det M - B\widetilde{B} = \Lambda^{2N}. \tag{4.10}$$

The right-hand side indeed has the correct power for an instanton effect ($\Lambda^b$). The effect of this quantum modified constraint is that the origin of moduli space—which previously had a conical singularity—has cordoned off from the theory. The singularities are smoothed out[16]. A direct consequence of this is that some global symmetries are necessarily broken. $M$, $B$, and $\widetilde{B}$ all carry charges. Eliminating the origin of the moduli space means that at least some of these are broken.

We have at least checked that for $F = M$ the low-energy description of the mesons and baryons is correct as long as we impose the quantum modified constraint. We can now consider actual vacua of this theory. We want to preserve as much symmetry as possible. For a somewhat relevant discussion of why this is the case, see [49]. First consider the branch of moduli space where only the meson gets a vev, $M = \Lambda^2 \cdot \mathbb{1}$. Baryon number and $R$-symmetry are preserved, but $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ symmetry is broken down to $\mathrm{SU}(N)_D$; we have chiral symmetry breaking, just like ordinary QCD. The particle content now has the charges

| | $\mathrm{SU}(N)$ | $\mathrm{SU}(N)_D$ | $\mathrm{U}(1)_B$ | $\mathrm{U}(1)_R$ |
|---|---|---|---|---|
| $Q$ | $\square$ | $\square$ | 1 | 0 |
| $\widetilde{Q}$ | $\overline{\square}$ | $\overline{\square}$ | -1 | 0 |
| $M$ | 1 | (Ad + 1) | 0 | 0 |
| $B$ | 1 | 1 | $N$ | 0 |
| $\widetilde{B}$ | 1 | 1 | $-N$ | 0 |

[16]For those who are keeping up with all the hip and cool review literature, these conical singularities should be familiar from the $XYZ$ model described in Strassler's unorthodox review [3]. In fact, if you're not sure what these cones are all about, I suggest looking over that review.

where (Ad+1) refers to the adjoint plus trace decomposition of a bifundamental. Now let's get to the matter at hand. Let's use the 't Hooft anomaly matching conditions to check that the global anomalies indeed match. It is instructive to do a few examples

- $\mathrm{SU}(N)_D^3$. In the UV there are $N$ fundamental quarks and $N$ antifundamental antiquarks $\widetilde{Q}$ which sum to give no anomaly. In the IR the adjoint is automatically anomaly free.

- $\mathrm{SU}(N)_D^2\mathrm{U}(1)_B$. In the UV we have

$$T(\Box)^2(+1) + T(\overline{\Box})^2(-1) = 0 \tag{4.11}$$

while in the IR we have

$$T(\Box)^2(0) + T(\overline{\Box})^2(0) = 0. \tag{4.12}$$

- $\mathrm{SU}(F)_D^2\mathrm{U}(1)_R$. This one is non-trivial. In the UV we have

$$[NT(\Box)]^2(-1) + \left[NT(\overline{\Box})\right]^2 = -2N^2. \tag{4.13}$$

This just comes from $A(r_1 \otimes r_2) = \dim(r_1)A(r_2) + \dim(r_2)A(r_1)$. In the IR, the mesons make no contributions since $A(\mathrm{Ad}) = 0$. The baryons give

$$N^2(-1) + (-N)^2(-1) = 2N^2. \tag{4.14}$$

- $\mathrm{U}(1)_B^3$. Both equal zero.

- $\mathrm{U}(1)_B\mathrm{U}(1)_R^2$. Both equal zero.

- $\mathrm{U}(1)_B^2\mathrm{U}(1)_R$. Both equal $-2N$.

- $\mathrm{U}(1)_R^3$. In the UV we sum the quarks, antiquarks, and gauginos:

$$(-1)N^2 + (-1)N^2 + (N^2 - 1) = -N^2 - 1. \tag{4.15}$$

On the IR side we sum the meson, baryon, and anti-baryon:

$$(-1)N^2 - 1 - 1 = -N^2 - 2. \tag{4.16}$$

It seems like 't Hooft has failed! However, we must remember that for the IR calculation we must impost the quantum modified constraint so that one degree of freedom can be expressed in terms of the others. This means that we really have $-N^2 - 1$ and the results match.

And that's the anomaly matching game. It's not exactly Starcraft 2, but it's probably more interesting than Wheel of Fortune. Let's see how this works for the baryonic branch where $\langle M \rangle = 0$ and

$$\langle B \rangle = \langle \widetilde{B} \rangle = \Lambda^N. \tag{4.17}$$

In this region of moduli space the $\mathrm{SU}(F)$ flavor symmetries are left intact, but $\mathrm{U}(1)_B$ is broken. Chiral symmetry is preserved, but baryon number is now spontaneously broken. The particle representations under the remaining symmetries are

| | SU($N$) | SU($F$)$_1$ | SU($F$)$_2$ | U(1)$_R$ |
|---|---|---|---|---|
| $Q$ | $\square$ | $\square$ | 1 | 0 |
| $\widetilde{Q}$ | $\overline{\square}$ | 1 | $\overline{\square}$ | 0 |
| $M$ | 1 | $\square$ | $\overline{\square}$ | 0 |
| $B$ | 1 | 1 | 1 | 0 |
| $\widetilde{B}$ | 1 | 1 | 1 | 0 |

We've labelled by flavor symmetries by 1 and 2 rather than $L$ and $R$ to avoid confusion with the $R$-symmetry. Now let's do a few anomaly matching examples.

- SU$(N)_1^3$. The anomaly coefficient is $N$. In the UV theory this comes from the SU($N$) multiplicity of states, while in the IR this comes from SU$(N)_2$ multiplicity.
- SU$(N)^2$ U$(1)_R$. The anomaly coefficient is $-N$ coming from $R$-charge. (Recall that the $R$ charge differs for each element of a supermultiplet so that even though all of the listed $R$ charges are zero, they include fermion zero modes that carry nonzero $R$ charge.)
- U$(1)_R$ and U$(1)_R^3$. These have anomaly coefficient $-N^2-1$.

## 4.4 The ADS superpotential from nothing

Now that we're comfortable matching anomaly coefficients, it turns out that there is yet another rabbit we can pull out of our hats. We can derive $W_{\text{ADS}}$ from this theory. The strategy should be obvious: integrate out a flavor to go from $F=N$ to $F=N-1$ and hopefully recover the ADS superpotential. In fact, consistency of our entire discussion of SQCD requires that this should be true. That being said, it does feel like we're doing a bit of hocus pocus; our $F=N$ theory has *no* superpotential, just a quantum modified constraint.

Our first step, then, is to put in a superpotential that we can later mold into $W_{\text{ADS}}$. To do this, we introduce an *auxiliary* chiral superfield $X$ which will act as a Lagrange multiplier to enforce the quantum modified constraint.

$$W = X(\det M - B\widetilde{B} - \Lambda^{2N}). \tag{4.18}$$

Thus we've swapped our quantum constraint for an actual superpotential. That was a good first step. Now let's cut to the chase and add a mass term to the $N^{\text{th}}$ flavor,

$$\Delta W = mM_{NN}. \tag{4.19}$$

It helps to write out the meson matrix as

$$M = \begin{pmatrix} \widetilde{M} & N \\ \widetilde{N} & M_{NN} \end{pmatrix}. \tag{4.20}$$

We can now take the equation of motion with respect to all fields containing the heavy flavor. First, the baryons:

$$\frac{\partial W}{\partial B} = 0 \quad \Rightarrow \quad X\widetilde{B} = 0 \tag{4.21}$$

$$\frac{\partial W}{\partial \widetilde{B}} = 0 \quad \Rightarrow \quad XB = 0. \tag{4.22}$$

In other words, we have $B = \widetilde{B} = 0$. Let's move on to the meson fields.

$$\frac{\partial W}{\partial M_{NN}} = 0 \quad \Rightarrow \quad m + X \det \widetilde{M} = 0 \tag{4.23}$$

$$\frac{\partial W}{\partial \widetilde{M}_{Ni}} = 0 \quad \Rightarrow \quad X M^{-1}_{Ni} \det M = 0. \tag{4.24}$$

This gives us the constraints

$$X = \frac{-m}{\det \widetilde{M}} \tag{4.25}$$

$$M^{-1}_{Ni} = 0, \tag{4.26}$$

where the last condition sets $N = \widetilde{N} = 0$. Finally, there's one more equation of motion to take: that of the auxiliary field $X$.

$$\frac{\partial W}{\partial X} = 0 \quad \Rightarrow \quad M_{NN} \det \widetilde{M} = \Lambda^{2N}, \tag{4.27}$$

which sets $M_{FF} = \Lambda^{2N} / \det \widetilde{M}$. Substituting these into the superpotential,

$$W = X\left(\det M - B\widetilde{B} - \Lambda^{2N}\right) + mM_{NN} \tag{4.28}$$

$$= X \det \widetilde{M}\, M_{FF} - X\Lambda^{2N} + mM_{NN} \tag{4.29}$$

$$= -X\Lambda^{2N}, \tag{4.30}$$

where we've used $X \det \widetilde{M}\, M_{NN} = -mM_{NN}$. This boils down to

$$W = \frac{m\Lambda^{2N}}{\det \widetilde{M}}. \tag{4.31}$$

Now recall our favorite scale matching condition (3.27), which in this case is

$$\left(\frac{\Lambda}{m}\right)^{2N} = \left(\frac{\widetilde{\Lambda}}{m}\right)^{3N-(N-1)}. \tag{4.32}$$

Plugging this into $W$, we obtain a familiar result,

$$W = W_{\text{ADS}} = \frac{\widetilde{\Lambda}^{2N+1}}{\det \widetilde{M}}. \tag{4.33}$$

This is the result which more technically minded people derived by an honest instanton calculation. Even the coefficient is correct. We derived using only the quantum modified constraint and a few slick moves. Everything really fits together nicely.

## 4.5 $F = N + 1$: $s$-confinement

Now we meet a second special case, which also turns out to be the simplest case. In fact, one can find many theories which exhibit similar behavior as the $F = N + 1$ scenario. Here the baryons and mesons perfectly match the anomalies; there is no additional constraint on the moduli space, quantum or otherwise. The light degrees of freedom (moduli) match the UV degrees of freedom.

There's a new feature in this theory: ***s*-confinement**. This is a really stupid name where the 's' appears to refer to screening. This is a phase of a gauge theory where there are massless degrees of freedom. Compare this to QCD where there are no massless fundamental quarks to screen; there is a linear potential until you hit the lightest quark mass, at which point the QCD flux tubes break. If there were massless quarks, as in the $s$-confining case, the flux tubes break immediately. As a handy summary, $s$-confinement carries the following implications:

- Confinement does *not* break chiral symmetry (contrast this with the $N = F$ case)
- All IR degrees of freedom are gauge-invariant composites of the fundamental fields
- It smoothly[17] interpolates between the Higgs and confining phases; there is no gauge-invariant order parameter which distinguishes these phases and no phase transition
- A non-vanishing confining superpotential is dynamically generated
- At the origin of moduli space all global symmetries are unbroken and the global anomalies in the UV and IR theories match

A very neat feature of this theory is the complementarity between the Higgs and confining phases. We already saw this in $F = N$ theories; consider the mesons and baryons of the low energy theory. Near the origin of moduli space the mesons and baryons act like composite states. On the other hand, in the semiclassical region of large moduli, these fields are Higgsed. We have a smooth transition between these two phases without a phase transition, and so these phases are identical.

For $F = N + 1$ no global symmetries are broken (e.g. chiral symmetry) and there are no quantum modified constraints. $B$ and $\widetilde{B}$ are no longer flavor singlets since there are too many flavors. The antisymmetrization of the color indices by the $\epsilon$-tensor antisymmetrizes all but one flavor index. We know that this is equivalent to the antifundamental representation; e.g. the Young tableaux relation

$$\ydiagram{1,1,1} = \overline{\ydiagram{1}}$$

Alternately, we can see this explicitly from the indices:

$$B_{i_1 \cdots i_N} = \epsilon^{\alpha_1 \cdots \alpha_N} Q_{i_1 \alpha_1} \cdots Q_{i_N \alpha_N} \tag{4.34}$$

$$= \epsilon^{i i_1 \cdots i_N} B_{i_1 \cdots i_N} \equiv B^i. \tag{4.35}$$

[17]For this reason my adviser proposes that the 's' can be interpreted as 'smooth,' though I think he's the only one to claim this. On the other hand, he seems to have naming rights [50, 51].

The classical constraints are trivially satisfied. Using

$$M^i{}_j B_i = \epsilon_{i_1\cdots i_N} Q^i Q^{i_1} \cdots Q^{i_N} \bar{Q}_j = 0, \tag{4.36}$$

and the analogous identity for $\widetilde{B}$, we find (using $\epsilon$ identities)

$$\left(M^{-1}\right)^i{}_j \det M = \widetilde{B}^i B_j. \tag{4.37}$$

We should check whether these classical constraints are quantum modified, as they were in the $F = N$ case. We will do this by using the same trick of adding a mass term and then taking the mass term to zero. (4.5) still holds. We end up with

$$M^{-1} \det M = m\Lambda^{2N-1}, \tag{4.38}$$

where the right-hand side vanishes as $m \to 0$, which is exactly what we expect from the classical constraint since the mass term sets the baryon moduli to zero. Thus we suspect that the classical constraints are not quantum modified. This is shown in [52]. We'll get back to this as soon as we write down our superpotential and anomaly matching.

There's one more difference from the $F = N$ case that we should highlight. In the case $F = N$, we were able to assign all superfields (including $\Lambda$) to have zero $R$-charge (where of course we mean the lowest components). For the $F = N + 1$ case, we cannot do this for the holomorphic scale.

| | SU($N$) | SU($F$)$_1$ | SU($F$)$_2$ | U(1)$_B$ | U(1)$_A$ | U(1)$_R$ |
|---|---|---|---|---|---|---|
| $Q$ | $\square$ | $\square$ | 1 | 1 | 1 | 0 |
| $\widetilde{Q}$ | $\overline{\square}$ | 1 | $\square$ | -1 | 1 | 0 |
| $M$ | 1 | $\square$ | $\square$ | 0 | 2 | 0 |
| $B$ | 1 | $\overline{\square}$ | 1 | $N$ | $N$ | 0 |
| $\widetilde{B}$ | 1 | 1 | $\overline{\square}$ | $-N$ | $N$ | 0 |
| $\Lambda^{2N-1}$ | 1 | 1 | 1 | 1 | $2(N+1)$ | $-2$ |

where the U(1)$_R$ charge of the $\Lambda^{2N-1}$ comes from the expression $2N-2(N+1)$, which in turn comes from the contribution of the gauginos, mesons, and baryons to the 't Hooft instanton operator. This reflects the anomaly in the canonical U(1)$_R$ charge, not to be confused with the anomaly-free $R$-charge which may mix the canonical charge with the other U(1)s in the theory. (This depends on the number of flavors.) This table gives us all of the possible terms for the superpotential. The most general thing we can write that satisfies these symmetries is

$$W = \frac{1}{\Lambda^{2N-1}} \left[ \alpha B M \widetilde{B} + \beta \det M + \det M \, f\left(\frac{\det M}{B M \widetilde{B}}\right) \right]. \tag{4.39}$$

We should be curious about the behavior of the overall $1/\Lambda^{2N-1}$ prefactor in the weak coupling limit, where we expect $W = 0$. In fact, we know that the *raison d'être* of this superpotential is *only* to impose the classical constraint as a Lagrange multiplier. If this is the case, then in the classical limit $W = 0$ indeed vanishes an there's no question about the weak limit. We can just calculate the equations of motion and require that they yield the classical constraints. This fixes $f = 0$, $\beta = -\alpha$ so that we finally have

$$W_{F=N+1} = \frac{1}{\Lambda^{2N-1}} \left( \det M - B M \widetilde{B} \right). \tag{4.40}$$

Great. Now we can go and check our 't Hooft anomaly matching conditions. Now everything is non-trivial. To do this we should re-write our particle table in terms of the anomaly-free symmetries. In particular, we should now write the anomaly-free $R$-symmetry by combining U(1)s so that $\Lambda^{2N}$ carries no $R$-charge.

| | SU($N$) | SU($F$)$_1$ | SU($F$)$_2$ | U(1)$_B$ | U(1)$_R$ |
|---|---|---|---|---|---|
| $Q$ | $\square$ | $\square$ | 1 | 1 | $1/(N+1)$ |
| $\widetilde{Q}$ | $\overline{\square}$ | 1 | $\square$ | -1 | $1/(N+1)$ |
| $M$ | 1 | $\square$ | $\square$ | 0 | $2/(N+1)$ |
| $B$ | 1 | $\overline{\square}$ | 1 | $N$ | $N/(N+1)$ |
| $\widetilde{B}$ | 1 | 1 | $\overline{\square}$ | $-N$ | $N/(N+1)$ |

Note that in the $F = N$ case we had a quantum modified constraint that told us that there was a vev to expand about. In particular,

$$\det M - B\widetilde{B} = \Lambda^{2N}. \tag{4.41}$$

So along the baryonic $F = N$ branch, for example, we could expand $B \to \langle B\rangle + \delta B$ (and similarly for the $\widetilde{B}$) so that we end up with linear pieces in the superpotential with respect to the dynamical fields $\delta B$. Once you have a linear expression for a field in the superpotential, you can take the equation of motion to solve for that field. On the baryonic branch we could solve $\delta B$ in terms of $\delta\widetilde{B}$, for example. This is also related to the fact that the origin has been removed from the moduli space; there *has* to be a vev that we can expand about and hence we must be able to eliminate a field using its equation of motion. Back to our present $F = N+1$ case, there's nothing keeping us away from the origin. And so the origin is part of the anomaly matching. This shows up non-trivially in the anomaly matching.

Let's demonstrate a few anomaly coefficients.

- SU$(F)^3$. Here the UV theory gives $N$ while the IR theory gives $(N+1) - 1$.
- SU$(F)^2$ U(1)$_B$. The UV and IR theories both give $N$.
- SU$(N+1)^2$U(1)$_R$. This is a non-trivial one. Recall that we're really counting the $R$-charge of the fermion of each superfield. In the UV we have

$$N\left(\frac{1}{N+1} - 1\right) = -\frac{N^2}{N+1}. \tag{4.42}$$

  In the IR we sum the meson and baryon

$$(N+1)\left(\frac{2}{N+1} - 1\right) + \left(\frac{N}{N+1} - 1\right) = \frac{-N^2}{N+1}, \tag{4.43}$$

  after a little bit of elbow grease.

You can see that this can get pretty non-trivial.

Let's get back to checking the quantum modified constraint. As one last plausibility check, let's see if we get the correct quantum modified constraint when we flow from $F = N+1$ to $F = N$ by adding a mass to the last flavor. We then take the equation of motion to get a quantum modified constraint. In particular,

$$\frac{\partial}{\partial M_{FF}} W = -\frac{1}{\Lambda^{2N-1}} \left( \det \widetilde{M} - B_F \widetilde{B}^F \right) + m = 0, \tag{4.44}$$

where $F = N + 1$. Note that the $B_F \widetilde{B}^F$ term is independent of the heavy flavor (which we can see since it came from the derivative of a $BM\widetilde{B}$ term). We end up with

$$\det \widetilde{M} - B\widetilde{B} = m\Lambda^{2N-1} = \widetilde{\Lambda}^{2N}, \tag{4.45}$$

where the right-hand side is the dynamical scale of the theory. This is a non-trivial relation among the remaining degrees of freedom. We have recovered our old quantum modified constraint for $F = N$. Looking back at the procedure, we notice that $M_{FF}$ (where $F = N + 1$) has played the role of a Lagrange multiplier to enforce the $F = N$ quantum modified constraint.

These types of $s$-confining theories are quite nice and are the easiest SQCD theories to find. They don't suffer from any singularities on the moduli space and are generally well-behaved. In fact, in the past a young theorist could spend some portion of their life writing a list of all such theories [50, 51].

# 5 $F > N + 1$: Seiberg Duality

Now that we've built up some sophistication with our methods, we can get to the good stuff. No more special cases. On to one of the great theoretical discoveries of the 1990s. While the main result—Seiberg duality—can be plainly state and used as is, the intuition leading to this result is firmly rooted in a solid understanding of the renormalization group. There are many good references for this, but Strassler's 'unorthodox introduction' [3] is perhaps the most intuitive and pedagogical. As it is highly unlikely that I can improve on these lectures, they are strongly recommended as background reading. For those who already have a strong background, Strassler's follow up lectures on the duality cascade [4] offer a good summary in his style.

**Some good lectures**. A personal note from the author: seriously, Strassler's write up on this subject are among the best lecture notes I have read for any subject.

## 5.1 The Banks-Zaks Fixed point

For $F > N + 1$ we need to know about the RG fixed points of the theory. As a starting point, we know that the one-loop $\beta$-function coefficient takes the form $b = 3N - F$. For $F > 3N$ the theory flows to a weakly coupled theory at low energies, i.e. to a 'boring' trivial fixed point. For $F < 3N$, on the other hand, the theory is asymptotically free. Now we are curious about what happens in the region around $F = 3N$, where the one-loop beta function vanishes. Could this be a fixed point?

**Fixed points in SUSY theories.** One thing that we know about SUSY fixed points is that they typically come from the cancellation with a higher-order term in a loop expansion. For example, for dimensionful couplings there is a 'classical' $\beta$-function associated with the 'engineering' dimensions of the tree-level interaction. In the rich phase structure of 3D $\mathcal{N} = 2$ models [53], non-trivial fixed points appear when tree-level $\beta$ function for a relevant coupling cancels against the loop-level (quantum) $\beta$ function. (Due to the change in the dimensions of each field, marginal operators in 4D are relevant in 3D.) In this case the gauge coupling runs at one-loop order, and we seek to understand a cancellation from the two-loop contribution. These examples should raise all sorts of concerns since they represent, by definition, a non-perturbative regime where we don't have much control. Fortunately, SUSY (essentially holomorphy) comes to the rescue and preserves this control.

The NSVZ exact $\beta$ function provides a very powerful tool for checking the existence of a fixed point. Recall from Section 2.6 that this takes the form

$$\beta(g) = \frac{-g^3}{16\pi^2} \frac{3N - F(1 - \gamma(g^2))}{1 - N\frac{g^2}{8\pi^2}}, \tag{5.1}$$

where the anomalous dimension is

$$\gamma(g^2) = \frac{-g^2}{8\pi^2} \frac{N^2 - 1}{N} + \mathcal{O}(g^4). \tag{5.2}$$

The $\gamma$ term in the numerator and the $g^2/8\pi^2$ term in the denominator are higher loop corrections to the one-loop running. In fact, things are looking pretty optimistic since we can see that the $F\gamma$ term in $\beta$ appears to carry the opposite sign as he one-loop contribution.

Let's now do this a bit more carefully. We shall take a slightly-weird limit where we take both $F$ and $N$ to infinity while staying very close to the $3N - F = 0$ region where the one-loop $\beta$ function vanishes. We'd like to fall into the asymptotically free region, so while taking $F, N \to \infty$, let us fix

$$\frac{F}{N} = 3 - \epsilon \tag{5.3}$$

for arbitrarily small, but positive, $\epsilon$. We will see shortly that for small $\epsilon$ the coupling is at weak coupling so that perturbative methods are valid. Expanding $16\pi^2\beta$ in powers of $g^2/8\pi^2$,

$$\begin{aligned}
16\pi^2\beta(g) &= -g^4\left(3N - F(1\gamma)\right)\left[1 - N\frac{g^2}{8\pi^2}\right]^{-1} &(5.4)\\
&= -g^3(3N - F) + N\frac{g^2}{8\pi^2}(-g^3)(3N - F) - g^3 F\gamma &(5.5)\\
&= -g^3(3N - F) - N\frac{g^5}{8\pi^2}(3N - F) - \frac{g^5}{8\pi^2}F\frac{N^2 - 1}{N} &(5.6)\\
&= -g^3(3N - F) - \frac{g^5}{8\pi^2}\left(3N^2 - 2NF + \frac{F}{N}\right), &(5.7)
\end{aligned}$$

where we've dropped terms of $\mathcal{O}(g^7)$. Here we can plug in our large-$N$ limit with $F/N = 3 - \epsilon$,

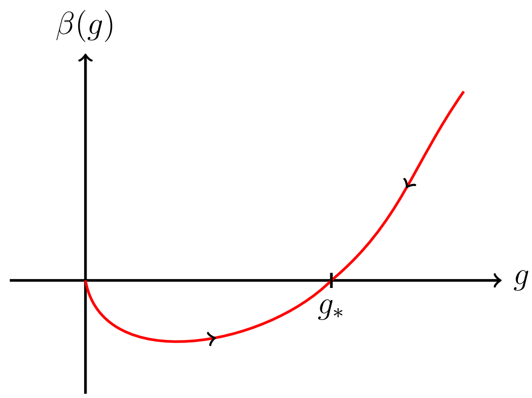
$$16\pi^2 \beta(g) = -g^3 \epsilon N + \frac{3g^5}{8\pi^2}(N^2 - 1) + \mathcal{O}(g^7, \epsilon^2). \tag{5.8}$$

We see that there is indeed a fixed point coming from the cancellation of a one loop and a two loop effect:

$$g_* = \frac{8\pi^2}{3} \frac{N}{N^2 - 1} \epsilon. \tag{5.9}$$

This is the SQCD version of the celebrated **Banks-Zaks** fixed point[18] of QCD [54]. Though the two-loop $\beta$-function had been known since 1974, Banks and Zaks were the first to seriously consider the zero of the $\beta$-function at $F_* = 16.5$. By performing an expansion in $(F - F_*)$, where $F$ is a physically meaningful natural integer. They found that a non-trivial fixed point indeed exists in the $F > F_*$ regime, and along with it the found many unexpected features. For example, chiral symmetry was not broken in the strongly coupled regime as was previously expected.

The phase diagram looks like this:




Recall that the fixed point indicates a scale-invariant theory. For a sensible quantum field theory of particles with spin less than 2, scale invariance implies a much larger symmetry, conformal invariance. And when we slap on supersymmetry, we have **superconformal invarance**.

**The 't Hooft coupling**. What is the meaning of the odd $N, F \to \infty$ while $F/N = 3 - \epsilon$ limit which we took? This is just the famous 't Hooft 'large $N$ limit' (see e.g. [25]). We should have known that this would have shown up. The 't Hooft limit originally came about when physicists studied non-supersymmetric $O(N)$ models where it was found that an expansion in $1/N$ can control loop-level effects by allowing the $\beta$-function to cancel in a perturbative

[18] For a nice set of slides about the history of the Banks-Zaks phase, see `http://scipp.ucsc.edu/Symposium/Peskin_BanksZaks.pdf`

regime. The key insight is that perturbation theory is *not* an expansion in $g^2/4\pi$, but rather in the 't Hooft coupling,

$$\lambda = \frac{g^2 N}{4\pi^2}. \tag{5.10}$$

(By the way, there are times when the loop factor of $4\pi$ turns out to be very important [55].) The large $N$ expansion corresponds to fixing $\lambda$ and taking $N \to \infty$. The $\beta$-function for $\lambda$ is

$$\beta_\lambda = -\frac{\lambda^2}{8\pi^2}\frac{3 - \frac{F}{N}(1-\gamma)}{1 - \frac{\lambda}{8\pi^2}}. \tag{5.11}$$

Perturbation theory breaks down then $\lambda \sim 1$, not necessarily when $g^2 \sim 1$. At the SQCD Banks-Zaks fixed point we find that $\lambda_* \sim 1/N$, so that the large $N$ limit indeed pushes this to the perturbative regime.

## 5.2 Some facts about superconformal theories

We shall invoke a few very useful theorems about superconformal field theories. We'll be terse, mostly because doing otherwise would take us a bit far afield.

**Fact 5.1.** *At a superconformal fixed point, there exists a unique anomaly-free R-symmetry which is part of the superconformal algebra. It is conventional to call this $R_{sc}$.*

Compare this to the case of the *canonical* $R$-symmetry, which is not well-defined since it can mix with any other U(1) in the theory. In fact, $R_{\text{sc}}$ is precisely the anomaly-free $R$-symmetry under which we assigned quantum numbers to our fields when we analyzed the $F = N$ and $F = N + 1$ theories.

**Fact 5.2.** *The dimension of a chiral operator $\mathcal{O}$ and its $R_{sc}$ charge are related by*

$$\mathit{dim}\ \mathcal{O} = \frac{3}{2}\left|R_{sc}[\mathcal{O}]\right|. \tag{5.12}$$

*Further, $D = \pm 3R_{sc}/2$ where the $+$ $(-)$ sign corresponds to (anti-)chiral superfields. More generally, in $d$ dimensions, dim $\mathcal{O} = (d-1)\left|R_{sc}[\mathcal{O}]/2\right|$.*

This is an extremely powerful result that comes from the fact that $R_{\text{sc}}$ is part of the supercurrent along with the stress energy tensor, which includes the generator of dilatations. The dimension of an operator is information about its transformation under scaling, so it is unsurprising that in a superconformal theory this information should be linked by SUSY to the $R_{\text{sc}}$ charge.

The really neat thing is that the particular dimension that is related to the $R_{\text{sc}}$ charge is the full quantum dimension of the chiral field: the canonical ('engineering') dimension plus the loop-induced anomalous dimension $\gamma$. Thus the real power of this supercurrent relation is that we can get direct information about $\gamma$, even in non-perturbative regions. These observations become even more powerful when we throw in one more useful fact:

**Fact 5.3.** *Near any conformal fixed point all spin-zero, gauge-invariant operators $\mathcal{O}$ must have [scaling] dimension greater than or equal to 1. More generally for d [spacetime] dimensions, $\mathcal{O}$ must have [scaling] dimension $(d-2)/2$. Further, saturation of this bound implies that the operator is a free field.*

This very bold statement is proved in [29]. These are very 'deep' statements about the structure of field theory and SUSY; we'll be sure to get good mileage out of them.

Let's do a couple of quick warm ups from [3]

- Suppose that all gauge couplings are set to zero. Then the chiral superfield $Q$ is gauge invariant (gauge redundancy has been turned off) and so it satisfies the conditions for Fact 5.3. If $Q$ has any residual superpotential interactions, then by (2.58), dim $\phi = 1 + \gamma/2 > 1$, note the 'strictly greater than' symbol. This tells us that $\gamma > 0$.

- In 4D, as we assume throughout this document, we know that an operator in the superpotential is relevant if its coupling has dimension greater than 3 and irrelevant if its coupling is less than 3. We can read this off by checking whether the $R_{\rm sc}$ charge of the operator is greater or less than 2.

**The chiral ring**. This is a phrase which pops up fairly often, so it's worth addressing its meaning. The product of chiral operators ('left chiral superfields') is also a chiral operator. Since the $R_{\rm sc}$ charges of the product of chiral operators is just the sum of the individual $R_{\rm sc}$ charges, then the dimension of the product of operators is also just a sum of the individual dimensions. (Recall that in the usual OPE, a composite operator picks up its own anomalous dimension.) A set with defined addition and multiplication operations is called a ring.

Now let's do a little more with all this. In particular, let's take a look at the role of $R_{\rm sc}$-symmetry in SQCD in $F > N$. Recall the table of quark charges,

| | $\mathrm{SU}(N)$ | $\mathrm{SU}(F)_1$ | $\mathrm{SU}(F)_2$ | $\mathrm{U}(1)_B$ | $\mathrm{U}(1)_{R_{\rm sc}}$ |
|---|---|---|---|---|---|
| $Q$ | $\square$ | $\square$ | 1 | 1 | $(F-N)/F$ |
| $\widetilde{Q}$ | $\overline{\square}$ | 1 | $\square$ | -1 | $(F-N)/F$ |

The $\mathrm{U}(1)_R$ charges are found by taking linear combinations of the U(1)s so that the holomorphic scale (now acting as a spurion for instanton contributions) $\Lambda^{3N-F}$ is uncharged under it, i.e. so that it is anomaly-free. We can now calculate the dimension of the meson in the UV theory,

$$\dim(Q\widetilde{Q}) = \frac{3}{2}R_{\rm sc}(Q\widetilde{Q}) = 3\frac{F-N}{F}. \tag{5.13}$$

We know that this is equal to the canonical scaling dimension plus an anomalous dimension, $\dim(Q\widetilde{Q}) = 2+\gamma$, where the 2 comes from the fact that there is a product of two fields. Thus the facts laid out in this section tell us that at a superconformal fixed point, $\gamma \to \gamma_*$,

$$\gamma_* = 1 - \frac{3N}{F}. \tag{5.14}$$

As a check, we can plug this into the NSVZ numerator (5.1) to find

$$\beta \propto 3N - F(1-\gamma_*) = 0, \tag{5.15}$$

as expected. Thus our technique is consistent.

## 5.3 $\frac{3}{2}N < F < 3N$: the conformal window

Our first significant step with our new superconformal tools will be to go back to the meson operator $Q\widetilde{Q}$. Our $R_{\text{sc}}$ analysis above told us that the dimension of this field is $3 - 3N/F$. The unitarity bound then requires

$$3 - \frac{3N}{F} \geq 1, \tag{5.16}$$

i.e. $F/N \geq 3/2$. This sets a *lower* bound on the region where we expect a superconformal fixed point, the so-called **conformal window**. Previously we only knew that for $F > 3N$ the theory goes to a trivial (weakly coupled) fixed point, and that the cases $F \leq N+1$ had been already discussed. Now it seems like the conformal window is smaller than we might have originally thought and has a lower bound so that it is constrained to exist in the regime

$$\frac{3}{2}N < F < 3N. \qquad \text{(conformal window)} \tag{5.17}$$

In fact, there's something interesting about the conformal window. Because $\dim(M) > 1$ (strictly), this theory flows to an *interacting* superconformal theory. It is in the interacting **non-Abelian Coulomb phase**. As we showed it is asymptotically free so that the coupling grows as we flow into the IR, but unlike 'real world QCD,' the coupling doesn't diverge. Instead, it asymptotically approaches the fixed value $g_*$. The superconformal theory at the IR fixed point is interacting and *not* confined.

Seiberg posited that the Banks-Zaks fixed point really exists in the conformal window [56]. This result (to the best of my knowledge) has not been proven rigorously. Our arguments above are valid for the large $N$ limit with $F$ near the upper limit of the window; for now we shall take on faith that this fixed point survives even away from these particular limits. The RG flow for a theory in this window thus looks like


The lower limit of the conformal window begs the following question: what happens in the regime $N+1 < F \leq 3N/2$? We'll get to this shortly. What we already know, however, is that our $R_{\text{sc}}$ analysis showed and the requirement that $\beta = 0$ at a fixed point shows that the Banks-Zaks point does not occur in this regime. It will turn out that this regime will be dual to conformal window by a duality, the famous Seiberg duality to which we now turn.

## 5.4 Seiberg Duality

We shall motivate Seiberg's remarkable proposal in Section 5.7. First, let's just dive in and explain the proposal. Let us call the conformal window SQCD theory by a suggestive name, the **electric** theory. Seiberg proposed that there is an equivalent description of this theory, which we shall call the **magnetic** theory. The remarkable aspect of this "electric-magnetic" duality is that the magnetic theory looks very different. First, it is a theory with an $\mathrm{SU}(n)$ gauge symmetry with $F$ flavors, were $n = F - N$. This is weird! Our usual 'real-world QCD' intuition from strongly coupled gauge theories is that quarks confine into mesons which are gauge singlets. Now the low energy theory contains **dual quarks** $q$ and $\widetilde{q}$ which are charged under a new gauge group. Further, while the electric theory is an 'honest' SQCD model with no superpotential, the magnetic theory has an additional superpotential

$$W_{\text{mag}} \sim M q \widetilde{q}, \tag{5.18}$$

where $M$ is a field independent of $q$ and $\widetilde{q}$ to be associated with the $Q\widetilde{Q}$ meson relative to the electric theory. It is sometimes customary to call this dual theory SQCD+M.

To be more precise, this is an **infrared duality** in which two different theories flow to the same fixed point in their IR limits. We will see that while the SQCD electric theory is asymptotically free, the SQCD+M magnetic theory is IR free and and that the magnetic theory can be understood to be the low-energy effective theory of the electric theory.

This duality is certainly unexpected. The two theories seem to be very different beasts with no obvious path connecting them. However, the fact that the two theories have different gauge groups should not discourage us: recall that the gauge symmetry is just a redundancy of how we describe the theory. In principle, one is perfectly free to describe the same physics with different redundancies. On the other hand, while gauge symmetries needn't match, the global symmetries of UV and IR theories should match. (Recall our 't Hooft anomaly matching discussion.) Thus let us write out the symmetries of our fields in the UV and IR theories:

| | $\mathrm{SU}(N)$ | $\mathrm{SU}(F)$ | $\mathrm{SU}(F)$ | $\mathrm{U}(1)_B$ | $\mathrm{U}(1)_{R_{\text{sc}}}$ |
|---|---|---|---|---|---|
| $Q$ | $\square$ | $\square$ | $1$ | $1$ | $\frac{F-N}{F}$ |
| $\widetilde{Q}$ | $\overline{\square}$ | $1$ | $\square$ | $-1$ | $\frac{F-N}{F}$ |
| $M$ | $1$ | $\square$ | $\square$ | $0$ | $2\frac{F-N}{F}$ |
| $q$ | $\square$ | $\overline{\square}$ | $1$ | $\frac{N}{F-N}$ | $\frac{N}{F}$ |
| $\widetilde{q}$ | $\overline{\square}$ | $1$ | $\overline{\square}$ | $\frac{-N}{F-N}$ | $\frac{N}{F}$ |

The meson field $M$ exists as a degree of freedom of the magnetic theory, but we identify it as well with the $Q\widetilde{Q}$ operator of the electric theory. This seems a little odd, since the $Q\widetilde{Q}$ bound state has canonical dimension 2 while the 'fundamental' $M$ field in the magnetic theory has canonical dimension 1. Of course, there's nothing strange about this: the canonical dimensions only hold in the UV of each theory. The Banks-Zaks superconformal fixed point exists as the IR limit of these theories[19] Thus during the RG flow from the UV to the IR, the $Q\widetilde{Q}$ field in the electric theory and the $M$ field in the magnetic theory pick up anomalous dimensions so that they end up with dimension

$$\dim M = \dim Q\widetilde{Q} = 3\frac{F-N}{F}. \tag{5.19}$$

To be precise we should define a separate scalar field $M_{\text{m}}$ with dimension 1 in the UV so that $M = Q\widetilde{Q} = \mu M_{\text{m}}$ for some characteristic scale $\mu$. It is conventional to write everything in terms

[19]This isn't *quite* true, as we'll discuss in Section 5.7 when we consider the RG flow of the SQCD+M theory.

of the meson $M$ and the scale $\mu$ so that

$$W_{\text{mag}} = \frac{1}{\mu} M q \widetilde{q}. \tag{5.20}$$

Thus $(1/\mu)$ plays the role of a coupling constant.

Now we can match the holomorphic scales of the two theories. We can assign $R$-charge by treating $\Lambda^b$ as a spurion for the instanton-generated 't Hooft operator. This can then be used to determine the scale matching of the (UV) electric theory $\Lambda$ and the (IR) magnetic theory $\widetilde{\Lambda}$

$$\Lambda^{3N-F}\widetilde{\Lambda}^{3(N-F)-F} = (-1)^{F-N}\mu^F, \tag{5.21}$$

where $\mu$ shows up once again to soak up scaling dimensions. This tells us something very important: since we have the relation $\Lambda^b\widetilde{\Lambda}^{\widetilde{b}} = \text{const}$, as one holomorphic scale increases, the other decreases. This tells us that this is a strong-weak duality, as one theory become strongly coupled the other is weakly coupled. This is the origin of the 'electric-magnetic' nomenclature, since this is reminiscent of a $g \leftrightarrow 1/g$ duality. (Unlike the $\mathcal{N} = 4$ case, this isn't quite a 'true' electric-magnetic $S$-duality.) *This* is the real power of Seiberg duality: just when one description of the theory is becoming non-perturbative, the other is returning to perturbativity. (Otherwise there's no point to a duality which relates one incalculable regime to an equally incalculable regime.)

**That funny 'duality' minus sign**. The minus sign in the relation between the holomorphic scales is a familiar sight from more familiar duality transformations. For example, taking the derivative of the action with respect to $\log\Lambda \sim \tau$ tells us that the electric and magnetic field strengths are related by

$$\mathbb{W}_\alpha \mathbb{W}^\alpha = -\widetilde{\mathbb{W}}_\alpha \widetilde{\mathbb{W}}^\alpha, \tag{5.22}$$

which should be familiar from the usual electric-magnetic duality in Maxwell theory, wherein

$$E^2 - B^2 = -(\widetilde{E}^2 - \widetilde{B}^2). \tag{5.23}$$

This sign should be reminiscent of the signs in Fourier and Legendre transforms. The reason why this minus sign must be here will be seen in Section 5.5.4. It is necessary in order to be able to identify the quarks of the dual-of-the-dual theory with the original quarks and to have the dual-of-the-dual superpotential vanish. In other words, this minus sign is there so that the dual-of-the-dual theory is the same as the original theory.

## 5.5 Checks of Seiberg Duality

### 5.5.1 't Hooft anomaly matching

The global anomalies match between the magnetic and electric theories. Here's a summary:

- $\mathrm{SU}(F)^3$: $N$

- $\mathrm{SU}(F)^2\mathrm{U}(1)_B$: $N$
- $\mathrm{SU}(F)^2\mathrm{U}(1)_R$: $-N^2/F$
- $\mathrm{U}(1)_B^2\mathrm{U}(1)_R$: $-2N^2$
- $\mathrm{U}(1)_R$: $-N^2-1$
- $\mathrm{U}(1)_B^3$, $\mathrm{U}(1)_B$, $\mathrm{U}(1)_B\mathrm{U}(1)_R$: 0

### 5.5.2 Moduli space

If the two theories indeed describe two regions of the same physics, then their flat directions (parameterized by gauge invariant polynomials) must match. Writing $B$ and $\widetilde{B}$ for the electric baryons and $b$ and $\widetilde{b}$ for the magnetic baryons,

| | $\mathrm{SU}(F)$ | $\mathrm{SU}(F)$ | $\mathrm{U}(1)_B$ | $\mathrm{U}(1)_R$ |
|---|---|---|---|---|
| $M$ | $\square$ | $\square$ | $0$ | $2\frac{F-N}{F}$ |
| $B$ | $\boxminus$ | $1$ | $N$ | $N\frac{F-N}{F}$ |
| $\widetilde{B}$ | $1$ | $\boxminus$ | $-N$ | $N\frac{F-N}{F}$ |
| $M$ | $\square$ | $\square$ | $0$ | $2\frac{F-N}{F}$ |
| $b$ | $\bar{\boxminus}$ | $1$ | $\frac{(F-N)N}{F-N}$ | $\frac{(F-N)N}{F}$ |
| $\widetilde{b}$ | $1$ | $\bar{\boxminus}$ | $\frac{-(F-N)N}{F-N}$ | $\frac{(F-N)N}{F}$ |

where we've written $\boxminus$ to mean the antisymmetric representation with $N$ indices and $\bar{\boxminus}$ to mean the conjugate antisymmetric representation with $(F-N)$ indices. The moduli indeed match because these two objects both have the same dimensionality, $N(1-N/F)$. There is a handy way to relate the moduli of both theories,

$$M_{\text{elec}} = M_{\text{mag}} \tag{5.24}$$

$$B = *b \tag{5.25}$$

$$\widetilde{B} = *\widetilde{b}, \tag{5.26}$$

where $*$ is the Hodge dual operator.

### 5.5.3 Navigating the moduli space

Another check of consistency comes from exploring the moduli space. We already learned how to do this when we studied the ADS superpotential. We have two tools: Higgsing both sides (recall Section 3.4) and integrating out a flavor from each side (recall Section 3.5).

[**To do:** write this up. See slides and Seiberg.]

This works out, and we are left with a commutative diagram

$$\begin{tikzcd}
\mathrm{SU}(N)_F \arrow[r, leftrightarrow, "\text{dual}"] \arrow[d, "\langle Q\widetilde{Q}\rangle_{FF}"'] & \mathrm{SU}(F-N)_F \arrow[d, "Mq\widetilde{q}"] \\
\mathrm{SU}(N-1)_{F-1} \arrow[r, leftrightarrow, "\text{dual}"'] & \mathrm{SU}(N)_{F-1}
\end{tikzcd}$$

### 5.5.4 Dual of the dual

Another check of consistency is that the duality transform takes the magnetic theory back into the electric theory. The first sign of potential trouble is the relation of the holomorphic scales, (5.21). Performing the duality transform again and identifying the dual-of-the-dual scale with the original electric scale, we find

$$\widetilde{\Lambda}^{3(F-N)-F}\Lambda^{3N-F} = (-)^N\widetilde{\mu}^F, \tag{5.27}$$

where $1/\widetilde{\mu}$ is the superpotential coupling for the dual-of-the-dual theory. The left-hand side is indeed the same as (5.21). Setting the right-hand sides equal give the relation $\widetilde{\mu} = -\mu$. We know that the dual-of-the-dual theory has gauge group $\mathrm{SU}(N)$ with $F$ flavors. We now want to see that the superpotential vanishes. Let us quite the dual-of-the-dual quarks as $d$ and $\widetilde{d}$ (soon to be identified with the original electric quarks $Q$ and $\widetilde{Q}$). The independent meson, formed from the magnetic theory's dual quarks, will be written as $N$. The duality-induced superpotential is thus

$$W_{\text{dual}^2} = \frac{1}{\widetilde{\mu}}Nd\widetilde{d}. \tag{5.28}$$

This doesn't look like the original theory, but we must also remember to include the superpotential that was generated when we first when from the original electric theory to the $\mathrm{SU}(F-N)$ magnetic theory. This looks like

$$W_{\text{dual}} = \frac{1}{\mu}MN, \tag{5.29}$$

where we've written the magnetic quarks in terms of the dual-of-the-dual meson $N$. The complete dual-of-the-dual superpotential is thus

$$W = \frac{1}{\mu}N(M - d\widetilde{d}). \tag{5.30}$$

This is just a linear (mass) term for $N$. As we learned in Section 3.9, this means that we can just use the $N$ equation of motion to integrate it out of our theory by a Legendre transform. In other words, $N$ is just a Lagrange multiplier. It sets $N = 0$ and $M = d\widetilde{d}$. The first relation tells us that $W = 0$, while the second relation tells us that $d\widetilde{d} = Q\widetilde{Q}$, so that we have indeed recovered the original electric theory.

## 5.6 $N+1 < F < 3N/2$: compositeness

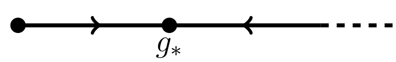
Now let's get back to the curious case of $N+1 < F < 3N/2$. Recall that when we identified the conformal window, unitarity forced us to impose a lower limit of $F > 3N/2$. This left a swath of $(F, N)$ space unaccounted for: we know that it must exist in a different phase, but how do we characterize it? This is the real problem. In the case of the conformal window we we were able to go into the corner of parameter space, $F \approx 3N$, where the theory is perturbative. On the other hand, the regime around $F \approx 3N/2$ is badly non-perturbative; what can we do?

Ah! Well, it just so happens that in the conformal window we had this magnetic theory which was weakly coupled in precisely the same regime where the electric theory is strongly coupled! Now we have a trick: while we're in the conformal window, go to the $F \approx 3N/2$ limit where the magnetic theory is weakly coupled. The key is to stay in the magnetic picture where we are weakly coupled and make the jump across $F = 3N/2$. Since $3n = 3(F-N) < F$, the theory is not asymptotically free. In particular, the superpotential is *irrelevant* and so becomes weakly coupled in the IR ($y$ flows to zero). This leaves us with an SU$(F-N)$ gauge theory with massless magnetically charged quarks and a singlet $M$. This is the **free magnetic** phase.

## 5.7 RG motivation for Seiberg duality

Seiberg duality is often presented somewhat miraculously, as if the details of the duality were collected by Seiberg on two stone tablets (one describing SQCD in the conformal window and the other describing the SQCD+M dual!). While this is all that is necessary for a 'working knowledge' for most phenomenological applications, one can have a fuller appreciation of the insights involved by better motivating why Seiberg originally conjectured his duality.

First let us recall the RG flow of an SQCD theory in the conformal window:




There are two fixed points: the trivial fixed point at $g = 0$ and the Banks-Zaks fixed point at $g = g_*$, where $g_*$ is perturbative in the large-$N$ limit. We have assumed that $g_*$ exists more generally in the conformal window. We've argued that the trivial fixed point is unstable and all theories in the conformal window eventually flow to the Banks-Zaks fixed point.

Note that if the electric theory is in the conformal window, then the gauge group of the proposed dual theory, $SU(n)$ with $n = F - N$, is also in the conformal window. (This is trivial to check.) The dual theory, however, has an additional field $M$. In the limit where the magnetic superpotential vanishes, $W_{\text{mag}} = 0$, then this field is free and is completely decoupled from the SQCD sector. The magnetic SQCD sector thus still flows to the Banks-Zaks fixed point so that the IR theory is Banks-Zaks plus a decoupled free scalar field with dimension 1. From Fact 5.2 we know that $R_{\text{sc}}[M] = 2/3$. Further, because $M$ is a free field, we can transform it independently of the dual quarks with respect to its flavor symmetries. In particular, this $W_{\text{mag}} = 0$ theory enjoys a larger flavor symmetry group,

$$SU(F)_L \times SU(F)_R \times U(F)_L \times U(F)_R. \tag{5.31}$$

Now let's turn on the superpotential. For simplicity, let's define the coupling to be $y = 1/\mu$. We note that $W_{\text{mag}}$ breaks the global symmetry above to $SU(F)_L \times SU(F)_R$, the same symmetry as the electric theory. In the special case where $g = 0$ (trivial SQCD fixed point) and $y \neq 0$, we are left a theory with no gauge symmetry and a trilinear superpotential. This is just the $XYZ$ model investigated for pedagogical reasons in [3]; we know this flows to the trivial fixed point. We know that this fixed point is unstable since small perturbations in $g$ will send it to the Banks-Zaks fixed point.

In fact, things are more interesting at the Banks-Zaks fixed point. A good question to ask is whether the superpotential destabilizes this fixed point. In fact, it does. Let's look at the $\beta$-function for $y$ in the neighborhood of $(g_*, y = 0)$. Suppose $y$ takes a very small value. We know that the dimension of $M$ is very close to 1 since

$$\dim\, M = 1 + \mathcal{O}(y^2). \tag{5.32}$$

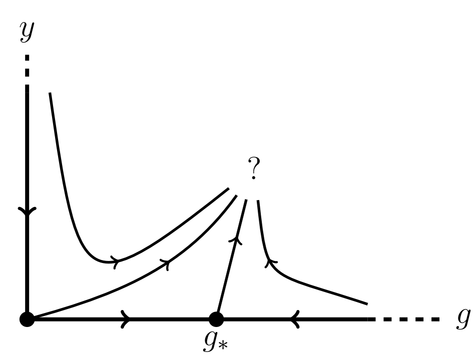
Further, we know that the $\beta$-function is given by the sum of anomalous dimensions,

$$\beta_y = y\left(\gamma_M + \gamma_q + \gamma_{\widetilde{q}}\right). \tag{5.33}$$

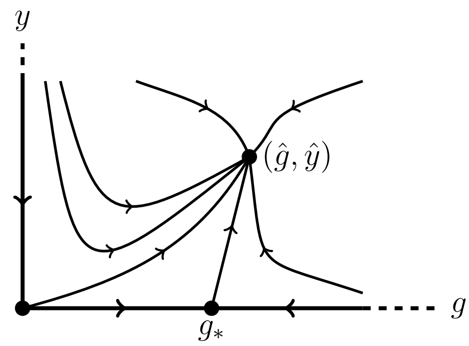
This is simply because the coupling $y$ is protected by holomorphy so that the only renormalization comes from wavefunction renormalization associated with non-canonical scaling dimensions. (See Strassler's notes for an excellent introduction [3].) Thus

$$\begin{aligned}\beta_y &= y\left((\dim\, M - 1) + \left(\dim\, q - \frac{3}{2}\right) + \left(\dim\, \widetilde{q} - \frac{3}{2}\right)\right) & (5.34)\\ &= y\left(1 - \frac{3n}{F}\right). & (5.35)\end{aligned}$$

We've dropped the higher-order terms in $y$ coming form $\gamma_M$. The key point about $\beta_y$ is that it is *negative* for any $F < 3n$ (i.e. in the conformal window). Thus $y$ is a *relevant* operator! (If you're worried about the effect of the higher order terms in $y$, then unitarity constraints should convince you that this doesn't invalidate our argument.) The relevance of $y$ means that the Banks-Zaks fixed point is an unstable fixed point. Now the RG structure of the theory has become very interesting, indeed:




Both the trivial and Banks-Zaks fixed points are unstable. Where, oh where, can our little theory go? It may be that there is no fixed point and that all couplings flow to infinity, this means that our theory must be defined with a cutoff. That would be boring. Fortunately, Seiberg imagined a more interesting scenario. The crux of Seiberg duality is that there is a *new* fixed point at $(\hat{g}, \hat{y})$ that is stable and is the IR limit of the RG flow:




Clearly this is different from the Banks-Zaks fixed point. The key point is that the new fixed point for the magnetic theory at $(\hat{g}, \hat{y})$ is to be identified with the Banks-Zaks fixed point of the electric theory. Why should we believe this? consider the dimension of the field $M$ at the IR fixed point. From $R_{\text{sc}}$-symmetry we know that $2\gamma_{\widetilde{q}} = 1 - 3n/F$. At a fixed point $\beta = 0$, therefore $\sum_i \gamma_i = 0$. This fixes $\gamma_M = (2F - 3N)/F$. Thus in the magnetic theory the meson has dimension

$$\dim M = 1 + \gamma_M = \frac{3F - 3N}{F}. \tag{5.36}$$

This is indeed exactly what we get from calculating the $Q\widetilde{Q}$ dimension at the Bank-Zaks fixed point of the electric theory, as we saw in (5.13). Thus the field $M$ in the magnetic theory can plausibly be associated with the meson $Q\widetilde{Q}$ of the electric theory. Similarly, one can check the dimension of the baryon operators in both theories. In fact, we have already shown that they have the same $R_{\text{sc}}$ charge.

## 5.8 The importance of the scale $\mu$

Let us return to the scale matching condition (5.21). To avoid (or perhaps generate) confusion, in this section we will explicitly refer to the dynamical scales as $\Lambda_{\text{el}}$ for the electric theory, $\Lambda_{\text{mag}}$ for the magnetic theory, and $\Lambda$ for the matching scale which we previously called $\mu$. The scale matching relation can then be written

$$\Lambda_{\text{el}}^{b_{\text{el}}} \Lambda_{\text{mag}}^{b_{\text{mag}}} = (-)^N \Lambda^{b_{\text{el}} + b_{\text{mag}}}. \tag{5.37}$$

This, in turn, can be written as

$$\frac{1}{g_{\text{el}}^2(|\Lambda|)} = \frac{b_{\text{el}}}{8\pi^2} \log\left(\frac{|\Lambda|}{\Lambda_{\text{el}}}\right) = \frac{-b_{\text{mag}}}{8\pi^2} \log\left(\frac{|\Lambda|}{\Lambda_{\text{mag}}}\right) = \frac{-1}{g_{\text{mag}}^2(|\Lambda|)}. \tag{5.38}$$

This shows us that $|\Lambda|$ may be interpreted as the scale at which the two dual couplings are equal up to a sign. As shown in Fig. 1, the size of $|\Lambda|$ determines the structure of the duality. For $\Lambda > \Lambda_{\text{el}}$, there is an energy range for which there exists no weakly coupled description of the dynamics. On the other hand, for $\Lambda < \Lambda_{\text{el}}$, it is possible to have a weakly coupled magnetic description of the dynamics which matches on to a weakly coupled description of the electric dynamics. Since Seiberg duality is actually a statement about the far infrared, the ambiguity in the 'correct' value of $\Lambda$ is not usually relevant. The ability to work with two weakly coupled theory, however, can be used in model building, e.g. [57], from which this section is based.

Naive dimensional analysis suggests that all three scales $\Lambda$, $\Lambda_{\text{el}}$, and $\Lambda_{\text{mag}}$ have similar magnitude. One way of understanding Seiberg duality is the statement that $\Lambda$ is a parameter and that the naive relation of scales needn't hold. In particular, having a small $\Lambda$ would allow one to consider weakly coupled magnetic dynamics.

The condition $|\Lambda| < \Lambda_{\text{el}}$ can also be written as $\Lambda_y < \Lambda_{\text{mag}}$ where $\Lambda_y$ is the Landau pole scale for the magnetic Yukawa coupling. This is because... [**To do:** see [57]].

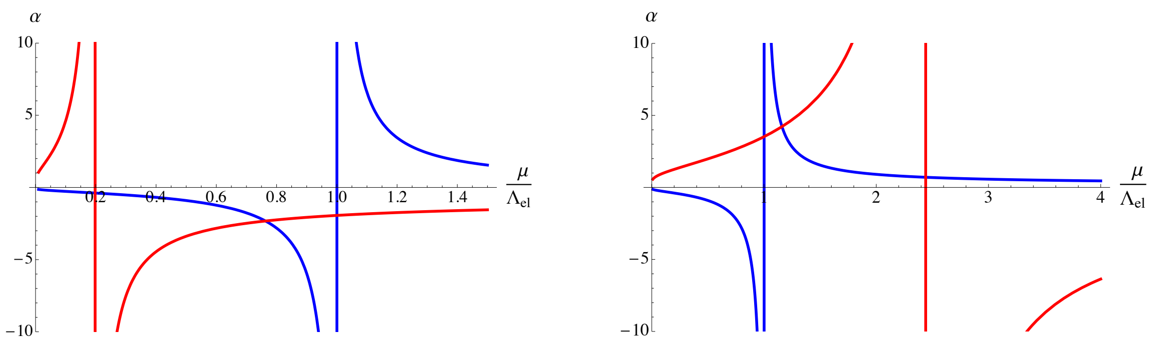



Figure 1: Values of $\alpha = g^2/4\pi$ as functions of the renormalization scale $\mu$ for $\Lambda = 1.5\Lambda_{\text{el}}$ on the left and $\Lambda = 0.8\Lambda_{\text{el}}$ on the right. The 'electric' coupling (blue) is positive for $\mu > \Lambda_{\text{el}}$ whereas the 'magnetic' coupling (red) is positive for $\mu < \Lambda_{\text{mag}}$. We use $N = 6$ and $F = 8$. Image from [57]. [**To do:** redraw in TikZ.]

## 5.9 Remarks on the meaning of the duality

[**To do:** flesh this out.] Seiberg duality is an **infrared duality**. This means that the electric and magnetic theories flow to the same IR fixed point. We have taken the additional step of identifying the magnetic theory as the effective IR theory after the electric UV theory becomes strongly coupled. Indeed, the effective field theory paradigm is a story of infrared duality. The perturbative low-energy effective theories which are valid the IR are typically non-renormalizable and require a UV cutoff. For a related discussion of the nonlinear sigma model and the Higgs sector, see Nima Arkani-Hamed's lectures at PiTP 2010[20].

In their UV limits, the magnetic and electric theories are indeed very different and have no deep relation. In particular, the duality appears to only exist at the IR fixed point in the continuum limit of the RG flow. This is often accompanied by the statement that such an infrared duality (as is typical in field theory) should be contrasted with the 'exact' dualities which appear in string theory. This perspective is not quite accurate, since one can indeed take a limit where there is a finite region in which the two theories actually have the same RG flow rather than just approaching the same fixed point. This is discussed in detail by Strassler in his two reviews [3, 4] and forms the basis for the duality cascade.

[20] `http://video.ias.edu/pitp-2010`

We will return to the idea of understanding Seiberg duality from an effective theory perspective in Section 8, where we describe recent work by Komargodski to relate the dual theories to nonlinear sigma models from the 60s.

# 6 The phase structure of SQCD

Here's a summary of Sections 3 through 5.

| | |
|---|---|
| | Free electric, $V \sim \frac{1}{\log(\Lambda R) R}$ |
| $F = 3N$ | |
| | Non-Abelian Coulomb, $V \sim \frac{1}{R}$ |
| $F = \frac{3N}{2}$ | |
| | Free magnetic, $V \sim \frac{\log(\Lambda R)}{R}$ |
| $F = N + 2$ | |
| $F = N + 1$ | $s$-confining, $V = 0$ |
| $F = N$ | Quantum modified constraint, $V = 0$ |
| $F = N - 1$ | Higgs, $V =$ const. |
| | ADS superpotential |
| $F = 1$ | |
| $F = 0$ | Confining, $V \sim R$ |

# 7 Siblings and Cousins of Seiberg Duality

- SO, Sp gauge groups.
- Kutasov Dualities.

# 8 Crouching Seiberg, Hidden Gauge Group

*F*cking magnets, how do they work?* – Insane Clown Posse, Miracles (2009)[21]

[21] http://knowyourmeme.com/memes/fcking-magnets-how-do-they-work

Despite being rigorously tested, the structure of the Seiberg magnetic dual is surprising. Why should the infrared effective theory describing confined degrees of freedom have some new emergent $SU(N-F)$ gauge symmetry? In 2010 Komargodski presented an alternate motivation for the duality from the point of view of 1960s meson physics [58].

## 8.1 Mini review: nonlinear sigma model

From QCD we are familiar with the idea that gauge degrees of freedom can confine and disappear in an effective theory. For example, in the SU(3)×SU(3)→SU(3) nonlinear sigma model, the quark degrees of freedom are confined leaving only the (pseudo)-Goldstone excitations in the light spectrum. Let us quickly review the steps for the nonlinear sigma model for a breakign pattern $G \to H$. This is a digression, but it will be useful in what follows.

1. Pick a vacuum state, $\langle\phi\rangle = \phi_0$. In the case of the pion NLΣM, this is $\langle U(x)\rangle = f\mathbb{1}_{3\times 3}$.
2. Transform it by one of the *broken* generators, $g \in G/H$. We thus have $U(x) \to g_L U(x) g_R^\dagger$. For the broken axial transformations, we have $g_R^\dagger = g_L \equiv g$, where we may write $g = e^{i\epsilon^a T^a}$.
3. Promote the transformation parameter to a field, call it a Goldstone boson. We thus take $\epsilon^a \to \pi^a(x)/f$. Thus the pion appears as $U(x) = \exp(2i\pi^a(x)T^a/f)f\mathbb{1}$.

The original field $U(x)$ is nonlinear in the low energy degrees of freedom, $\pi^a(x)$. The reason for this is that we had to constrain our field to live along the non-trivial vacuum manifold from the $G \to H$ breaking pattern. The cost of imposing that our low energy degree of freedom always points in the Goldstone direction is that it had to come in a rather nasty exponential.

This isn't the only way to impose the constraint. There are *many* ways to do this, see for example the texts by Donoghue [59] or Cheng & Li [60]. The key point, however, is that the low energy *physics* is completely independent of how we choose to represent the NLΣM. This elegant result was first presented by Haag [61] and more completely by Coleman et al. [62, 63].

## 8.2 Hidden gauge group

Let us return to the original question: how is it that we have an emergent gauge group in the low energy theory? In our pion Lagrangian, all traces of the QCD gauge group disappeared because they were confined. From this point of view, it seems ridiculous that the low energy theory should have any new gauge degrees of freedom. It seems like there is no analog to the magnetic SU$(F-N)$ gauge symmetry in Seiberg duality. *Or is there?*

Back in the old days, when QCD was young and the Beatles were still on topping charts, physicists wondered what to make of the higher resonances in the hadronic spectrum. We

Georgi papers on $\rho$... [64, 65]

Current algebra, reviewed in [66]... see also refs in flavor notes

# 9 Relation to AdS/CFT

Mention Steve's Papers. Mention Csaba's work. Important: Zohar's paper (above)

From Adam: how to think about this: AdS/CFT takes your strongly coupled theory to a 5D theory. The 5D theory can be deconstructed. This deconstruction is related to a chiral lagrangian (breaking of gauge group between links). This chiral lagrangian can be understood as the chiral lagrangian in Zohar's $\rho$ meson paper, which relates it to the magnetic gauge field. What's not obviously rigorous: relating the 5D theory and the deconstruction... do you require non-local couplings? Also, is Zohar's result rigorous in all cases?

# 10 Some current directions of interest

- In a rather elegant confluence, two of the remarkable dualities of the 1990s, Seiberg duality and the AdS/CFT correspondence, come together in the so-called "warped deformed conifold" construction. This is stringy construction wherein the supergravitational theory is dual to a superconformal field theory which, itself, can be understood as a sequence of Seiberg dualities. The latter has been dubbed the duality cascade, and is elegantly reviewed in [4]. The gravitational side was described in the author's A-examination[22].
- In 2006, Seiberg duality was employed to find a way to make generic models of dynamical SUSY breaking in the ISS model [67]. This had been one of the elusive goals of SUSY model building for two decades and came at the cost of the SUSY-breaking vacuum being metastable. This model launched an entire model building industry that continues to thrive. We present some aspects of this field when we discuss SUSY breaking.
- A very interesting question from a formal as well as model-building perspective is whether unification survives across a Seiberg duality when the low-energy theory suffers from a Landau pole before it unifies. This was examined by Abel and Khoze [68, 69].
- The relation between Seiberg duality and AdS/CFT has also recently been applied to model-building. This is a particularly interesting direction which the author would like to pursue. See: [70, 71].
- Recently a group of formal theorists have applied methods from algebraic geometry to better understand the classical moduli space of SQCD [72]. The author's ignorance in this subject has left him to pronounce the paper's title, much less understand it.

# 11 Unsorted ISS and SUSY breaking notes

## 11.1 Questions originating from Yael's chiral ISS model

(Thanks to Yael for really clarifying things that I was confused about. She points out 0705.1074 for a discussion of some of the details of ISS, especially incalculability)

**s-confinement**: while the phenomenon is interesting physically, this has nothing to do with why this a nice playground for model building. (cf TY and John's paper.) The reason why we like it is that it is easy. Csaba wrote out all s-confining theories and the dualities are easy. In general,

[22] `http://www.lepp.cornell.edu/~pt267/files/documents/A_ks.pdf`

it's difficult to write out the dual theory models that aren't s-confining. For example, there are tricks like deconfinement (see Pouliot) that one has to use to get these duals. (Deconfinement: trick to use s-confinement to engineer dualities.)

The definition of 'chiral' that Yael uses is that one cannot write a mass term for all the fields. In this sense it furnishes a chiral Weyl fermion. For example, the tensor in her SU(6) model cannot obtain a mass term and so has a chiral fermion. Note that these chiral fermions cannot be used to furnish the SM matter particles since this is prohibited by tree-level no-go theorems for single sector susy breaking.

Why are these models interesting, then? The point is that the ISS susy breaking sectors are vectorlike. One might like to check to see if anything interesting happens if the sector is chiral. For example, in old-style (global SUSY breaking minimum) models, the chiral models were very different from the vector models because they weren't affected by the Witten Index theorem.

**Incalculability**: Incalculability has to do with higher order corrections to the Kahler potential. In old-school models with global SUSY-breaking minima, it was otfen the case that one could construct a model where one knew that a SUSY-breaking global minimum existed, but nothing more. For example, the ADS potential with its flat directions lifted by a tree-level potential. In general the region where the minimum occurs has no well controlled expansion, though one can see that the F-term equations are not satisfied so that SUSY is broken. On the other hand, one can usually tune the tree-level potential to consider regions where vevs are large and the theory is weakly coupled. (See the 3-2 model.)

In ISS like models, incalculability is a question of whether one can be sure that all mass-squareds at the SUSY-breaking local minimum are positive. In other words, the local minimum is stable. In ISS this is done by controlling the SUSY breaking effects through a parameter $\mu$. All of the field vevs are proportional to a power of $\mu$ so that the higher order corrections to the Kahler potential are also controlled by $\mu$. One can then tune $\mu$ so that the higher order corrections are under control and the theory is perturbative.

**Caveats**: Yael warns that this is not necessarily a fruitful business to enter at this time, not only because one should be data-driven. She points out the gaugino mass problem which is still has not been solved in a satisfactory way. Further, she notes that while ISS has open up a wide range of models to play with, as a whole they haven't yet added anything in terms of fully viable models. She says that we already have a decent model, MGM. (She did say that it's still valid to look for a toy project to learn how to use these tools.)

# $\mathcal{N} = 2$ Duality

## 12 Seiberg-Witten

Reviews: Bilal [22], Alvarez-Gaume [20]

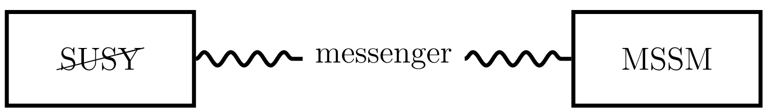
## 13 Gaiotto Dualities

# Breaking SUSY

## 14 SUSY Breaking: history

### 14.1 SUSY breaking

If you're reading this then you're already familiar with the theoretical features of supersymmetry as an appealing model for physics beyond the TeV scale. At low energies, however, SUSY is clearly broken. In order to preserve the good features of SUSY, we would like SUSY to be broken spontaneously (i.e. 'softly' in terms of phenomenology) rather than explicitly. Attempts to write down realistic SUSY-breaking models for the Standard Model are immediately confronted with problems associated with the supertrace rule, i.e. that the sum of all boson masses minus the sum of all fermion masses must vanish. Since we haven't discovered any very light scalar partners to the leptons or quarks, this imposes the usual modular structure for SUSY breaking.




The two questions are (1) how to build a model for the SUSY-breaking sector, and (2) how to mediate this to the MSSM. We will not say anything about this second question and, for the remainder of this document, assume gauge mediation which we review in Appendix 16. For the first question, any student of beyond the Standard Model physics will tell you that breaking supersymmetry is not as easy as one might otherwise think. The simplest model of spontaneous SUSY breaking presented in the literature is the O'Raifeartaigh model which contains three superfields and a very specific superpotential. It turns out that one really has to work hard to kill supersymmetric vacua! In other words, SUSY-breaking vacua appear to be highly-non-generic. We will return to this momentarily.

### 14.2 Dynamical supersymmetry breaking

Dynamical symmetry breaking (DSB) is an elegant idea in which spontaneous symmetry breaking is realized via the vaccum expectation value of a composite field, e.g. the BCS theory of superconductivity. The key point is that the field which obtains a vev is not a 'fundamental' field in the theory and is the result of strong coupling. Dynamical breaking of supersymmetry was first proposed by Witten in 1981 [73, 74] as a solution to the Hierarchy problem.

"Wait a second," you ask, "I thought that supersymmetry itself is a solution to the Hierarchy problem?" Indeed you are correct. The point is that naïvely imposing SUSY breaking at the TeV scale—where it provides a reasonably natural solution to the Higgs mass hierarchy ($m_H \ll M_{\rm Pl}$)—still leaves one with the question of why the SUSY breaking scale should be so much lower than the fundamental scale, $M_{\text{SUSY}} \ll M_{\text{Pl}}$. Obtaining this hierarchy suggests that supersymmetry is not broken at tree-level, but rather by quantum corrections. In other words, the theory's vacuum is manifestly supersymmetric at tree-level but is only rendered non-supersymmetric through the dynamics of the theory itself. Further, the powerful non-renormalization theorems in supersymmetry state that if a theory is supersymmetric at tree-level, then it is supersymmetric at all orders in perturbation theory. Thus the only way for SUSY to be broken dynamically is through *nonperturbative* effects. Since these effects go as $e^{-8\pi^2/g^2}$, we see that they are strongly suppressed and we can hope to explain the hierarchy between $M_{\text{SUSY}}$ and $M_{\text{Pl}}$.

It was quickly discovered, however, that it is very difficult to build a straightforward and realistic theory of dynamical supersymmetry breaking. Models end up requiring a special structure on top of a chiral gauge theory. Models based on supersymmetric $SU(N)$ gauge theories ("super-QCD," or SQCD), i.e. what one's first option for building a DSB sector, run into problems with strong coupling leading to 'non-calculability.' The nail in the coffin for such theories is the observation that the Witten index for such theories is $N$, i.e. there exists $N$ supersymmetric vacua[23]. To add insult to injury, we note that gauge mediation, for all of its features, tends to restore supersymmetry somewhere in field space based on the above Witten index argument. How could one hope to write down a calculable theory with a *non*-supersymmetric vacuum? There exist several famous exceptional cases with silly names like (3-2), (4-1), and ITIY. It was clear, however, that models realizing Witten's original hope for dynamical SUSY breaking are very non-generic.

## 14.3 What makes SUSY-breaking non-generic?

We've now twice used the pejorative phrase 'non-generic.' Intuitively this means that models of a given type don't require any special constructions and are, in a sense, easy to create from simple building blocks. In other words, models don't require fine-tuning (in theory space). A particularly useful notion of genericness that we will make use of is the statement that $n$ equations for $n$ unknowns *generically* have a solution. This is of course not *always* true, but one would expect this for equations that don't have special relations (degeneracy).

In 1993 Nelson and Seiberg proved a very powerful theorem that showed that the conditions for a SUSY breaking vacuum are connected to the existence of an $R$-symmetry [75]. The Nelson-Seiberg theorem stated that

$$\begin{aligned}
\cancel{\text{SUSY}} &\Rightarrow \exists\ R\text{-symmetry} \\
\text{SSB } \cancel{R} &\Rightarrow \cancel{\text{SUSY}},
\end{aligned}$$

or, in other words: $R$-symmetry is a necessary condition for the existence of a SUSY-breaking vacuum and a spontaneously broken $R$-symmetry is a sufficient condition for such a vacuum.

This was able to shed light on the difficulty of constructing generic SUSY breaking models. In order to have a SUSY breaking vacuum, Nelson and Seiberg tell us that the theory must have an $R$-symmetry. We know, however, that gaugino masses explicitly break $R$-symmetry. We must thus consider the case where the $R$-symmetry is spontaneously broken, which is fine since this automatically implies the existence of a SUSY-breaking vacuum. On the other hand, we know that spontaneously breaking the $R$-symmetry would give us a Goldstone boson. Thus either

[23] This Witten index argument also imposes a chiral structure theories with SUSY breaking global minima.

we have a preserved $R$-symmetry and massless gauginos or a spontaneously broken $R$-symmetry and a massless $R$-Goldstone boson. No matter what we end up with a massless particle which is unobserved! We could try to be more sophisticated and appeal to gravity: since gravity hates continuous symmetries[24], we might expect that gravitational effects will give mass to the Goldstone and save us [76]. Unfortunately, such effects would usually be far too small to give phenomenologically acceptable masses (though see Dine, Nelson, and Shirman [77] for a counterexample). Alternately, formal theorists might argue with a result by Banks and Dixon that there can be no such thing as an exact global symmetry in string theory [78]. We are finally forced into the conclusion that 'generic' theories of supersymmetry have supersymmetric vacua. This explains why it was so damn hard to construct pretty SUSY-breaking models.

## 14.4 Metastable SUSY breaking

The 'modern' (after 2006) approach to this problem is that "when theory gives you lemons, make models based on lemonade." We shall squeeze out of the problem of non-generic SUSY-breaking *global* vacua by looking instead at models with a *local* SUSY-breaking vacuum. Until this point we had assumed that all the vacua we were considering were global, i.e. the 'true' vacuum of the potential. This is sensible since otherwise we'd have to worry about tunneling to the true vacuum[25] [79, 80]. However, if we want to hold on to the principle of being able to construct *generic* models, then metastable SUSY breaking is *inevitable*. If you are particularly high-brow—which the author is not—then you might also appeal to gravitational motivations such as the cosmological constant ("it's hard to make sense of deSitter space," a string landscape).

It is worth noting that models with metastable SUSY-breaking vacua have been around for some time. Ellis, Llewellyn Smith, and Ross constructed a model with a *classically* metastable vacuum in 1982 [81]. Around fifteen year later, Dimopoulos, Dvali, Rattazzi, and Giudice developed the first metastable SUSY-breaking models based on pseudomoduli space, i.e. the "quantum-modified" moduli space [82]. Unencumbered by the burden of finding a SUSY-breaking vacuum, these models were much simpler to construct: one could use the framework of SQCD straight out of the box. These theories have an explicit $R$-symmetry breaking which is realized as an accidental $R$-symmetry in the low-energy theory[26]. Finally, there is generally a small parameter $\epsilon$ which parameterizes the explicit $R$-symmetry breaking and the separation of the SUSY and SUSY-breaking vacua in field space. One can dial in a convenient value for $\epsilon$ to evade vacuum tunneling problems.

These models, however, were faced with the practical problem of calculability. Supersymmetric QCD in the asymptotically free regime is IR confining. Thus it is not obvious what the low-energy

[24] For example, a black holes might eat something charged under a global symmetry and then forget about that charge. Only if symmetry is gauged would the black hole take on the charge of the object it ate. Compare this to the popular Kirby video games by Nintendo.

[25] There's a lot of interesting physics involved in such a tunneling. If a bubble of space tunneled to the true vacuum, there would be a boundary which is stretched over the potential barrier. The volume ($\sim R^3$) would live in a lower-energy state, but its boundary ($\sim R^2$) lives in a higher-energy state. Thus for small bubbles the boundary energy would dominate and would cause the bubble to collapse, while for larger bubbles the volume would dominate and eventually expand to eat the entire universe.

[26] Note that typically instanton effects only break $R$-symmetry to a discrete subgroup so that more work needs to be done to allow explicit gaugino masses.

degrees of freedom are in this nonperturbative regime and it's not clear if we can even say that SUSY is broken since higher order terms in the Kähler potential might ruin one's construction. It seems like we're at a loss for constructing useable models.

In 2006, Intriligator, Seiberg, and Shih (ISS) found a way around calculability by [67] using the powerful and surprising electromagnetic duality in SQCD discovered by Seiberg [56] (see Intriligator and Seiberg's lectures [2] for a review). This so-called Seiberg duality connects an $SU(N)$ 'electric' theory with $F$ flavors to an $SU(n) = SU(F-N)$ 'magnetic' theory in the same universality class so that the two theories flow to the same IR fixed point and so describe the same low-energy physics. These two Seiberg dual theories can be chosen so that one is UV-free while the other is IR-free, thus allowing us to work perturbatively both in the UV and IR limits. Intriligator, Seiberg, and Shih were then able to construct a dynamical supersymmetry breaking ultraviolet theory while maintaining the existence of a long-lived metastable vacuum in the IR. (Technically the model-building proceeded the other way around, starting with the IR theory and then making it dynamical in the UV. We will go over this in Section 20.)

The ISS model opened the floodgates for a new wave of model building for metastable SUSY-breaking vacua. Using Seiberg duality and simple gauge groups, theorists could produce generic, calculable models and pit them against each other in a beauty pageant of which model is more elegant than the next. The swimsuit competition of such a beauty pageant, however, is always the ability of a model to reproduce realistic phenomenology, and it turned out that there were still a few features of ISS-type models that physicists were having a hard time ironing out.

## 14.5 Problems with gaugino masses

The main problem was that ISS builders ended up having to use every trick they could muster to create models with phenomenologically acceptable gaugino masses. Even though $R$-symmetry is broken in these metastable models, it turned out that the gaugino masses of any semi-realistic direct mediation model couldn't be made to be very heavy without, in turn, pushing the scalar masses to be dramatically heavier. See Ookuchi for a recent review [83]. This appears to be true even after the residual discrete $R$-symmetry (which still protects against gaugino masses) is broken. This would be nearly automatic disqualification from the SUSY model beauty pageant since any large scalar masses would contribute to corrections to the Higgs and would thus re-introduce our old nemesis the Hierarchy problem (defeating the whole point of having SUSY to begin with).

Now the lesson here is that history tends to repeat itself. Just as it seemed unsettling 15 years ago that it was so hard to find generic models of SUSY breaking, it was unsettling more recently that it was so hard to find ISS models of metastable SUSY breaking without 'anomalously small' gaugino masses. Once again physicists were faced wondering why such a thing would not be generic, and once again a clever duo was able to devise a theorem based on deeper feature of the theory to explain this.

Just as Nelson and Seiberg's 1993 theorem made clear that the difficulty in constructing SUSY-breaking vacua was rooted in $R$-symmetry, Komargodski and Shih wrote down a theorem in 2009 that explained why gaugino masses were so small in ISS-like models [84]. We will thoroughly dissect their derivation in Section 18. The result, however, is that gaugino masses vanish identically at tree level if the pseudomoduli space is everywhere non-tachyonic (i.e. stable). It is rather surprising that the pseudomodulus topology is related to the gaugino mass, but the result is able to shed light on the 'anomalously small' gaugino masses encountered by those who tried to construct realistic ISS models.

# 15 Dynamical Supersymmetry Breaking

We now remind ourselves of the basics of dynamical supersymmetry breaking (DSB). We shall roughly follow David Shih's lectures from TASI 09[27] Review articles on dynamical SUSY breaking models before ISS include those by Poppitz and Trivedi [85], and Shadmi [86]. During the [long] time that this document was being prepared, Dine and Mason also published an excellent review that touches on many current topics [8].

## 15.1 Motivation

The main motivation for dynamical SUSY breaking is understanding why there should be such a hierarchy between the SUSY breaking scale $M_{\cancel{\text{SUSY}}}$ and the Planck Scale $M_{\text{Pl}}$. Naturalness[28] suggests that that $M_{\cancel{\text{SUSY}}} \sim \text{TeV} \ll M_{\text{Pl}}$. As quantum field theorists we can conjecture that maybe this is because SUSY is broken at a higher order in perturbation theory, e.g. radiatively by multi-loop effects. This does not work in supersymmetry since holomorphy tells us that the superpotential $W$ is not perturbatively renormalized. Thus if SUSY is unbroken at tree level, it is left unbroken at all orders in perturbation theory.

The hope, then, is that one might be able to *dynamically* generate the TeV scale from a much higher UV scale using *nonperturbative* effecits in much the same way that the confinement scale $\Lambda_{\text{QCD}}$ is generated in QCD. In other words, we hope to construct a model where

$$M_{\cancel{\text{SUSY}}} = M_{\text{UV}} e^{-1/g^2} \ll M_{\text{UV}}.$$

The primary analogue here is the breaking of chiral symmetry in QCD from the condensation of quarks to form a QCD vacuum that [spontaneously] breaks the axial $SU(3)_A$ symmetry dynamically.

Our ultimate goal is to be able to find a nice model of DSB and use it to build a model of gauge mediation to write down a viable theory of nature.

## 15.2 Toy example: SUSY QM

This section is borrowed from the review [8].

Consider a quantum mechanical system with two operators

$$Q_{1,2} = \frac{1}{2}\left(\sigma_{1,2} P \pm \sigma_{2,1} W(x)\right). \qquad (15.1)$$

[27] Recordings available at `http://www.colorado.edu/physics/Web/tasi09_annc.html`.

[28] Technically we mean 'technical naturalness.'

By construction these satisfy $\{Q_i, Q_j\} = \delta_{ij} H$ where

$$H = \frac{1}{2}\left(p^2 + W^2 + \sigma_3 \frac{dW}{dx}\right). \tag{15.2}$$

This turns out to already exhibit many of the features of 4D SUSY field theories. For example, if $W$ has a zero then the system has a supersymmetric ground state which is preserved to all orders in perturbation theory. Ref. [8] gives the example of a harmonic oscillator, $W = \omega x$ so that $V = \frac{1}{2}\omega^2 x^2$. The ground state energy gets a bosonic zero-point contribution $\frac{1}{2}\hbar\omega$, but also 'fermionic' contributions from the $\hbar\sigma_3$ terms which cancel: $\Delta E = \pm\hbar\omega$.

Now let's start to play with what could happen non-perturbatively. The condition for unbroken SUSY is $Q_i|\psi\rangle = 0$. For $Q_1$,

$$\left(i\frac{d}{dx} + i\sigma_3 W\right)\psi = 0. \tag{15.3}$$

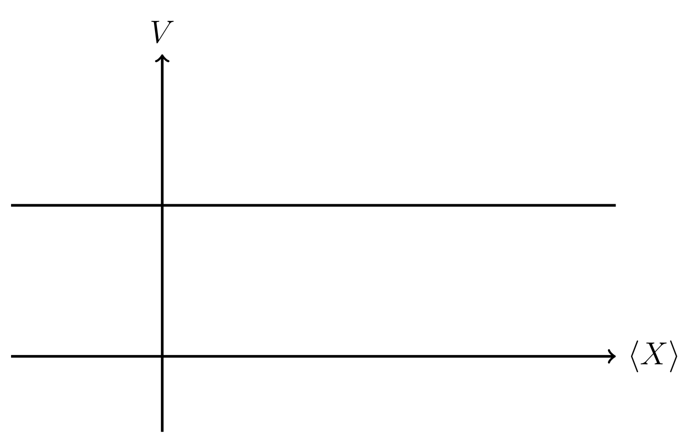
We can even solve this:

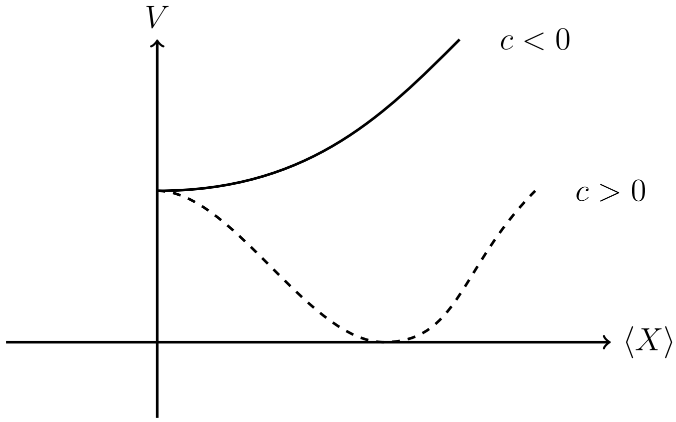
$$\psi = e^{\pm\int_{-\infty}^{x} dx'\, W(x')\sigma^3}\psi_0. \tag{15.4}$$

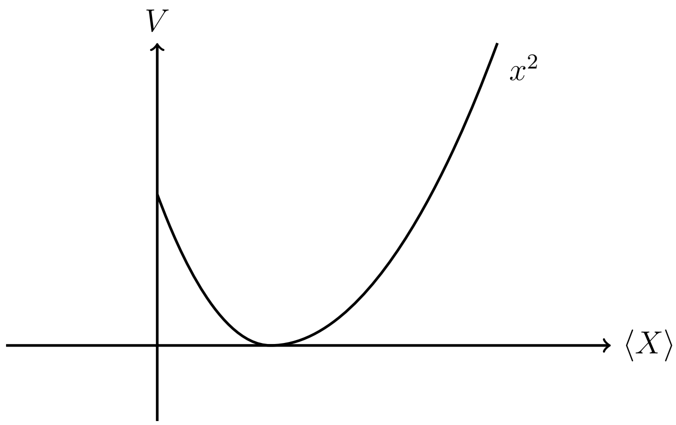
If $|W| \to \infty$ as $x \to \infty$, the state is only normalizable for $W$ odd.

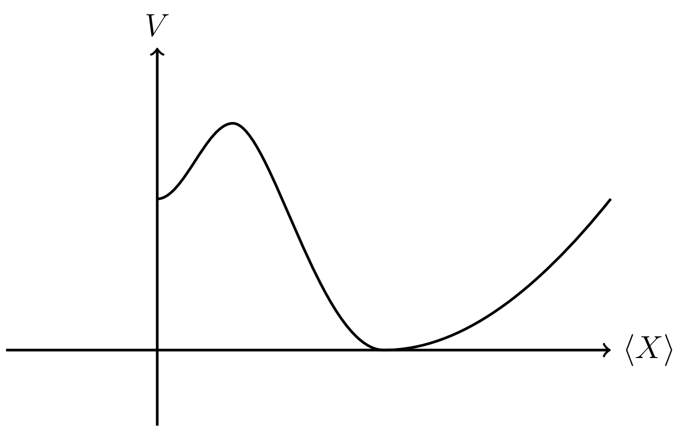
## 15.3 Basics

Let us start by being very up-front. There are three types of DSB... WHAT?






## 15.4 The 3-2 Model

Reviewed in Nelson, Dine, ...

## 15.5 The ITIY Model

## 15.6 Tools for Noncalculable Models

See ADS phenomenology paper: [87] See Hitoshi's paper using vectors: [88]

# 16 Gauge Mediated SUSY Breaking

In this appendix we review features of gauge-mediated supersymmetry breaking (GMSB). The canonical reference for traditional gauge mediation models and their phenomenology is the review by Giudice and Rattazi [89]. Over the decade since the review was written, however, there has been significant model-building progress in this direction. After briefly reminding ourselves of the key features of gauge mediation, we mention relevant pieces of these recent developments. Finally, a very powerful tool for calculating masses in many gauge mediation models, so-called analytic continuation into superspace, is reviewed in Appendix E. The material in this section is loosely based on Patrick Meade's lectures on gauge mediation from TASI 2009[29].

Meade divides the history of gauge mediation into three eras.

1. **1982 – 1993**. Gauge mediation originated with the papers by Dine and Fishler [90] and Nappi and Ovrut [91]. It is now an old and simple idea but built a huge amount of literature from models of dynamical supersymmetry breaking, e.g. the Affleck-Dine-Seiberg (ADS) model. The are, however, not many working examples of such models.

2. **1993 – 2006**. A watershed moment for gauge mediation was the Dine-Nelson-Shirman model that built in the now-standard modular structure for supersymmetry breaking [92]. Combined with experimental hints for supersymmetry in the 1990s, this led to a boom in the number of gauge mediation models.

3. **2006 – present**. The Intriligator-Seiberg-Shih (ISS) model demonstrated new ways to break supersymmetry dynamically with simple, generic models based on metastable vacua [67]. This led to another boom in the number of gauge-mediated and dynamically broken models. In 2008 Meade, Seiberg, and Shih generalized the definition of gauge mediation and described the features that are common to all possible models with this structure [93]. For the purposes of the present paper we will not explore general gauge mediation (GGM) any further.

## 16.1 SUSY Breaking refresher

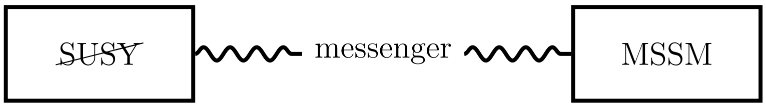
Recall that the **supertrace rule** tells us that

$$\mathrm{STr}M^2 \equiv \sum_j (-)^{2j}(2j+1)m_j^2 = 0, \tag{16.1}$$

where $j$ labels the spin of a particle and $m_j$ is that particle's mass. This means that in a (spontaneously-broken) supersymmetric theory, the sum of the masses of the fermion masses and

[29] Recordings available at `http://www.colorado.edu/physics/Web/tasi09_annc.html`.

the sum of the boson masses, weighted by the number of degrees of freedom for each field, must vanish. In particular, this implies that naïvely supersymmetrizing the Standard Model would imply the existence of new scalars lighter than, e.g. the up and down quarks. This is a tree-level relation and could be modified by loop effects (though such effects are small), but the typical solution is a modular structure in which a hidden sector breaks supersymmetry and mediates such breaking to the unbroken MSSM.




In the SUSY-breaking sector we can parameterize spontaneous supersymmetry breaking by the vacuum expectation value of a higher component of a superfield[30]. For example, we may give a vev to the $F$ term of a chiral superfield, $\langle X \rangle = \theta^2 F$. This SUSY breaking is communicated to the MSSM via messenger fields at some mass scale $M$. At low energies in the visible (MSSM) sector one would observe that supersymmetry is broken at the scale $M$. In this regime one can write down effective operators for the masses of the sfermions and gauginos,

$$m_Q^2 \sim \int d^4\theta \frac{X^\dagger X}{M^2} Q^\dagger Q \sim \frac{F^2}{M^2} \tag{16.2}$$

$$m_\lambda \sim \int d^2\theta \frac{X}{M} \mathbb{W}_a \mathbb{W}^a \sim \frac{F}{M}. \tag{16.3}$$

This gives us a wide range of scales to play with. At the end of the day we want $(F/M) \sim 100\text{GeV}$ for a natural solution to the Hierarchy Problem. The naïve choice is to use $M = M_{\text{Pl}}$ and assume gravity mediation. However, this leads to severe problems with flavor since gravity doesn't respect global symmetries. Further problems can arise due to incalculability in the regime where gravity is strongly coupled. We thus want to find a mediator with a lower scale.

A general feature of SUSY breaking is the appearance of a goldstino field. When we promote 'rigid' SUSY to local supersymmetry, i.e. supergravity, the goldstino is eaten by the gravitino. In most theories we don't have to worry about this since the resulting gravitino mass is

$$m_{3/2} \sim \frac{F}{M_{\text{Pl}}}, \tag{16.4}$$

so that even though this is $\sim 100$ GeV in gravity mediation, it is gravitationally (i.e. very weakly) coupled and would only be relevant for early universe cosmology. As we dial down the messenger mass scale, however, we must correspondingly dial down $F$ to maintain $(F/M) \sim 100\text{GeV}$. Thus for $\sqrt{F} \sim 1000$ TeV, we are already left with

$$m_{3/2} \sim \mathcal{O}(\text{eV}). \tag{16.5}$$

We thus end up with lighter particles and stronger couplings, and we have to consider the decay of our otherwise-LSP to gravitinos ('gravintii').

[30]Recall that vevs for the lowest component of a superfield do not break SUSY.

## 16.2 Set up and features

In the gauge mediation scenario SUSY-breaking is transmitted to the MSSM via gauge fields such that SUSY is restored in the MSSM sector in the limit when the gauge coupling is taken to zero. These are naturally flavor-blind so we avoid many of the tight flavor-constraints in the general MSSM. We ensure calculability by choosing a low messenger scale. Let's go ahead and build a gauge mediation model as a concrete example.

### 16.2.1 Write down a SUSY-breaking sector

Let's start with the original O'Raifeartaigh model.

$$W_{\cancel{\text{SUSY}}} = fX + m\Phi_1\Phi_2 + yX\Phi_1^2. \tag{16.6}$$

We know that at the minimum of the resulting potential there must exist a nonzero $F$ term, which we shall take to be $F = F_X \neq 0$. We further take the limit where $m$ is the largest scale and $\phi_1 = \phi = 2 = 0$. We note that

$$\langle X \rangle = M + \theta^2 F \tag{16.7}$$

is undetermined, i.e. it is a flat direction.

### 16.2.2 Have this sector talk to the MSSM via gauge fields

Now that we're armed with $\langle X \rangle$, we would like to couple this SUSY-breaking field to a messenger sector that is charged under the MSSM. We'll populate our messenger sector with two left-chiral superfields, $\varphi$ and $\widetilde{\varphi}$ which transform as a $\mathbf{5}$ and $\overline{\mathbf{5}}$ of $SU(5)$. These have to be chosen to form a vectorlike representations since this allows them to have large Dirac masses that can become heavy[31]. We introduce the hidden sector superpotential

$$W_{\text{hidden}} = X\varphi\widetilde{\varphi}. \tag{16.8}$$

From Eq. (16.7) we know that this gives a mass of $m_\Psi = M$ to the Dirac spinor $\Psi$ formed from the Weyl spinors in $\varphi$ and $\widetilde{\varphi}$. The scalar potential is

$$V = \langle X \rangle^2 \varphi^\dagger\varphi + \langle X \rangle^2 \widetilde{\varphi}^\dagger\widetilde{\varphi} + \langle X \rangle_{\theta^2}\varphi\widetilde{\varphi} = M^2(\varphi^\dagger\varphi + \widetilde{\varphi}^\dagger\widetilde{\varphi}) + F\varphi\widetilde{\varphi}.$$

This leads to a mass matrix

$$M_\varphi^2 \sim \begin{pmatrix} M^2 & F \\ F^* & M^2 \end{pmatrix},$$

which acts on $(\varphi, \widetilde{\varphi}^\dagger)^T$. The eigenvalues of this matrix are

$$m_{\varphi_\pm}^2 = M^2 \pm F. \tag{16.9}$$

[31]Otherwise, purely chiral fermions would be protected by chiral symmetry and would be unacceptably light.

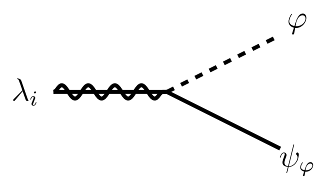
We can see that the breaking of SUSY in the hidden sector from an $F$-term vev is transmitted as a splitting in the masses of the messenger sector scalars. The fields $\varphi$ and $\widetilde{\varphi}$ are charged under the MSSM. We want to ensure that these fields don't obtain a vev that would break color or spoil electroweak symmetry breaking, so we require

$$m^2_{\varphi_i} \geq 0 \Rightarrow F \leq M^2.$$

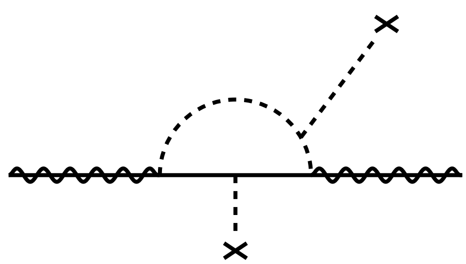
This now completes the information that we need for the messenger sector.

### 16.2.3 Feed this into the MSSM via gauge interactions.

The messenger fields can then transmit SUSY-breaking effects into the visible MSSM sector through couplings to the gauginos and scalars, inducing SUSY-breaking masses in these fields. These are, of course, precisely the Standard Model superpartners whose masses we want to lift. The tree-level gaugino coupling takes the form:




From this we can build loop-level diagrams that contribute to the gaugino mass:



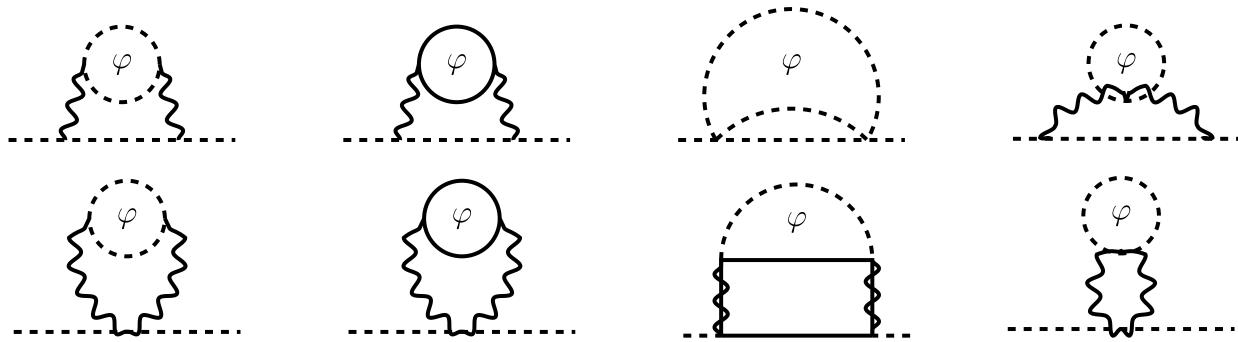
For this diagram explicitly drawn the SUSY-breaking mass insertions. From now on we'll be rather lazy and leave that implicit. To get the right insertions one can check that the Weyl arrows take the right form. One may check that this gives a gaugino mass of

$$m_{\lambda_i} = \frac{\alpha_a}{4\pi} n \frac{F}{M} \left[ \frac{1}{x^2}(1+x)\log(1+x) + (1-x)\log(1-x) \right], \tag{16.10}$$

where $x = F/M^2 \leq 1$ and $n$ is the Dynkin index for the *pair* $\varphi, \varphi$ (for example, $n = 1$ for the $\mathbf{N} + \overline{\mathbf{N}}$ of $SU(N)$). We can now take the convenient limit $x \ll 1$ for our minimal $SU(5)$ model,

$$m_{\lambda_a} = n \frac{\alpha_a}{4\pi} \frac{F}{M}. \tag{16.11}$$

The scalar masses are much more difficult. The scalars have no direct coupling to the messenger fields. These couplings are only induced at one-loop, thus the SUSY-breaking masses given to the scalars only occur at *two*-loop order. If you're reading this document, you're interested in BSM model-building and probably never intend to calculate a two-loop *anything*. Fortunately, the technique of **analytic continuation into superspace** (reviewed in Appendix E) will allow us to calculate these masses to leading order in the SUSY-breaking in a slick and elegant way. Just for the heck of it, here are the diagrams.




The result of those calculations are scalar masses

$$\widetilde{m}^2 = \frac{F^2}{M^2} C_a n \left(\frac{\alpha_i}{4\pi}\right)^2 f(x) \xrightarrow{x \ll 1} 2n \frac{F^2}{M^2} C_a \left(\frac{\alpha_i}{4\pi}\right)^2, \tag{16.12}$$

where $C_a$ are the quadratic Casimirs of the scalar representation[32]. We've written $f$ to mean a complicated function of dilogarithms,

$$f(x) = \frac{1+x}{x^2}\left[\log(1+x) - 2\mathrm{Li}_2\left(\frac{x}{1+x}\right) + \mathrm{Li}_2\left(\frac{2x}{1+x}\right)\right] + (x \to -x).$$

We won't care[33] about this function or the overall prefactor of 2. What *is* relevant for us is that Eq. (16.11) and Eq. (16.12) tell us that our theory has $\widetilde{m} \sim m_\lambda$, even the loop factors match. What this tells us is that the mass splittings come from the factors of $C_a$ and the strength of the gauge coupling.

This gives us a distinct phenomenology where the heaviest superpartners are those charged under the largest $SU(N)$, e.g. colored superpartners are the heaviest, followed by those with $SU(2)_L$ charge, and so forth. The above formulae tell us

$$M_{\lambda_1} : M_{\lambda_2} : M_{\lambda_3} = \alpha_1 : \alpha_2 : \alpha_3 \tag{16.13}$$

and

$$m^2_{\widetilde{q}} : m^2_{\widetilde{\ell}} : m^2_{\widetilde{E}} = \frac{4}{3}\alpha_3^2 : \frac{3}{4}\alpha_2^2 : \frac{3}{5}\alpha_1^2. \tag{16.14}$$

A few general remarks are in order,

[32] $C_1 = 0$ for singlets, $C_2 = 3/4$ for weak doublets, $C_3 = 4/3$ for color triplets.

[33] Those who would like to show off their calculational prowess can follow the calculation in the appendix of Martin's paper on generalized (gauge) messengers [15].

1. These relations are independent of the details of the SUSY-breaking sector and even those of the messenger sector.

2. The gaugino, squark, and slepton masses are all described by the vev of the spurion $X$.

3. Flavor-changing neutral currents are automatically suppressed and CP violation is conserved since each of the mass matrices are proportional to the identity and the $A$ terms are highly suppressed.

4. The $\mu$ term (see below) is protected by symmetries so that further model-building is required.

If one is particularly clever, one would object that we appear to have missed something in our above analysis: one-loop diagrams coming from non-zero (due to SUSY-breaking) hypercharge $D$-terms. This is protected, however, by an accidental, approximate symmetry

$$q \leftrightarrow \overline{q} \qquad\qquad \ell \leftrightarrow \overline{\ell} \qquad\qquad V_Y \leftrightarrow V_Y.$$

This symmetry is broken by the MSSM interactions, but the effects of this breaking only occurs at high-loop order. For details in a more involved model, see Giudice and Dimopoulos [94].

### 16.2.4 Basic phenomenology

Let us now discuss the features of these [ordinary] gauge mediation models. If we have a set of $n$ messengers in $SU(5)$, that is to say $n$ fields $\varphi$ in the $\mathbf{5}$ and $n$ fields $\widetilde{\varphi}$ in the $\overline{\mathbf{5}}$, then in the limit $F \ll M^2$ we have the $SU(3)_c - SU(2)_L - U(1)_Y$ hierarchy

$$M_{\lambda_i} = n\frac{\alpha_i}{4\pi}\left(\frac{F}{M}\right) \tag{16.15}$$

$$m_\phi^2 = 2n\left(\frac{F}{M}\right)^2 \sum_i \left(\frac{\alpha_i}{4\pi}\right) C_i[\phi]. \tag{16.16}$$

Note that as $n \to \infty$, $M_\lambda^2/m_\phi^2 \to \infty$, so the characteristic scale of the gauginos verses the scalars can be very different. *Within* the gauginos and (separately) *within* the scalars, however, the 3-2-1 hierarchy is preserved. The real danger of $n \to \infty$ are the presence of Landau poles in the MSSM sector due to a large contribution to the running of the MSSM gauge couplings. A good rule of thumb for $SU(5)$ models is that $n \lesssim 5$.

For the low SUSY-breaking scales in our gauge mediation models, the gravitino mass $m_{3/2} \sim F/M_{\rm Pl}$ matters since this is generally the lightest particle in the theory. We then have to recognize that the field that we would otherwise call the LSP is actually the NLSP and will eventually decay into the gravitino, $\widetilde{G}$. Phenomenologically we need to figure out how the gravitino couplings. These are predominantly due to the goldstino (which is eaten by the gravitino via the Higgs mechanism) whose couplings come from the conservation of the supercurrent. The goldstino Lagrangian takes the form

$$\mathscr{L} = -\frac{1}{F}J^\mu_Q\partial_\mu\widetilde{G} \tag{16.17}$$

$$= \frac{1}{F}\left[(m_\psi^2 - m_\phi^2)\overline{\psi}\phi + m\lambda^i\overline{\lambda}_i\sigma^{\mu\nu}F^i_{\mu\nu}\right]\widetilde{G} + \cdots . \tag{16.18}$$

Note that the mass terms in the brackets also depend on $F$ so that the expression on the right-hand side is well defined in the $F \to 0$ limit. This gives us two types of NLSP decay modes, depending on the type of NSLP.

$$\chi_i^0 \to \begin{cases} \gamma\widetilde{G} \\ 2\widetilde{G} \\ h\widetilde{G} \end{cases} \qquad \boxed{\Gamma \sim \frac{m_{\chi_0}^5}{F^2}}$$

$$\widetilde{\tau} \longrightarrow \tau\widetilde{G} \qquad \boxed{\Gamma \sim \frac{m_{\widetilde{\tau}}^5}{F^2}}$$

For $F \gtrsim 1000$ TeV one would expect the NLSP to be collider-stable. For $F < 100$ TeV one gets a prompt decay to the gravitino. For intermediate scales one gets a decay inside the detector which may be measurable as a displaced vertex. The take-home phenomenological lesson is that the 'smoking gun' signal for ordinary gauge mediation models are photons plus missing energy[34].

In practice this is enough to go an talk to your favorite experimentalist. It's important to talk to an experimentalist who can tell you about the actual assumptions going into what *they* call gauge mediation since experimental collaborations typically make assumptions about parameters. For example, CDF and D0 assume

$$\begin{aligned} M_{\text{mess}} &= 2\frac{F}{M} \\ n_{\text{mess}} &= 1 \\ \tan\beta &= 15 \\ \mu &> 0. \end{aligned}$$

## 16.3 EWSB and the $\mu$-$B_\mu$ problem

It is worth mentioning the well known $\mu$-$B_\mu$ problem of gauge mediation, first identified by Kim and Nilles before I was born [96]. The $\mu$ parameter is the only SUSY-preserving parameter with dimensions of mass and hence its natural lives at the Planck scale, $M_{\text{Pl}}$, while the $B_\mu$ parameter is a soft SUSY-breaking term, $\mathcal{L}_{\text{soft}} = B_\mu H_u \cdot H_d + \text{h.c.}$. Note that there are a few different notations floating around in the literature

Recall the potential for the neutral scalar Higgs in the MSSM,

$$\begin{aligned} V = \; & \left(\mu^2 + m^2_{H_u^0}\right)|H_\mu|^2 + \left(\mu^2 + m^2_{H_d^0}\right)|H_d|^2 - \left(B_\mu H_u^0 H_d^0 + \text{h.c.}\right) \\ & + \frac{1}{8}\left(g^2 + g'^2\right)\left(\left|H_u^0\right|^2 - \left|H_d^0\right|^2\right)^2 \end{aligned} \tag{16.19}$$

[34]Though this shouldn't be taken *too* seriously since one can cook up non-supersymmetric models of new physics that mimic this signature, e.g. [95]

This can be found in any self-respecting MSSM phenomenology review or textbook. Nepotism leads us to suggest the MSSM review written in the mid '90s by a promising young graduate student [50]. We note in particular that there is no quartic potential in the direction $|H_u^0| = |H_d^0|$.

In order to obtain electroweak symmetry breaking, the origin of the neutral Higgs potential must be destabilized without introducing a run-away direction. In other words, there should be one direction with a negative $(\text{mass})^2$, but we cannot have this both directions or else the lack of a quartic potential in the $|H_u^0| = |H_d^0|$ direction will lead to a run-away. We can ensure this by taking the determinant of the mass matrix in Eq. (16.19) and imposing that it is negative,

$$\begin{vmatrix} |\mu|^2 + m_{H_u}^2 & -B_\mu \\ -B_\mu & |\mu|^2 + m_{H_d}^2 \end{vmatrix} < 0.$$

This imposes

$$B_\mu^2 > \left(|\mu|^2 + m_{H_\mu}^2\right)\left(|\mu|^2 + m_{H_d}^2\right). \tag{16.20}$$

In order to ensure stability, i.e. to avoid the run-away direction, we want to impose that the $(\text{mass})^2$ is positive along $|H_u^0| = |H_d^0|$. This gives the constraint

$$2|\mu|^2 + m_{H_\mu}^2 + m_{H_d}^2 - 2B_\mu > 0. \tag{16.21}$$

These two equations relate supersymmetric $\mu$ term and the soft SUSY-breaking $B_\mu$ term which naïvely have nothing to do with each other. This is a first hint of the $\mu - B_\mu$ problem. One can check explicitly that there is no solution to Eqs. (16.20-16.21) for $m_{H_u}^2 = m_{H_d}^2$. The natural choice is to have $m_{H_u}^2 < 0$ and $m_{H_u}^2 > 0$. This can be seen by looking at the running of the soft-breaking scalar masses, from which we obtain at leading order

$$m_{H_u}^2 = \left(m_{H_u}^2\right)_0 - \frac{6y_t^2}{16\pi 2}\ln\left(\frac{\Lambda^2}{m^2}\right)\left(\widetilde{m}_t^2 - m_t^2\right). \tag{16.22}$$

The up-type Higgs couples to the top (s)quark and so the *negative* renormalization has a large coefficient. A more detailed discussion along with remarks about fine-tuning can be found in Section 11.3 of Dine's textbook [10] or Section 4.5 of Terning's textbook [5]. Let us assign the vevs $\langle H_{u,d}^0\rangle = v_{u,d}/\sqrt{2}$ with the relations $v_u = v\sin\beta$ and $v_d = v\cos\beta$. Minimizing the Higgs potential gives us the famous equations,

$$\sin 2\beta = \frac{2B_\mu}{2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2} \tag{16.23}$$

$$\frac{M_Z}{2} = -\mu^2 + \frac{m_{H_d}^2 - m_{H_u}^2\tan^2\beta}{\tan^2\beta - 1}. \tag{16.24}$$

One can already tell that there's something strange in neighborhood of Eq. (16.24). In order for the terms in this equation to avoid fine-tuning, each term must be roughly of the same order. Thus this equation tells us that *naturally*,

$$M_Z^2 \sim \mu^2 \sim m_{H_{u,d}}^2.$$

Are you unhappy yet? The first term is the physical $Z$ mass which lives at a well-investigated scale, the second term is a supersymmetric term that appears in the superpotential, and the third terms are part of the soft SUSY-breaking Lagrangian. Why should these scales all have to be at roughly the same order? This is a manifestation of the Little Hierarchy problem.

We can play with the $\mu$ and $B_\mu$ parameters to see what we can do. Since we obtained electroweak symmetry breaking radiatively (i.e. from the running of $m^2_{H_0}$), one might hope that we could play the same game and set $\mu = B_\mu = 0$ and generate them radiatively such that they exist at the electroweak scale as EWSB seems to require. One natural symmetry that prohibits both the $\mu$ and $B_\mu$ terms is a Peccei-Quinn-type symmetry [97] that sends

$$H_{u,d} \to e^{i\alpha} H_{u,d}.$$

We then assume that the SUSY-breakign sector breaks this symmetry and we cross our fingers that this produces the necessary values for $\mu$ and $B_\mu$ at the weak scale. It turns out that this works out perfectly in *gravity* meduation and is called the **Giudice-Masiero mechanism** [98]. The $\mu$ term is generated by an effective operator of the form

$$\int d^4\theta \frac{X^\dagger H_u H_d}{M_{\mathrm{Pl}}},$$

where $\langle X \rangle \sim F\theta^2$ and we get an effective $\mu$ term at the scale $\mu \sim F/M_{\mathrm{Pl}}$ which is at the order of the soft SUSY-breaking terms. The $B_\mu$ term is generated from

$$\int d^4\theta \frac{X^\dagger X}{M_{\mathrm{Pl}}} H_u H_d$$

from which we get $B_\mu \sim F^2/M^2_{\mathrm{Pl}} \sim \mu^2$. That's great.

Finally we return to gauge mediation, where this $\mu - B_\mu$ problem is not so easy to solve. Recall that we now have $F \ll 10^{11}$ GeV, from which we get $\mu$ and $B_\mu$ terms that are much too small. At the very least, we know from bounds on the Higgsino mass that $\mu \geq 100$ GeV. If we forbid the tree level $\mu$ and $B_\mu$ terms, then then natural value for a radiatively generated $\mu$ from the hidden sector is

$$\mu \sim \frac{1}{16\pi^2}\frac{F}{M}. \tag{16.25}$$

We will see that this is not a problem to attain. The problem in gauge mediation is to simultaneously obtain a value of $B_\mu$ of the same order of magnitude. If we have a direct coupling to the SUSY-breaking sector $W = \lambda X H_u H_d$, where $\lambda$ generally can encode loop factors, we end up with $\mu = \lambda M$ and $B_\mu = \lambda F$ such that $B_\mu/\mu \sim 10 - 100$ TeV. One generically has the problem that $\mu$ and $B_\mu$ show up at the same order in $\lambda$ (e.g. loop order), which means they cannot both end up at the weak scale[35]. We know from our discussion of EWSB above that a large ratio of $B_\mu$ to $\mu$ destabilizes the electroweak symmetry-breaking vacuum, or at least introduces fine tuning.

Dvali, Giudice, and Pomarol presented a model in 1996 [99] that highlighted both the nature of the $\mu - B_\mu$ problem in gauge mediation and provided a somewhat elaborate strategy to combat

[35] If this statement doesn't make sense then check it with the simple example we just presented.

it. The model introduces additional singlets to cook up a scenario where $\mu$ is generated by a more complicated operator which manifestly cannot simultaneously generate a $B_\mu$ term, which must then be generated at a higher-loop order. Such models additionally require a mechanism to prohibit the operators above that would otherwise generate $\mu$ and $B_\mu$ simultaneously at a lower scale.

A second strategy is based on the next-to-minimally supersymmetric Standard Model (NMSSM) which is reviewed by Maniatis [100]. This involves throwing in a new weak-scale singlet whose vev produces the $\mu$ and $B_\mu$ terms, but requires some extra structure to maintain electroweak symmetry breaking.

A third approach is to use have large renormalization effects suppress $B_\mu$ while leaving $\mu$ relatively unaffected. One such model by Roy and Schmaltz used the dynamics of the SUSY-breaking sector to impose this suppression [101]. The model, however, relies on assumptions about incalculable anomalous dimensions in the hidden sector.

A final approach is to live with the 'natural' $\mu^2 \ll B_\mu$ hierarchy of gauge mediation to see if there is another way out. Csáki, Falkowski, Nomura, and Volansky presented this idea by showing that if $\mu^2 m_{H_u}^2 \ll B_\mu \ll m_{H_d}^2$, then one can still obtain electroweak symmetry breaking, Eqs. (16.20-16.21) [55]. Such a relation can be engineered if the Higgs fields are directly coupled to the SUSY-breaking sector.

## 16.4 Direct, semi-direct, extraordinary, and general

Gauge mediation was born in 1993 with Dine and Nelson's model [92] which is more or less what we've presented above. We refer to our simple SU(5) $\mathbf{5} \oplus \overline{\mathbf{5}}$ messenger model as **minimal gauge mediation** (MGM), or the general class of such models as **ordinary gauge mediation** (OGM). In particular, this is a narrow umbrella where the hidden sector has a single field $X$ obtaining a SUSY-breaking vev (the mechanism is unimportant) which couples to the vector-like (meaning they come as conjugate pairs, $\mathbf{N} \oplus \overline{\mathbf{N}}$) messengers $\phi$, $\widetilde{\phi}$ via Yukawa interactions,

$$W_{\text{OGM}} = \lambda_{ij} X \phi^i \widetilde{\phi}^j. \tag{16.26}$$

This model was born roughly at the same time as the World Wide Web (at CERN), and just as we've seen a remarkable growth in the Internet, gauge mediation model-building has come a long way.

Back in the 90s, along with denim jackets and the TV show *Friends*, the big question was whether one could further simplify the modular structure that Dine and Nelson had established. The messenger sector was valuable to 'insulate' the MSSM from the SUSY-breaking sector. One is able to avoid strict constraints from the supertrace rule and flavor-changing neutral currents. The cost, however, is a rather arbitrary messenger sector. Theorists were thus driven to try to construct more elegant models that did away with the messenger sector by completely by allowing the messenger fields to participate in the SUSY-breaking mechanism, i.e. to incorporate the messenger sector into the SUSY-breaking ('hidden') sector. This is called **direct gauge mediation** (DGM).

Note the historical logic here: the original attempts to build dynamical SUSY breaking models were also 'single sector' but were considered unpalatable since it was so hard to find a realistic model. This was because naïvely building the 'simplest' models invoking only the paradigm of DSB would never have led one to consider what would (again, naïvely) seem like a very arbitrary set up. Dine and Nelson demonstrated a new paradigm where a messenger sector is introduced to insulate the MSSM from the 'dirty laundry' of the DSB sector. People then took this as a lesson and went back to the 'old-style' DSB models but set them up in such a way that there is still an effective separation between the MSSM and the DSB fields.

This brought back problems that were already apparent in the original DSB attempts of the 80s. In particular, having a DSB gauge group which generates the SUSY-breaking scale is effectively a very large flavor group for the Standard Model gauge fields. The running of the Standard Model couplings is then enhanced by this flavor factor and they can become nonperturbative before they unify. This is the so-called **Landau pole problem**.

The first viable direct mediation model was presented by Poppitz and Trivedi [102] based on $SU(N) \times SU(N-2)$ gauge group. The gauge messengers of this model are charged under the Standard Model, which is embedded in an unbroken flavor symmetry of the SUSY-breaking sector. The model has a very large SUSY-breaking scale, $\sim 10^{10}$ GeV, because of the large $N$ require to embed the Standard Model[36]. At such scales the effects of gravity mediation must be taken into account, making this a kind of 'hybrid' model. One then has to do a lot of work to rule out flavor-changing neutral currents.

Shortly after Arkani-Hamed, March-Russel, and Murayama developed an alternative model closer to what we recognize as gauge mediation [103]. Their model utilizes a pseudomodulus $X$ which is lifted by a non-renormaliziable operator in the superpotential. The field can then get a very large lowest-component vev while maintaining a small vacuum energy, i.e. $\langle X \rangle = M + F\theta^2$ with $M^2 \ll F$ which suppresses supergravity contributions. They arranged for the Standard Model-charged fields to get masses on the order of $\langle X \rangle$ so that their contribution to gauge coupling renormalization only appears above the large scale $M$. This avoids the Landau pole problem and saves perturbative unification. However, there was a leftover problem that afflicts both this and the Poppitz-Trivedi model: there are Standard Model-charged fields below $10^5$ GeV whose scalar components get soft-masses on the order of $10^4$ GeV. This contributes to the renormalization of the squark and slepton masses at two-loop order and actually drive them to negative values at low energies.

A third model by Murayama which appeared in short succession was the 'first phenomenologically viable' model of direct mediation and was the gold-standard for direct mediation for about a decade afterward [104]. The light SM-charged fields in this model do not have large soft masses so do not make large negative contributions to the squark and slepton mass renormalizations. Further, the model is completely chiral and one does not have to forbid mass terms for the messenger fields by hand, as one had to in the previous models.

The modern era of gauge mediation (post-ISS) has brought more diverse directions, returning to the modular structure of OGM model (and how this can again teach us about building DGM models). The first ISS-type models based on vacua whose metastability are established near the origin via Seiberg duality were developed before Christmas of 2006. Murayama and Nomura highlighted the role that metastable vacua play in relieving the Nelson-Seiberg $R$-symmetry condition for model-building [105]. Kitano, Ooguri, and Ookouchi presented a direct mediation model with

[36]SUSY breaking occurs due to non-renormalizable operators whose dimension grows with $N$ and which are suppressed by factors of $M_{\text{Pl}}$. This leads to the large SUSY-breaking scale.

string-inspired deformations [106]. Days afterward, the Three Musketeers developed a low-scale direct mediation model based on the ISS framework [107].

With metastability making gauge mediation vogue once again, the IAS-Harvard axis started thinking about jazzing up the framework itself. Seiberg, Volansky, and Wecht developed **semi-direct gauge mediation** (sDGM) in which the messenger field exists in the SUSY-breaking sector but does not itself participate in the breaking of supersymmetry [108]. Cheung, Fitzpatrick, and Shih explored the consequences of generalizing the messenger sector by allowing its superpotential to include all renormalizable couplings to any number of hidden sector singlets $X_k$. They called their framework **(extra)-ordinary gauge mediation** (EOGM) since their results can be understood as a generalization of the ordinary gauge mediation (OGM) formulae. Since one can perform unitary rotations on the $X_k$ fields so that only one field, $X$, obtains an $F$-term vev, the superpotential coupling SUSY-breaking to the messengers is given by

$$W_{\text{EOGM}} = (\lambda_{ij}X + m_{ij})\phi^i\widetilde{\phi}^j, \tag{16.27}$$

where we've written the scalar (lowest-component) vevs of the supersymmetric fields $X_k$ into $m_{ij}$. The resulting formulae can be cast in terms of quantities identified with *effective* number of messengers, by analogy to the OGM formulae Eqs. (16.15-16.16). They classified three types of models within the EOGM framework:

1. $\det m \neq 0$
2. $\det \lambda \neq 0$
3. $\det m = \det \lambda = 0$.

Theories based on generalized O'Raifeartaigh models, including ISS-type models, fall under the first class and will be our primary interest.

Next, Meade, Seiberg, and Shih [93] defined a framework for **general gauge mediation**, i.e. the 'essence' of gauge mediation that is common to all known gauge mediation models (including DGM). They used current correlators to generate sum rules that characterized the phenomenology possible gauge mediation models. Under some of the models within gauge mediation one can actually break the 3-2-1 hierarchy of sparticle masses, leading to very different phenomenology from ordinary gauge mediation. The technique of using current correlators has since been used to develop closed formulae for the soft masses of extraordinary gauge mediation [109] and semi-direct gauge mediation [110].

# 17 The Nelson-Seiberg $R$-symmetry theorem

We would like to review the Nelson-Seiberg $R$-symmetry theorem [75] and provide a sketch of a proof from Argyres' well-written 2001 SUSY notes[37] The theorem formally states:

**Theorem 17.1** (Nelson-Seiberg)**.** *If one has a supersymmetric model so that the effective Lagrangian is generic and the theory calculable at low energies, then (1) the existence of an R-symmetry is a necessary condition for a SUSY-breaking vacuum and (2) a spontaneously broken R-symmetry is a sufficient condition for a SUSY-breaking vacuum.*

[37]These are available at `http://www.physics.uc.edu/~argyres/661/index.html`.

The first condition means that the theory is assumed to not have any special relations between its parameters. We shall use the definition of 'generic' provided in Section 14.3, namely that a system of $n$ equations with $n$ unknowns *generically* has a solution. The second condition of calculability is more precisely phrased by saying that the low-energy theory must be described by a Wess-Zumino model with no gauge fields. Such a theory of only chiral superfields would not suffer from the problems of nonperturbative dynamics that appear in $SU(N)$ gauge theories.

In such a theory the scalar potential is given by the square of the $F$-term, $F_i = \partial_i W$. As we know the minimum of the scalar potential tells us whether or not SUSY is broken. If $\min V = \min |F|^2 = 0$ then SUSY is preserved in the vacuum, otherwise SUSY is broken. If we label our chiral superfields by $i$ such that our low-energy Wess-Zumino model is composed of fields $\Phi_i$, $i = 1, \cdots, n$ then the condition for a SUSY-preserving vacuum is

$$\partial_i W(\Phi_1, \cdots, \Phi_n) = 0 \quad \forall\, i. \tag{17.1}$$

This is a system of $n$ complex analytic equations for $n$ complex unknowns[38]. Thus the system generically has a solution and hence the theory has a supersymmetric vacuum. Boring. What else can we do? The only tool that is really at our disposal is to play with global symmetries[39]. We can argue that Eq. (17.1) didn't take into account the global symmetries that our theory might have. Under such a global symmetry the superfields each have some charge, $Q[\Phi_i] = q_i$. Typically the superpotential must be invariant under this symmetry, imposing a further constraint on the theory and naïvely giving us hope that perhaps we can get generic SUSY-breaking. Suppose for simplicity that the symmetry is a $U(1)$ and assume without loss of generality that the charge $q_1 \neq 0$ (at least one such field must be charged in order for the symmetry to be nontrivial).

If the $U(1)$ is preserved, then the vacuum is given by the state where all of the charged fields must have vanishing vevs,

$$\langle \Phi_i \rangle = 0 \qquad \text{if } q_i \neq 0. \tag{17.2}$$

If the first $k$ fields $\Phi_i, \cdots \Phi_k$ have nonzero charges $q$ and the rest have vanishing charge, then this imposes $k$ constraints. Restricted to the remaining subspace of unknown field vevs, superpotential is still gives $(n-k)$ generically independent equations for $(n-k)$ unknowns. Thus the case for a preserved global symmetry does not work.

We can consider what happens when the global symmetry is broken spontaneously, in which case some of the charged fields are allowed to have nonzero vev. The superpotential as a term in the Lagrangian, must still be neutral. We may incorporate this constraint by writing our superpotential as a function of only $n-1$ superfields,

$$W(\Phi_1, \cdots, \Phi_n) \equiv w(\Phi_2 \Phi_1^{-q_n/q_1}, \cdots, \Phi_n \Phi_1^{-q_2/q_1}), \tag{17.3}$$

where all we have done is absorbed the $\Phi_1$ dependence into the condition that the superpotential can be expressed in terms of variables that are uncharged under the global symmetry. Now,

[38]Note that we have made use of the standard, but sometimes confusing, notation where we write the vev of a field using the same notation as the field itself, i.e. $\Phi = \langle \Phi \rangle$ when it is clear from context that we are discussing the vev. This saves a lot of clutter in the notation, but the reader must be a little more careful.

[39]Even this is arguable under the banner of genericness, but the point is that we will be interested in $R$-symmetries which *are* generic features in SUSY models.

however, we've just written everything in terms of a system of $(n-1)$ equations with $(n-1)$ unknowns. SUSY vacua are still generic.

At this point it may look like we've exhausted our options, but there is a way out. We assumed that the superpotential had to be neutral under this symmetry since it is part of the Lagrangian,

$$\mathscr{L} = \cdots + \int d^2\theta \, W(\Phi_i).$$

We note that if the superspace coordinate $\theta$ were charged under the symmetry, then $W$ must also be charged. This is precisely what occurs in the $R$-symmetry which is present in SUSY theories, the superpotential has $R$-charge $R[W] = 2$. Thus, for the case of an $R$-symmetry, Eq. (17.3) must be modified to

$$W(\Phi_1, \cdots, \Phi_n) \equiv \Phi_1^{2/r_1} w(\Phi_2 \Phi_1^{-r_2/r_1}, \cdots, \Phi_n \Phi_1^{-r_n/q_1}), \tag{17.4}$$

where we've written $r_i$ as the $R$-charge of the lowest-component field in $\Phi_i$. The overall factor of $\Phi_1^{2/r_1}$ must be included to maintain $R[W] = 2$. Now we can see that Eq. (17.1) implies

$$\frac{2}{r_1}\Phi_1^{2/r_1 - 1} w(\Phi_2 \Phi_1^{-r_2/r_1}, \cdots, \Phi_n \Phi_1^{-r_n/q_1}) = 0 \tag{17.5}$$

$$\partial_{i\neq 1} w(\Phi_2 \Phi_1^{-r_2/r_1}, \cdots, \Phi_n \Phi_1^{-r_n/q_1}) = 0. \tag{17.6}$$

The second equation is just the usual system of $(n-1)$ equations for $(n-1)$ unknowns, but the first equation is an *additional* constraint imposing $w(\cdots) = 0$. This gives us a total of $n$ equations for $(n-1)$ unknowns and thus the system is overconstrained and generically does not have a system. We thus conclude that supersymmetry must be broken. This concludes the simple proof of the Nelson-Seiberg theorem.

# 18 The Komargodski-Shih Gaugino Mass Theorem

In Komargodski and Shih's 2009 paper, 'Notes on SUSY and $R$-Symmetry Breaking in Wess-Zumino Models,' they collect a series of theorems[40] about 'general O'Raifeartaigh' models that can appear as the low-energy effective theories of dynamical SUSY breaking models [84]. They introduces a new technique for 'generic' model building by introducing "tree-level $R$-symmetry breaking" and, most importantly for our present purposes, they elucidate the nature of the anomalously light gauginos that appear in these models. This latter result is what we shall refer to as the Komargodski-Shih theorem. In this section we shall review their derivation, following the structure of their paper very closely.

## 18.1 Basics of SUSY breaking in Wess Zumino Models

We start with a general, weakly-coupled, renormalizable Wess-Zumino (WZ) model with a canonical superpotential,

$$W = f_i \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{6} \lambda_{ijk} \Phi_i \Phi_j \Phi_k, \tag{18.1}$$

[40] For proper attribution, some of their results were first discussed by S. Ray [111].

that breaks supersymmetry through tree-level $F$-term vevs. We shall call such models **generalized O'Raifeartaigh models**. The conditions for a SUSY-breaking minimum are then[41]:

1. There exists some $i$ such that $W_i \neq 0$. This just says that there is a non-vanishing $F$ term that causes the vacuum energy not to vanish, $\langle V \rangle > 0$. Note that the fields $\Phi_j$ which preserve SUSY still have $W_j = 0$.
$$W_i \begin{cases} = 0 & \text{if } \phi_i \text{ preserves SUSY} \\ \neq 0 & \text{if } \phi_i \text{ breaks SUSY} \end{cases} \tag{18.2}$$

2. The fields take their values at the minimum of the potential $V$. In other words,
$$W_{ij} W_j^* = 0. \tag{18.3}$$
Recall that the fermion mass matrix $(\mathcal{M}_F)_{ij} = W_{ij}$ at tree level, so this is just the familiar goldstino theorem that the spontaneous breaking of supersymmetry leads to a massless Goldstone fermion. (Recall that the only fields with $W_j \neq 0$ are those that participate in SUSY breaking.)

3. The boson (mass)$^2$ matrix $\mathcal{M}_{\mathrm{B}}^2$ must be positive definite, i.e. the vacuum is free of tachyons.

Let us recall that form of the boson (mass)$^2$ matrix is
$$\mathcal{M}_b^2 = \begin{pmatrix} \mathcal{M}_F^* \mathcal{M}_F & \mathcal{F}^* \\ \mathcal{F} & \mathcal{M}_F \mathcal{M}_F^* \end{pmatrix} \tag{18.4}$$
where $\mathcal{F}_{ij} = W_k^* W_{ijk}$. $\mathcal{M}_B^2$ is manifestly a positive semi-definite Hermitian matrix. For such a matrix we may always write $\mathcal{M}_B^2 = A^\dagger A$ for some $A$. This is obvious if we write
$$e_i^\dagger \left(\mathcal{M}_B^2\right)_{ij} e_j = \hat{e}^\dagger U^\dagger (\mathcal{M}_{\mathrm{Diag}}^2) U \hat{e}_j. \tag{18.5}$$
From this we arrive at a handy lemma,

**Lemma 18.1.** *In any SUSY-breaking vacuum of a generalized O'Raifeartaigh model, if there exists a massless fermion at tree-level, then its scalar superpartner must also be massless at tree-level.*

*Proof.* From the above observation, we see that
$$w^\dagger \mathcal{M}_B^2 w = 0 \Leftrightarrow \mathcal{M}_B w = 0. \tag{18.6}$$
Now suppose that $\mathcal{M}_F$ has a zero eigenvector, $v$. This is, of course, a vector in field space. We shall construct the bosonic vector $\begin{pmatrix} v & v^* \end{pmatrix}^T$. Then we observe that
$$\underbrace{\begin{pmatrix} v^* & v \end{pmatrix}}_{w^\dagger} \underbrace{\begin{pmatrix} \mathcal{M}_F^* \mathcal{M}_F & \mathcal{F}^* \\ \mathcal{F} & \mathcal{M}_F \mathcal{M}_F^* \end{pmatrix}}_{\mathcal{M}_B^2} \underbrace{\begin{pmatrix} v \\ v^* \end{pmatrix}}_{w} = v^T \mathcal{F} v + c.c. \tag{18.7}$$
Since $\mathcal{M}_B^2$ is positive semi-definite, this expression *must* vanish otherwise one may perform a phase rotation on $v$ to make the right-hand side negative and hence inconsistent. Thus the scalar is also massless. $\square$

[41] see Appendix A for our conventions if you are confused about expressions like $W_ij$.

Note that even though we define our generalized O'Raifeartaigh model to be renormalizable, the proof of this lemma never depended on this property and it turns out to actually hold for any general polynomial superpotential regardless of renormalizability. From this we can also write down two corollaries,

**Corollary 18.2.** *If $\mathcal{M}_F v = 0$, then $\mathcal{F}v = 0$.*

In other words, $\mathcal{M}_F$ and $\mathcal{F}$ have the same null eigenvector.

**Corollary 18.3.** *For a SUSY-breaking minimum,*

$$W_{ijk}W_i^*W_j^* = 0 \tag{18.8}$$

*Proof.* For a SUSY-breaking vacuum, we have Eq. (18.3), which can be written as $(\mathcal{M}_F)_{ij}W_j^* = 0$. To be precise, one can rotate the fields such that the SUSY-breaking linear combination is labelled $\hat{j}$ and Eq. (18.3) can be written as $(\mathcal{M}_F)_{i\hat{j}}W_{\hat{j}}^* = 0$ where there is no sum over $\hat{j}$. We thus have a massless fermion associated with the $W_{\hat{j}}^*$ direction. Applying the lemma above we then have

$$\mathcal{F}_{i\hat{j}}W_{\hat{j}}^* = - \Rightarrow W_{ij\hat{j}}W_i^*W_{\hat{j}}^* = 0. \tag{18.9}$$

Rotating back to the original field direction we get precisely Corollary 18.3. □

It turns out that not only are the scalar partners of the golstino massless, but it can be extended to an entire **pseudomodulus**, i.e. a tree-level flat direction emanating from a SUSY breaking minimum which obtains a potential from quantum corrections.

**Theorem 18.4.** *The direction $\phi_i = \phi_i^{(0)} + zW_i^*$ leaves the tree-level potential $V[\phi_i]$ unchanged for any $z \in \mathbb{C}$, in other words, it is a pseudomodulus.*

*Proof.* An earlier proof was provided by Ray in [111], but the notation is rather cumbersome so we'll follow the derivation by Komargodski and Shih. Under this field transformation,

$$\begin{aligned}\delta W_i &= \partial_j W_i \cdot \delta W_j + \frac{1}{2}\partial_k\partial_j W_i \cdot \delta W_k \delta W_j \\ &= W_{ij}(zW_j^*) + \frac{1}{2}W_{ijk}(zW_j^*)(zW_k^*).\end{aligned}$$

There are no other terms since $W$ is renormalizable, i.e. $W_{ijk\ell} = 0$. We know from above that

$$W_{ij}W_j^* = 0 \tag{18.3}$$
$$W_{ijk}W_i^*W_j^* = 0. \tag{18.8}$$

From this we deduce that $\delta W_i = 0$ and hence $W_i$ is constant along this direction. This proof is sufficient for our purposes, though Komargodski and Shih have a more general version of their theorem in their appendix [84]. □

Now that we've proved the existence of a pseudomoduli space, we would next like to show that we may perform a rotation on our fields such that we may write the superpotential in what Komargodski and Shih refer to as the **canonical form**,

$$W = X(f + \frac{1}{2}\lambda_{ab}\varphi_a\varphi_b) + \frac{1}{2}m_{ab}\varphi_a\varphi_b + \frac{1}{6}\lambda_{abc}\varphi_a\varphi_b\varphi_c. \tag{18.10}$$

In this basis the SUSY-preserving fields $\varphi$ have zero vev, $\langle\varphi_a\rangle = 0$ while the SUSY-breaking pseudomodulus field $X \sim W_i^*$ can be arbitrary.

*Proof.* We rotate our fields according to

$$\phi_i = U_{ix}X + U_{ia}\varphi_a$$

such that

$$W = f_i U_{ix} X + f_i U_{ia}\varphi_a + \cdots$$

Thus we may identify $f' = f_i U_{ix}$ and $f'_a = f_i U_{ia}$. Similarly we may define $m' = U_{ix}U_{jx}m_{ij}$ and so forth. Now expanding the $\phi$s about their vevs $\phi_a \to \langle\phi_a\rangle + \phi_a$ and reabsorb factors of $\langle\phi_a\rangle$ into the coefficients, e.g.

$$\frac{1}{3}\lambda'_{abc}\langle\phi_c\rangle \equiv \lambda_{ab}.$$

The factors of 1/2 and 1/6 are part of the definition of the new parameters and take care of permutations of the $\phi$ fields. We now only have to appeal to the equations above to explain the form of Eq. (18.10). First of all Eq. (18.2) tells us that $W_a = 0$ and so there are no terms in $W$ linear in $\varphi$. Next Eq. (18.3) tells us that $W_{xx}X = W_{aa} = 0$ and so $W$ cannot have any $XX$ or $\varphi X$ terms. Finally, Eq. (18.8) tells us $W_{axx}XX = W_{xxx}XX = 0$ so that $W$ cannot have any $XXX$ or $\varphi XX$ terms. This gives us the canonical form above. □

More generally, we will use what Komargodski and Shih refer to as the **generic form** of the generalized O'Raifeartaigh superpotential,

$$W = X_i f_i(\varphi_a) + g(\varphi_a). \tag{18.11}$$

Note the dependence on the genericness assumption: one could easily construct an O'Raifeartaigh theory that does not take this form, for example one can take a superpotential in the canonical form and do a rotation of the fields. Such a superpotential, however, would not be generic in that the couplings would not be independent since they would be related to the couplings of the original via the unitary transformation and hence would not be generic. For future reference, the original ISS model is based on the case $g = 0$.

## 18.2 Tree-level SUSY and $R$-symmetry breaking

Komargodski and Shih now shift gears a little and introduce the idea of SUSY and $R$-symmetry breaking 'at tree-level.' The main idea was to identify a set of models where one doesn't have to calculate the Coleman-Weinberg effective potential to check vacuum stability of the pseudomoduli with the hope that this would be a particularly nice place to do realistic model-building. The main result that we shall take from this, however, will be to identify an incompatibility of the assumption of vacuum stability with gaugino masses.

**Definition 18.5.** *A model **breaks supersymmetry at tree level** if*

1. *The pseudomoduli space is locally stable everywhere.*
2. *The Coleman-Weinberg potential on the pseudomoduli rises at infinity in every direction.*

In other words, tree-level SUSY-breaking models are those where we don't have to worry about checking the stability of states along the pseudomoduli. We gan go on to define $R$-symmetry breaking 'at tree level.'

**Definition 18.6.** *Further, a model **breaks $R$-symmetry at tree level** if, in addition to the above conditions,*

3. *The pseudomoduli space breaks R-symmetry everywhere.*

Thus for such models we would not have to calculate the details of the Coleman-Weinberg potential to be guaranteed that SUSY and $R$-symmetry are broken in the vacuum. The second condition requires some knowledge of the full potential, but only at large fields[42].

We now observe from the generic form of the generalized O'Raifeartaigh superpotential that if $g(\varphi) = 0$, then the model cannot break $R$-symmetry at tree level.

*Proof.* If $g = 0$ then the theory has an $R$ symmetry with $R[X_i] = 2$ and $R[\varphi_a] = 0$. Since we've written our variables such that only the $X_i$ fields have non-zero $F$-terms, $W_a = 0$. We note, however, that this means

$$\frac{\partial W}{\partial \varphi_a} = X_i \partial_a f_i(\phi)(\varphi) = 0,$$

in other words, $X_i$ must be a null eigenvector of $M_{ai} \equiv \partial_a f_i$. Rescaling $X_i$ then leaves the vacuum energy unchanged, as one can see explicitly from the form of the potential obtained form the generic form of the generalized O'Raifeartaigh superpotential,

$$V = \sum_i |f_i(\varphi)|^2 + \sum_a |X_i \partial_a f_i(\varphi) + \partial_a g(\varphi)|^2 .$$

This freedom to rescale $X_i$ tells us that the origin $\{X_i = 0\}$ is a connected element of any pseudomodulus. Since we've shown that the $X_i$ are the only $R$-charged fields, there is then always a point on any pseudomodulus where $R$-symmetry is unbroken (the origin). Hence, by the definition of "broken $R$-symmetry at tree level," we see that for $g = 0$, $R$-symmetry cannot be broken at tree level. □

[42]In this limit it can generally be computed using the techniques developed by Intriligator, Shih, and Sudano [112]. For the case of a single pseudomodulus, $X$, it one show that the potential rises like $\log X$ times the [positive] anomalous dimension of $X$ [113].

That is the main result that we'd like to use to start discussing gaugino masses. Before proceeding let us first make a brief aside since part of the purpose of this document is to collect a set of tools for metastable model building. Even for a $g = 0$ generalized O'Raifeartaigh model, we may *engineer* it to have tree-level SUSY and $R$-symmetry breaking. The general idea is to add new fields $\widetilde{\varphi}$ and a $g(\varphi, \widetilde{\varphi})$ term to the superpotential that set the $R$-charges to 'exotic' values. For simplicity, let us assume that the model in question respects an additional $U(1)$ symmetry in addition to $R$-symmetry that is spontaneously broken in the vacuum via $\langle\varphi\rangle \neq 0$. To construct a 'tree-level $R$-breaking' model, we may add 'by hand' additional fields $\widetilde{\varphi}$ and an additional superpotential term $g(\varphi, \widetilde{\varphi})$ such that both the $U(1)$ and $U(1)_R$ are broken explicitly while maintaining a nontrivial combination $U(1)'_R \subset U(1)_R \times U(1)$. As long as the $F$-terms associated with the new $\widetilde{\varphi}$ fields can be all be set to zero, this doesn't spoil our tree-level SUSY breaking. The $\langle\varphi\rangle \neq 0$ vevs then breaks $R$-symmetry 'at tree level.' This is illustrated schematically in Figure 2. Komargodski and Shih give an explicit example of such a construction in their paper [84]

$$\begin{tikzcd}
U(1)_R \times U(1) \arrow[r, "g(\varphi,\widetilde{\varphi})"] \arrow[d, "\langle\varphi\rangle"'] & U(1)'_R \arrow[d, "\langle\varphi\rangle"] \\
U(1)_R \arrow[r, "g(\varphi,\widetilde{\varphi})"] & \mathbb{1}
\end{tikzcd}$$

Figure 2: Schematic representation of how to build a $R$-symmetry breaking model out of an $R$-symmetry preserving $g = 0$ model.

## 18.3 Application to Gaugino Masses and Model Building

At the risk of alienating the reader with re-writing the generalized O'Raifeartaigh superpotential once again, let us write its canonical form Eq. (18.10) with relabeled variables that will make things easier in the future:

$$W = fX + \frac{1}{2}(\widetilde{\lambda}_{ab}X + \widetilde{m}_{ab})\varphi_a\varphi_b + +\frac{1}{6}g_{abc}\varphi_a\varphi_b\varphi_c. \tag{18.12}$$

This is nothing other than the relabelling $\lambda_{ab} \to \widetilde{\lambda}_{ab}$, $m_{ab} \to \widetilde{m}_{ab}$, and $\lambda_{abc} \to g_a bc$. We will rotate the matrices $\widetilde{\lambda}$ and $\widetilde{m}$ such that they can be written as

$$\widetilde{\lambda} = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \qquad\qquad \widetilde{m} = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}$$

so that the 'reduced determinant' $\det(\lambda X + m) \neq 0$ for generic $X$. This brings us to our very important result,

**Theorem 18.7** (Komargodski-Shih)**.** *For models with tree-level SUSY and $R$-symmetry breaking, the reduced determinant is a constant function of $X$,*

$$\det(\lambda X + m) = \det m. \tag{18.13}$$

*In particular, if this expression is to be violated, then there must exist tachyonic directions at values of $X$ where the reduced determinant vanishes.*

The value of this theorem is that the $X$-derivative expression on the left-hand side of Eq. (18.13) is precisely what appears in the expression for the gaugino mass in theories of gauge mediation, as we will show below.

*Proof.* Suppose Eq. (18.13) does not hold. Then we may write the right-hand side as a polynomial in $X$,

$$\det(\lambda X + m) = \sum_i c_i(\lambda, m) X^i. \tag{18.14}$$

Thus there exist values $X = X_0 \in \mathbb{C}$ where $\det(\lambda X_0 + m) = 0$. This means that there exists a direction in field space $v$ such that

$$(\lambda X_0 + m)v = 0. \tag{18.15}$$

This $v$ is a massless fermion direction. From Lemma 18.1, however, we know that this either implies the existence of the massless boson in the same direction or else, according to the proof of that lemma, there must be a tachyonic direction. This massless boson direction tells us that $\mathcal{F}_{ij}v_j = 0$, using the notation from Eq. (18.4) so that (e.g. see Corollary 18.3)

$$0 = W_k^* W_{ijk} v_j = W_x^* W_{ab} v_b = X_0 \lambda_{ab} v_b,$$

where we've used $W_a^* = 0$ from Eq. (18.2). Combined with Eq. (18.15), this tells us that $\lambda v = 0$ and hence $mv = 0$. This contradicts the assumption[43] that $\det(\lambda X + m) \neq 0$. Hence either $\det(\lambda X + m)$ cannot have zeroes at finite points in field space, i.e. it must be a constant function, *or* there must be a tachyonic direction at $X = X_0$. $\square$

This theorem has an immediate and important consequence in models of gauge mediation where the hidden sector is described by a generalized O'Raifeartaigh model. In such models a subset of the $\varphi_a$ fields are charged under the Standard Model gauge group and communicate the SUSY breaking from the $X$ field to the MSSM. Due to gauge invariance, the mass matrices for these messengers must factorize at quadratic order and so one can apply the Komargodski-Shih theorem to these fields independent of the rest of the hidden sector. This results in

$$\det(\lambda X + m)|_{\text{mess.}} = \text{constant}.$$

Using the techniques of analytic continuation into superspace reviewed in Appendix E (or traditional two-loop calculations), one knows that the leading order (in SUSY breaking) gaugino masses are given by

$$m_\lambda \sim F^\dagger \frac{\partial}{\partial X} \log \det(\lambda X + m)|_{\text{mess.}} = 0. \tag{18.16}$$

This leads the Komargodski and Shih to proclaim that, "This simple result shows that it is impossible to build viable theories of gauge mediation with tree-level SUSY breaking, unless one is prepared to accept an exacerbated little hierarchy problem and the attendant fine tuning coming from very heavy sfermions."

[43] i.e. the entire construction where we defined $\widetilde{\lambda}$ and $\lambda$ to ensure that $(X\lambda + m)$ is nondegenerate.

So the point is this: gaugino masses in gauge mediation are zero at leading order (in the SUSY-breaking parameter) unless the pseudomoduli space is *not* locally stable everywhere. In order to construct a realistic gauge mediation model, one requires a tachyonic direction *somewhere* on the pseudomoduli space. Of course, this 'somewhere' should be away from the vacuum that we populate, and this will be the game played by of most of this paper.

As a sanity check, we can ask ourselves about the models of gauge mediation that have been around for 15 years. In **minimal gauge mediation** (MGM) (see, e.g. Dine and Nelson [92] and the follow up paper with Shirman [77]), the superpotential takes the form

$$W \supset \lambda X \widetilde{\varphi}\widetilde{\varphi},$$

which is tachyonic at $X = 0$. A more recent manifestation, the **direct gauge mediation** (DGM) models recently studied in the **extraordinary gauge mediation** (EOGM) scenario by Cheung, Fitzpatrick, and Shih [114] also have tachyons at $X = 0$ required for $m_\lambda \neq 0$. This result is broadly applicable for dynamical SUSY breaking with gauge mediation since such models are often described by renormalizable Wess-Zumino models.

One can also wonder about models whose hidden sectors are not described by generalized O'Raifeartaigh models at low energies. Such models can be strongly coupled or have non-renormalizable Kähler- and superpotentials, and tend not to be calculable. Notable exceptions such as Seiberg, Volansky, and Wecht's **semi-direct gauge mediation** (sDGM) model [108] also still have gaugino masses vanishing at leading order. This leads Komargodski and Shih toopenly wonder if there is a way to generalize this result to non-canonical Kähler potentials, noting that a hint may be that the leading-order contribution to gaugino mass is a superpotential term in the effective action.

# 19 The Landau Pole Problem

See ADS phenomenology paper: [87]

Good summary in intro of: 0809.4437

# 20 ISS: Metastable SUSY Breaking

We now review the Intriligator, Seiberg, Shih model of metastable vacua [67]. The original paper is very readable, though it assumes a core set of background material which is not yet found in most textbooks. Fortunately, two of the authors provide excellent reviews in their lecture notes on Seiberg duality [2] and supersymmetry breaking [1]. Their SUSY breaking review should be taken as supplementary pedagogical notes that were tailor-made for understanding the ISS model, though the topics are presented in such a way that their significance is only illuminated in hindsight. Additional thoughts can be found in Dine's Cargese lectures [7] or Shirman's TASI lectures [9].

## 20.1 Summary in words

The construction of the ISS model proceeds in three steps.

1. Construct a theory of chiral superfields were supersymmetry is broken at tree-level. We will use a theory of matrix fields where supersymmetry is broken by the **rank condition**. We will call this the macroscopic model I.

2. Promote this model to one with gauge superfields by gauging a global symmetry. This generates new supersymmetric vacua, but we will do this in such a way that the vacuum of the previous theory is preserved as a metastable vacuum. We shall call this the macroscopic model II.

3. We then use Seiberg duality to realize this metastable supersymmetry breaking dynamically. This will give us our ISS model.

We will follow the structure of the original paper, including detours to check the consistency of what we are doing. We will ignore the generalization to SO($N$) and Sp($N$) groups.

## 20.2 Macroscopic Model I

We shall start by considering a theory where supersymmetry is broken by the rank condition. This is reviewed pedagogically in Section 2.7 of [1]. The theory will have a global symmetry,

$$\mathrm{SU}(N) \times \mathrm{SU}(F) \times \mathrm{SU}(F) \times \mathrm{U}(1)_B \times \mathrm{U}(1)' \times \mathrm{U}(1)_R,$$

where these are a soon-to-be gauge symmetry, flavor symmetries (for quarks and antiquarks), a baryon number charge, a U(1) that will be broken by the superpotential, and an $R$-charge. We will be particularly interested in the case $F > N$. The fields and their representation under the global symmetries of the theory are given by

| | SU($N$) | SU($F$)$_\mathrm{L}$ | SU($F$)$_\mathrm{R}$ | U(1)$_B$ | U(1)$'$ | U(1)$_R$ |
|---|---|---|---|---|---|---|
| $\Phi$ | 1 | $\square$ | $\overline{\square}$ | 0 | -2 | 2 |
| $\varphi$ | $\square$ | $\overline{\square}$ | 1 | 1 | 1 | 0 |
| $\widetilde{\varphi}$ | $\overline{\square}$ | 1 | $\square$ | -1 | 1 | 0 |

The Kähler potential is taken to be canonical for these matrix-valued fields,

$$K = \mathrm{Tr}\ \varphi^\dagger\varphi + \mathrm{Tr}\widetilde{\varphi}^\dagger\widetilde{\varphi} + \mathrm{Tr}\ \Phi^\dagger\Phi. \tag{20.1}$$

The most general superpotential compatible with the above symmetries is

$$W_0 = h\mathrm{Tr}\ \varphi\Phi\widetilde{\varphi}, \tag{20.2}$$

where $h$ is some dimensionless coupling constant. We will also add an additional term to this superpotential that explicitly breaks SU($F$)$_\mathrm{L}\times$ SU($F$)$_\mathrm{R}\times$ U(1)$'$ $\to$ SU($F$),

$$\Delta W = -h\mu^2\mathrm{Tr}\ \Phi, \tag{20.3}$$

where $\mu$ is a parameter with dimensions of mass and our resulting superpotential is $W = W_0 + \Delta W$. Our theory's global symmetry is now broken to

$$\mathrm{SU}(N) \times \mathrm{SU}(F) \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R, \tag{20.4}$$

where we recall that because the $\mathrm{SU}(F)_{\mathrm{L}} \times$ $\mathrm{SU}(F)_{\mathrm{R}} \times$ $\mathrm{U}(1)'$ symmetry is broken *explicitly*, there are no Goldstone bosons associated with it.

We can now check that supersymmetry is broken. The main idea is this: because $F > N$, the $F$-terms cannot all be set to zero by the rank of the relevant matrices. Consider, in particular, the $F_\Phi$ term,

$$-F_\Phi^\dagger = h\varphi\widetilde{\varphi} - h\mu^2 \mathbb{1}_F, \tag{20.5}$$

where this is understood to be an $F \times F$ matrix relation. If one is uncomfortable with this, it's easy to write out particular components of the $F$-term by taking derivatives of $W$ with respect to particular elements of the matrix fields. The first term, $h\varphi\widetilde{\varphi}$ is an object of rank $N$ while $h\mu\mathbb{1}_F$ is manifestly an object of rank $F$. Since $F > N$, these two terms cannot sum to zero and so the scalar potential is manifestly greater than zero,

$$V_{\min} = (F - N)|h^2\mu^4. \tag{20.6}$$

We call this supersymmetry breaking by the **rank condition**.

We can use the global symmetries to parameterize our classical moduli space by

$$\Phi = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_0 \end{pmatrix} \qquad\qquad \varphi = \begin{pmatrix} \varphi_0 \\ 0 \end{pmatrix} \qquad\qquad \widetilde{\varphi}^T = \begin{pmatrix} \widetilde{\varphi}_0 \\ 0 \end{pmatrix}, \tag{20.7}$$

where these are understood to be up to $\mathrm{SU}(N)\times\mathrm{SU}(F)\times\mathrm{SU}(F)$ rotations. Note that we've written the upper (left) blocks of these matrices to be $N \times N$, so that $\Phi_0$ is $(F - N) \times (F - N)$ while $\varphi_0$ and $\varphi_0$ are $N \times N$. We can now choose the vacuum that preserves as much of the global symmetry Eq. (20.4) as possible,

$$\Phi_0 = 0 \qquad\qquad \varphi_0 = \widetilde{\varphi}_0 = \mu\mathbb{1}_N. \tag{20.8}$$

This gives us a spontaneous breaking

$$\mathrm{SU}(N) \times \mathrm{SU}(F)^2 \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R \to \mathrm{SU}(N)_D \times \mathrm{SU}(F - N) \times \mathrm{U}(1)_{B'} \times \mathrm{U}(1)_R. \tag{20.9}$$

The next thing that we'd like to do is to determine the Coleman-Weinberg effective potential, which we introduce in some detail in Appendix B. The main question we want to answer is whether or not our SUSY-breaking vacuum is stable on the moduli space under quantum corrections. In order to do this, we know that we need the mass spectrum of of the fields. To figure this out, we expand about the vacuum Eq. (20.8):

$$\Phi = \begin{pmatrix} \delta Y & \delta Z^T \\ \delta\widetilde{Z} & \delta\hat{\Phi} \end{pmatrix} \qquad \varphi = \begin{pmatrix} \mu + \frac{1}{\sqrt{2}}(\delta\chi_+ + \delta\chi_-) \\ \frac{1}{\sqrt{2}}(\delta\rho_+ + \delta\rho_-) \end{pmatrix} \qquad \varphi = \begin{pmatrix} \mu + \frac{1}{\sqrt{2}}(\delta\chi_+ - \delta\chi_-) \\ \frac{1}{\sqrt{2}}(\delta\rho_+ - \delta\rho_-) \end{pmatrix}, \tag{20.10}$$

where the division into $N$ and $(F - N)$ blocks are as before. Our choice of parameterization will simplify (though not by much) some of the expressions for the mass eigenstate fields. We label the 'dynamical' fields with a $\delta$ prefix, which is meant to distinguish the field from the background value; i.e the $\delta\chi_\pm$ fields are perturbations about the $\phi_0 = \widetilde{\phi}_0 = \mu$ background value. Follow-up papers have dropped this cumbersome notation, but for the sake of bop-you-over-the-head clarity, we'll follow the original ISS conventions here.

Before working out some details about the spectrum, let's stop to discuss what we expect. Most fields should get tree-level masses $\sim |h\mu|$, since this is the only mass term in the superpotential. We also expect to find some tree-level massless scalars which come in two flavors: (1) the Goldstone bosons associated with the breaking in Eq. (20.9) and (2) the fluctuations about the pseudomoduli space. The Goldstones are protected against quantum mass terms, but the pseudomoduli generically get mass terms from the Coleman-Weinberg potential. Alright? *Allons-y*!

Let's get our hands a little dirty because it's good for us. Let's start by writing out the superpotential in all its indexed glory. This way we can convince ourselves that the derivatives we take to get the scalar potential actually work in the 'intuitive' way. (Then we can stare at it a little and the slap our foreheads because it was obvious to begin with.)

$$W = h{\varphi_c}^i \Phi_{ij} \widetilde{\varphi}^{jc} - h\mu^2 \Phi_{ij} \delta^{ij}. \tag{20.11}$$

We now take the appropriate derivatives,

$$\begin{aligned}
\frac{\partial W}{\partial {\varphi_c}^i} &= h\Phi_{ij}\varphi^{jc} \\
\frac{\partial W}{\partial \widetilde{\varphi}^{jc}} &= h{\varphi_c}^i \Phi_{ij} \\
\frac{\partial W}{\partial \Phi_{ij}} &= h\left({\varphi_c}^i \widetilde{\varphi}^{jc} - \mu^2 \delta^{ij}\right).
\end{aligned}$$

The scalar potential is

$$V = |W_\varphi|^2 + |W_{\widetilde{\varphi}}|^2 + |W_\Phi|^2,$$

where we mean

$$|W_\phi|^2 = \sum_{ij} \left(\frac{\partial W}{\partial \phi_{ij}}\right)\left(\frac{\partial W}{\partial \phi_{ij}}\right)^\dagger = \mathrm{Tr}\left|\frac{\partial W}{\partial \phi}\right|.$$

The factor of $|h|^2$ end up everywhere, so for simplicity we'll just set $h = 1$. Given the form of the superpotential, it's easy to put them back at the end.

### 20.2.1 Tree-level massive fields

Let's now work out the spectrum of massive fields at tree-level. We will explicitly derive the simple case of the spectrum at the origin of the pseudomoduli space. We will see later from the calculation of the Coleman-Weinerg potential that this is where the theory prefers to live. The scalar potential can be written as

$$V = |\Phi\widetilde{\varphi}|^2 + |\varphi\Phi|^2 + |\varphi\widetilde{\varphi} - \mu^2 \mathbb{1}|^2, \tag{20.12}$$

where we should clarify what we mean, e.g. in the first term

$$\begin{aligned}\Phi\widetilde{\varphi} &= \Phi_{ij}\widetilde{\varphi}^{jc} = \left(\Phi\widetilde{\varphi}\right)_i^{\ c} \\ |\Phi\widetilde{\varphi}| &= \left(\Phi\widetilde{\varphi}\right)_i^{\ c}\left[\left(\Phi\widetilde{\varphi}\right)^\dagger\right]_c^{\ i} = \Phi_{ij}\widetilde{\varphi}^j_{\ c}\widetilde{\varphi}^\dagger{}_k^{\ c}\Phi^{\dagger ki} = \mathrm{Tr}\,\Phi\widetilde{\varphi}\widetilde{\varphi}^\dagger\Phi^\dagger.\end{aligned}$$

In terms of our matrix fields, this gives us

$$\Phi\widetilde{\varphi} = \begin{pmatrix}\delta Y\left(\mu + \frac{1}{\sqrt{2}}(\delta\chi_+ - \delta\chi_-)\right) + \delta Z^T\frac{1}{\sqrt{2}}(\delta\rho_+ - \delta\rho_-) \\ \delta\widetilde{Z}\left(\mu + \frac{1}{\sqrt{2}}(\delta\chi_+ - \delta\chi_-)\right) + \delta\hat{\Phi}\frac{1}{\sqrt{2}}(\delta\rho_+ - \delta\rho_-)\end{pmatrix}. \tag{20.13}$$

We only care about the mass term inside $|\Phi\widetilde{\varphi}|^2$, i.e. terms that are bilinear in the fields. Thus we want terms in $\Phi\widetilde{\varphi}$ which are linear in the fields, i.e. the $\mu$ term. It is easy to see that

$$|\Phi\widetilde{\varphi}|^2\Big|_{\text{mass}} = |\mu\,\delta Y|^2 + |\mu\,\delta\widetilde{Z}|^2. \tag{20.14}$$

Analogously,

$$|\varphi\Phi|^2\Big|_{\text{mass}} = |\mu\,\delta Y|^2 + |\mu\,\delta Z|^2. \tag{20.15}$$

This covers the first two terms in Eq. (20.12). Let's sketch out the last term.

$$\varphi\widetilde{\varphi} - \mu^2\mathbb{1} = \begin{pmatrix}\frac{1}{2}(\delta\chi_+ + \delta\chi_-)(\delta\chi_+ - \delta\chi_-) & \left(\mu + \frac{1}{\sqrt{2}}(\delta\chi_+ + \delta\chi_-)\right)\frac{1}{\sqrt{2}}(\delta\rho_+ - \delta\rho_-) \\ \frac{1}{\sqrt{2}}(\delta\rho_+ + \delta\rho_-)\left(\mu + \frac{1}{\sqrt{2}}(\delta\chi_+ - \delta\chi_-)\right) & \frac{1}{2}(\delta\rho_+ + \delta\rho_-)(\delta\rho_+ - \delta\rho_-) - \mu^2\end{pmatrix}. \tag{20.16}$$

Boy, that's ugly looking. However, we know that we only care about the diagonal terms in the trace, so let's remind ourselves that

$$\begin{pmatrix}A & B \\ C & D\end{pmatrix}\begin{pmatrix}A^\dagger & C^\dagger \\ B^\dagger & D^\dagger\end{pmatrix} = \begin{pmatrix}AA^\dagger + BB^\dagger & \\ & CC^\dagger + DD^\dagger\end{pmatrix}.$$

Thus

$$\begin{aligned}\mathrm{Tr}|\varphi\widetilde{\varphi} - \mu^2|^2_{\text{mass}} &= \frac{1}{2}|\mu(\delta\rho_+ - \delta\rho_-)|^2 + \frac{1}{2}|\mu(\delta\rho_+ + \delta\rho_-)|^2 \\ &\quad - \frac{1}{2}(\mu^\dagger)^2(\delta\rho_+ + \delta\rho_-)(\delta\rho_+ - \delta\rho_-) - \frac{1}{2}\mu^2(\delta\rho_+ + \delta\rho_-)^\dagger(\delta\rho_+ - \delta\rho_-)^\dagger.\end{aligned}$$

*Ack!* It still looks really ugly, especially since $\mu$ comes in as $\mu^2$, $(\mu^\dagger)^2$, and $|\mu|^2$. However, upon further inspection, this is easy to fix. We just have to absorb the $\mu$ into our $\delta\rho$ fields:

$$\delta\rho_\pm \to \delta\rho'_\pm = \frac{\mu^*}{|\mu|}\delta\rho_\pm, \tag{20.17}$$

where the $|\mu|^{-1}$ is there to preserve canonical normalization. We can now drop the $'$ to clean up our notation. We end up with

$$\begin{aligned}\mathrm{Tr}|\varphi\widetilde{\varphi}-\mu^2|^2_{\mathrm{mass}} &= \frac{1}{2}|(\delta\rho_+-\delta\rho_-)|^2+\frac{1}{2}|(\delta\rho_++\delta\rho_-)|^2 \\ &\quad -\frac{1}{2}(\delta\rho_++\delta\rho_-)(\delta\rho_+-\delta\rho_-)-\frac{1}{2}(\delta\rho_++\delta\rho_-)^\dagger(\delta\rho_+-\delta\rho_-)^\dagger. \end{aligned} \tag{20.18}$$

Putting in some more elbow grease, we get

$$\begin{aligned}2\mathrm{Tr}|\varphi\widetilde{\varphi}-\mu^2|^2_{\mathrm{mass}} = \quad & |\delta\rho_+|^2-\cancel{\delta\rho_+\delta\rho_-^\dagger}-\cancel{\delta\rho_-\delta\rho_+^\dagger}+|\delta\rho_-|^2 \\ & +|\delta\rho_+|^2+\cancel{\delta\rho_+\delta\rho_-^\dagger}+\cancel{\delta\rho_-\delta\rho_+^\dagger}+|\delta\rho_-|^2 \\ & -\delta\rho_+^2\cancel{-\delta\rho_-\delta\rho_++\delta\rho_+\delta\rho_-}+\delta\rho_-^2 \\ & -(\delta\rho_+^\dagger)^2\cancel{+\delta\rho_-^\dagger\delta\rho_+^\dagger-\delta\rho_+^\dagger\delta\rho_-^\dagger}+(\delta\rho_-^\dagger)^2. \end{aligned} \tag{20.19}$$

This still requires some massage work. Let's split the $\delta\rho_\pm$ fields into its real and imaginary parts (as matrices),

$$\delta\rho_\pm = a_\pm + ib_\pm. \tag{20.20}$$

What does this buy us? recall that

$$\begin{aligned}(a+ib)(c+id) &= ac+iad+ibc-bd \\ (a-ib)(c-id) &= ac-iad-ibc-bd.\end{aligned}$$

In terms of these fields we get

$$\begin{aligned}2\mathrm{Tr}|\varphi\widetilde{\varphi}-\mu^2|^2_{\mathrm{mass}} = \quad & 2(a_+^2+b_+^2)+2(a_-^2+b_-^2) \\ & -2(a_+^2-b_+^2)+2(a_-^2-b_-^2) && (20.21)\\ = \; & 4b_+^2+4a_-^2 \\ = \; & 4\mathbb{I}\mathrm{m}(\delta\rho_+)^2+4\mathbb{R}\mathrm{e}(\delta\rho_-)^2. && (20.22)\end{aligned}$$

Good. Thus we've found that the fields $\delta Y$, $\delta\widetilde{Z}$, $\delta Z$, $\mathbb{I}\mathrm{m}(\delta\rho_+)$, and $\mathbb{R}\mathrm{e}(\delta\rho_-)$ have all acquired tree level masses on the order of $|h\mu|$. Still with us? Good, because that was the easy part.

Let us remark that later on we will calculate the Coleman-Weinberg effective potential to determine how the pseudomoduli are lifted. We will find that the vacuum of the theory lives at the origin of pseudomoduli space so that the tree-level spectrum above turns out to be accurate. Note, however, that this is *not* the spectrum that we plug into the Coleman-Weinberg formula. In order to calculate the effective potential for the pseudomoduli, we will have to determine the spectrum about an arbitrary point on the pseudomoduli space (we'll see that it is sufficient to restrict to a submanifold). In this case, the spectrum will become a function of the pseudomoduli and, in particular, one will obtain mass terms which mix the above fields.

Now let's work out the linear combinations that appear as pseudomoduli and Goldstones.

### 20.2.2 Pseudomoduli fields

The pseudomoduli are actually trivial. These are the directions that are associated Eq. (20.7), up to rotations by our global symmetries. Thus pseudomoduli are precisely

$$\delta\hat{\Phi} \qquad \text{and} \qquad \delta\hat{\chi} \equiv \delta\chi_- + \text{h.c.}, \tag{20.23}$$

where we've also rescaled the $\chi$ fields to absorb $\mu$:

$$\delta\chi_\pm \to \delta\chi'_\pm = \frac{\mu^*}{|\mu|}\delta\chi_\pm, \tag{20.24}$$

and then we again drop the $'$ for simplicity. Note that it is important that $\delta\hat{\chi}$ take the precise form above. The excitation has to be the $\delta\chi_-$ part of $\varphi$ and $\widetilde{\varphi}$ because this is the antisymmetric part: this is the part that will cancel in the last term of Eq. (20.12). Note that in the vacuum of the theory these excitations manifestly do not contribute to the first two terms. This cancellation only occurs for the real part of this field, which we isolate by summing with the Hermitian conjugate.

### 20.2.3 Goldstone bosons

Finally, let's identify the Goldstone bosons coming from the spontaneous breaking of global symmetries. Let's write down our symmetry table once again:

| | SU($N$) | SU($F$)$_\text{L}$ | SU($F$)$_\text{R}$ | U(1)$_B$ | U(1)$'$ | U(1)$_R$ |
|---|---|---|---|---|---|---|
| $\Phi$ | 1 | $\square$ | $\overline{\square}$ | 0 | -2 | 2 |
| $\varphi$ | $\square$ | $\overline{\square}$ | 1 | 1 | 1 | 0 |
| $\widetilde{\varphi}$ | $\overline{\square}$ | 1 | $\square$ | -1 | 1 | 0 |

Recall that there is an explicit breaking SU($F$)$_\text{L}\times$ SU($F$)$_\text{R}\times$ U(1)$'$ $\to$ SU($F$) by the $\Delta W$ term in Eq. (20.3). This means, in particular, that $\Phi$ transforms as an $\mathbf{Ad}\oplus\mathbb{1}$, i.e. an adjoint plus the trace. Because one typically doesn't work with spontaneous symmetry breaking of with multiple fields getting related vevs, let's work through this section somewhat carefully.

Let's review how fields transform under the fundamental and anti-fundamental representations of a Lie group[44].

$$\begin{aligned}\varphi^i &\to \varphi'^i = \varphi^i + i\epsilon^a (T^a)^i{}_j\varphi^j \\ \varphi_i &\to \varphi'_i = \varphi_i + i\epsilon^a(-T^{*a})_i{}^j\varphi^j,\end{aligned}$$

[44]This should be a very basic review, but it recently came to my attention that many students are unaware of handy references for the representation theory of Lie groups for particle physicists. I am not particularly enthused by the canonical text by Georgi [115]. Instead, for a quick introduction Cheng & Li do a good job [116] while the lecture notes for the Cambridge Part III course 'Symmetries and Particle Physics' are usually very good. Two recent sets of lecture notes can be found at `http://www.mth.kcl.ac.uk/~jbg34/Site/Dr._Jan_Bernard_Gutowski.html` and `http://www.damtp.cam.ac.uk/user/ho/GNotes.pdf`.

where $\varphi^i$ transforms as a fundamental $\Box$ and $\varphi_i$ transforms as an anti-fundamental $\overline{\Box}$. We may use anti-Hermiticity to relate the generators fundamental and anti-fundamental representations

$$T^a_{\overline{\Box}} = -(T^a_{\Box})^* = -(T^a_{\Box})^T,$$

from which we obtain

$$\varphi'_i = \varphi_i - i\epsilon^a (T^a)^j{}_i \varphi_j = \varphi_i - i\epsilon^a \varphi_j (T^a)^j{}_i.$$

The U(1)s are all generated by identity matrices, $\mathbb{1}$ with respect to the matrix Lie groups. This just means that they are the traces of the multi-dimensional matrices that generate our global symmetry. For now let's not worry about them because they're easy. The generators of our [broken] SU($N$)$\times$ SU$(F)_\mathrm{L}\times$SU$(F)_\mathrm{R}$ symmetry are

$$T^A = T^A_{\mathrm{SU}(N)} \otimes T^A_{\mathrm{SU}(F)_\mathrm{L}} \otimes T^A_{\mathrm{SU}(F)_\mathrm{R}}. \tag{20.25}$$

Let's see how this acts on a bifundamental $\mathbf{F}\times\overline{\mathbf{F}}$ field like $\Phi^i{}_j$.

$$\begin{aligned} i\epsilon^A T^A \Phi &= i\epsilon^a_L (T^a_L)^{i_L}{}_{k_L} \Phi^{k_L}{}_{j_R} + i\epsilon^b_R (T^b_R)_{j_R}{}^{k_R} \Phi^{i_L}{}_{k_R} && (20.26)\\ &= i\epsilon^a_L (T^a)^{i_L}{}_{k_L} \Phi^{k_L}{}_{j_R} - i\epsilon^b_R (T^b)^{k_R}{}_{j_R} \Phi^{i_L}{}_{k_R}, && (20.27)\end{aligned}$$

where for clarity we've labelled the SU$(F)_\mathrm{L}$ and SU$(F)_\mathrm{R}$ indices separately and used the above observation that since $\Phi$ is a fundamental under SU$(F)_\mathrm{L}$ and an anti-fundamental under an identical SU$(F)_\mathrm{R}$, we can write everything with respect to the fundamental generators of SU($F$). Now the main point is that the explicit breaking SU$(F)_\mathrm{L}\times$ SU$(F)_\mathrm{R} \to$ SU($F$) enforces

$$\epsilon_L = \epsilon_R. \tag{20.28}$$

This is just the analog of chiral symmetry breaking in QCD (only this is done explicitly).

Now let's get to the good stuff. We know that the Goldstone bosons are constructed by acting on the vev by the broken generators since this determines the flat directions in field space. The somewhat novel feature here relative to what is found in introductory field theory texts is that *two* fields ($\varphi$ and $\widetilde{\varphi}$) obtain vevs. The procedure is the same, but one must remember to act on *both* vevs simultaneously with each broken generator. The Goldstone directions in field space will then be a linear combination of both fields. This is obvious in retrospect, though a helpful mnemonic might be to imagine a single multi-component field $\varphi \oplus \widetilde{\varphi}$ which is transformed by generators $T^A \oplus \widetilde{T}^A$ and which obtains a vev $\langle\varphi\rangle \oplus \langle\varphi\rangle$. In this case it is clear that the correct procedure is to act on both $\varphi$ and $\widetilde{\varphi}$ simultaneously by the broken generators.

We already know what the vevs are from Eq. (20.8). We'll consider the spontaneous breaking in two steps. First we consider the breaking SU($F$) $\to$ SU$(N)_F\times$SU$(F-N)\times$U$(1)_{B'}$. The first factor is the upper-left $N\times N$ part of the SU($F$) generators, the second factor is the lower-right $(F-N)\times(F-N)$ part, and the U$(1)_{B'}$ corresponds to the diagonal generator diag$(a,\cdots,a,b,\cdots,b)$. Note that we are not yet considering that the vevs in Eq. (20.8) take the form $\mu\mathbb{1}$ and so break $SU(N)\times SU(N)_F \to SU(N)_D$. All we're considering for now is that we've broken $SU(F)$ into two disconnected blocks.

Let's remind ourselves what our set of generators look like:

$$\left(\begin{array}{c:c} T & \\ \hdashline & 0 \end{array}\right) \qquad \left(\begin{array}{c:c} & 1 \\ \hdashline 1 & 0 \end{array}\right) \qquad \left(\begin{array}{c:c} & i \\ \hdashline -i & 0 \end{array}\right) \qquad \left(\begin{array}{c:c} & \\ \hdashline & t \end{array}\right) \qquad \left(\begin{array}{ccc:c} a & & & \\ & \ddots & & \\ \hdashline & & & b \end{array}\right),$$

where clearly generators of the second and third type are broken since they mix the $N \times N$ block which obtains a vev $\mu \mathbb{1}_N$ with the lower $N \times (F - N)$ block in $\varphi$ and $\widetilde{\varphi}^T$. This makes it clear that the Goldstone directions are precisely these lower $N \times (F - N)$ blocks. We emphasize, once again, that the directions obtained by doing this for each field are *not* independent. Fortunately, we've already intelligently split up our fields in Eq. (20.10).

Acting with the real broken generators, we see that our Goldstone directions are (writing $T^A\langle\varphi\rangle + T^A\langle\widetilde{\varphi}\rangle$)

$$\frac{1}{\sqrt{2}}\mathbb{R}\mathrm{e}(\rho_+ + \rho_-) + \frac{1}{\sqrt{2}}\mathbb{R}\mathrm{e}(\rho_+ - \rho_-) \propto \mathbb{R}\mathrm{e}\rho_+. \tag{20.29}$$

Similarly, the imaginary broken generators give us the Goldstones

$$\frac{1}{\sqrt{2}}\mathbb{I}\mathrm{m}(\rho_+ + \rho_-) - \frac{1}{\sqrt{2}}\mathbb{I}\mathrm{m}(\rho_+ - \rho_-) \propto \mathbb{I}\mathrm{m}\rho_-, \tag{20.30}$$

where the minus sign comes from the fundamental versus the anti-fundamental representation.

Now let's move on. The particular form of the vevs in Eq. (20.8) break $\mathrm{SU}(N)\times\mathrm{SU}(F)_N\times\mathrm{U}(1)'_B$ to $\mathrm{SU}(N)_D$. Note that this is a different spontaneous breaking from the $\mathrm{SU}(F) \to \mathrm{SU}(N)_F\times\mathrm{SU}(F-N)\times\mathrm{U}(1)_{B'}$ we considered above: that had to do with breaking $\mathrm{SU}(F)$ into blocks. Now we're dealing with the actual form of the vev in the nontrivial block.

Let us write the upper $N \times N$ blocks of $\varphi$ and $\widetilde{\varphi}$ as $\varphi_N$ and $\widetilde{\varphi}_N$. Then recalling how $\varphi$ and $\widetilde{\varphi}$ transform under $\mathrm{SU}(N)$ and $\mathrm{SU}(F)_N$, we see

$$\begin{aligned} \varphi_N &\to U_N \varphi_N U_F^\dagger \\ \widetilde{\varphi}_N &\to U_F \widetilde{\varphi}_N U_N^\dagger. \end{aligned}$$

This is preserved by the vevs if the transformation parameters are such that $\epsilon_N = \epsilon_F$, i.e. we break to the diagonal subgroup. This breaking is precisely analogous to the pattern of chiral symmetry breaking in QCD. The broken generators then have $\epsilon = \epsilon_N = -\epsilon_F$, i.e. they are the axial generators. Let's work out the change in the $\varphi$ field after an axial transformation:

$$i\epsilon^A T^A \langle\varphi_N\rangle = i\epsilon^A (T_N^A)^{i_N}{}_{k_N} \langle\varphi_N\rangle^{k_N}{}_{j_F} - i\epsilon_F^A \langle\varphi_N\rangle^{i_N}{}_{k_F} (T_F^A)^{k_F}{}_{j_F}. \tag{20.31}$$

Recalling that $\langle\varphi_N\rangle^{i_N}{}_{j_F} = \mu\delta^{i_N}{}_{j_F}$, we have

$$i\epsilon^A T^A \varphi_N = 2i\epsilon^A (T^A)^{i_N}{}_{j_F}. \tag{20.32}$$

This is a basis of traceless anti-Hermitian matrices. The analysis for $\widetilde{\varphi}$ gives the same result (there's an overall minus sign). There's one missing piece: the $\mathrm{U}(1)_{B'}$ generator which is also broken. This gives a trace part to the Goldstone fields. Thus our Goldstones are the trace-included anti-Hermitian matrices,

$$\chi_- - \chi_-^\dagger. \tag{20.33}$$

That wraps up our summary of the spectrum of fields.

### 20.2.4 The Coleman-Weinberg potential

At one-loop order the pseudomoduli are lifted by the Coleman-Weinberg potential. One must check that the potential has positive curvature rather than negative curvature, or else our stable vacuum will be spoiled. Using the global symmetries (e.g. the unbroken U(1)), the fact that only single traces appear in the Coleman-Weinberg potential, and some dimensional analysis for the overall factor, the relevant piece of the effective potential is

$$V_{\text{CW}} = |h^4\mu^2| \left(\frac{1}{2}a\text{Tr }\delta\hat{\chi}^2 + b\text{Tr }\delta\hat{\Phi}^\dagger\delta\hat{\Phi}\right) + \cdots , \tag{20.34}$$

for some coefficients $a$ and $b$ which we'd like to establish are greater than zero. Because $h$ is marginally irrelevant in the IR, this one-loop contribution to the effective potential dominates over higher-order corrections.

[**Check** Why is $h$ marginally irrelevant?] Note that the way we've defined $h$ is precisely analogous to the use of $\hbar$ to count loops in Appendix B.3. If we take $h \to 0$ with $f, X, q \sim h^{-1}$ then the classical Lagrangian goes as $h^{-2}$, the one-loop corrections go as $h^0$, and higher loop contributions go as $h^{2n}$ for $n > 0$.

Now recall that it is not correct to simply plug in the tree-level spectrum that we've derived above. These masses are all dependent on the point on the pseudomoduli space in which we live. With some foresight, we calculated that spectrum at the origin of the pseudomoduli space. In order to determine the effective potential of the pseudomoduli, however, it is necessary to determine the spectrum for an *arbitrary point* on the pseudomoduli space so that the potential can be written as a function of the pseudomoduli. Thus, generally calculating the effective potential requires some work since there are so many pseudomoduli (counting each component of the matrix fields). Fortunately, we can simplify the analysis significantly since we don't care about the full effective potential: we only need the quadratic part which tells us about the local stability of a point. Thus we can be clever and only choose to move along specific directions along the pseudomoduli space. We will pick directions labelled by $X_0$ and $\theta$:

$$\Phi = \begin{pmatrix} \delta Y & \delta Z^T \\ \delta\widetilde{Z} & X_0\mathbb{1}_{(F-N)} + \delta\hat{\Phi}\end{pmatrix} \qquad \varphi = \begin{pmatrix}\mu e^{\theta}\mathbb{1}_N + \delta\chi \\ \delta\rho\end{pmatrix} \qquad \widetilde{\varphi}^T = \begin{pmatrix}\mu e^{-\theta}\mathbb{1}_N + \delta\widetilde{\chi} \\ \delta\widetilde{\rho}\end{pmatrix}, \tag{20.35}$$

where we can assume $X_0$ and $\theta$ are small. (If we determine $a$ and $b$ anywhere on the pseudomoduli space then we've determined it everywhere.) Plugging this into the formula for $V_{\text{CW}}$ (see Appendix B), we get

$$V_{\text{CW}} = \text{const} + h^4\mu^2\left(\frac{1}{2}aN\mu^2(\theta+\theta^*)^2 + b(F-N)|X_0|^2\right) + \cdots . \tag{20.36}$$

Our task is to determine $a$ and $b$. We need to find the tree-level masses associated with a point $(X_0, \theta)$ on the pseudomoduli submanifold, so we plug in our parameterization into the superpotential,

$$\begin{aligned} W = h\text{Tr }&\Big[\left(\mu e^{\theta}\delta\chi\right)\delta Y\left(\mu e^{-\theta} + \delta\widetilde{\chi}\right) + \delta\rho\delta\widetilde{Z}\left(\mu e^{-\theta}+\delta\widetilde{\chi}\right) \\ &+ \left(\mu e^{\theta}+\delta\chi\right)\delta\widetilde{Z}\delta\widetilde{\rho} + \delta\rho\left(X_0 + \delta\hat{\Phi}\right)\delta\widetilde{\rho}\Big] - h\mu^2\text{Tr }\left[\delta Y + X_0\mathbb{1}_{(F-N)} + \delta\hat{\Phi}\right] \end{aligned} \tag{20.37}$$

Keep in mind what's going on: fields prefixed with a $\delta$ are dynamical excitations, while fields without a $\delta$ are background fields (i.e. pseudomoduli). This is why the $\delta$ notation, while cumbersome, is handy.

We can now invoke a bit of a trick: we know that the contributions to the Coleman-Weinberg effective potential come from SUSY-breaking, since the manifestly supersymmetric parts cancel in the supertrace. We know how supersymmetry is broken in this model, so we can be clever and identify which fields have masses which actually couple to the SUSY-breaking $F$-terms. So, pop-quiz: which fields obtain $F$-term vevs?

We know that the fields $\varphi$ and $\widetilde{\varphi}^T$ obtain vevs along their upper $N \times N$ blocks. These vevs come from using the available global symmetries to cancel as much of the the $\mu^2$ term in $F_\Phi$, c.f. Eq. (20.5). The remaining $N \times (F - N)$ matrix of fields are those which could not cancel the remaining terms in the $\mu^2$ diagonal matrix and hence it is these fields which break supersymmetry. These are just the $\delta\rho$ and $\delta\widetilde{\rho}$ fields. It is easy to see in the superpotential that when the lower-right $(F-N)\times(F-N)$ block $\langle F_\Phi \rangle$ is nonzero, the $\delta\rho$ and $\delta\widetilde{\rho}$ fields obtain obtain SUSY-breaking scalar masses.

The SUSY-breaking $\delta\rho$ and $\delta\widetilde{\rho}$ scalars mix with other fields at tree level[45]. Thus the fields which make a nontrivial contribution to the Coleman-Weinberg potential are those which mix with the $\delta\rho$ or $\delta\widetilde{\rho}$ fields. Writing out the quadratic part of the superpotential, we get

$$W = h\mathrm{Tr}\,\left[\mu e^{\theta}\left(\delta Z^T\delta\widetilde{\rho} + \delta Y\delta\widetilde{\chi}\right) + \mu e^{-\theta}\left(\delta\widetilde{Z}^T\delta\rho + \delta Y^T\delta\chi\right) + \left(\delta\widetilde{\rho}\delta\rho^T - \mu^2\right)\left(X_0 + \delta\hat{\Phi}\right)\right] + \cdots .$$

For reasons that will become clear shortly, we've written out $(X_0 + \delta\hat{\Phi})$ even though this term includes non-quadratic terms. At this point the ISS paper (see their Appendix B) makes a cryptic remark that the off-diagonal components of $\delta\hat{\Phi}$ do not contribute to the mass matrix. This is not quite an accurate or relevant observation since we have chosen a submanifold of the pseudomoduli space where we are only expanding about the diagonal background (parameterized by $X_0$) of the $\delta\hat{\Phi}$ field. The $\delta\hat{\Phi}$ itself is not a background value but a physical excitation. At any rate, this is neither here nor there so we may move on. We can make the more important observation that the fields $\delta\chi$, $\delta\widetilde{\chi}$, and $\delta Y$ do *not* mix with the SUSY-breaking fields at tree-level. They certainly have higher-power couplings with $\delta\rho$ and $\delta\widetilde{\rho}$, but those could only contribute mixing at the one-loop level. Thus, these superfields have manifestly supersymmetric spectra and do not contribute at all to the one-loop effective potential. The remaining terms which are of interest may be written

$$W_{\text{mass}} = h\sum_{i=1}^{(F-N)}\left(X_0 + \delta\hat{\Phi}_{ii}\right)\left(\delta\rho\delta\widetilde{\rho}^T\right)_{ii} + \mu e^{\theta}\left(\delta\widetilde{\rho}\delta Z^T\right)_{ii} + \mu e^{-\theta}\left(\delta\rho\delta\widetilde{Z}^T\right)_{ii} - \mu^2\left(X_0 + \delta\hat{\Phi}_{ii}\right).$$

Now we make a very handy observation that also justifies our choice of writing out $(X_0 + \delta\hat{\Phi})$ explicitly. This looks precisely like $(F-N)$ decoupled copies of an O'Raifeartaigh-like model with superpotential

$$W = h\left(X\phi_1 \cdot \phi_2 + \mu e^{-\theta}\phi_1 \cdot \phi_3 + \mu e^{\theta}\phi_2 \cdot \phi_4 - \mu^2 X\right).$$

[45] Again, this is true for some generic point on the pseudomoduli space, but we calculated above that for the case of the origin of the pseudomoduli the imaginary and real parts of the $\delta\rho$ and $\delta\widetilde{\rho}$ fields are independent physical degrees of freedom.

The Coleman-Weinberg potential for this model can be worked out straightforwardly for homework. Contrary to my usual practice I won't work it out here[46], but helpful points for the derivation of the effective potential for simple O'Raifeartaigh model can be found in Intriligator and Seiberg's SUSY-breaking notes [1]. We may invoke these results to determine that the Coleman-Weinberg potential for this pseudomoduli submanifold is

$$V_{\text{CW}}^{(1)} = \text{constant} + \frac{h^4\mu^2(\log 4 - 1)N(F-N)}{8\pi^2}\left(\frac{1}{2}\mu^2(\theta+\theta)^* + |X|^2\right) + \cdots , \tag{20.38}$$

from which we determine the coefficients

$$a = \frac{\log 4 - 1}{8\pi^2}(F-N) \qquad\qquad b = \frac{\log 4 - 1}{8\pi}N. \tag{20.39}$$

We don't actually care what these precise values are, only that they are positive definite in the regime of interest and therefore our pseudomoduli are indeed stabilized about the origin as we assumed above.

## 20.3 Macroscopic Model II

## 20.4 Dynamical Realization: the ISS model

## 20.5 Immediate directions beyond the basic ISS model

## 20.6 Extensions and related work

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[46]I did a simpler case on scratch paper but decided that the calculation was so non-illuminating that it wasn't worth typing up.

# Appendix

## A Notation and Conventions

Here we present a set of self-consistent notation and conventions that we (try to) use in this document.

### A.1 Field labels

Chiral superfields are typically written with capital Roman letters, e.g., $S$, $N$, $X$. Complex conjugation is denoted by a star, $(a+ib)^* = (a-ib)$. A bar, on the other hand, is used to distinguish pairs of vector-like chiral superfields, e.g, $N$ and $\bar{N}$ have opposite charges under a particular symmetry. Do not confuse this bar with complex conjugate. To avoid confusion, it is typical to use a tilde to denote the vector-like pair, e.g., $N$ and $\widetilde{N}$. We denote the axino by $\chi$ rather than the usual $\widetilde{a}$ to avoid cumbersome notation and to reinforce its identity as dark matter. The dual gauge field strength $*F$ is defined in component notation relative to the field strength via

$$\widetilde{F}_{\mu\nu} = \frac{1}{2} F^{\alpha\beta} \epsilon_{\alpha\beta\mu\nu}. \tag{A.1}$$

### A.2 Spacetime and spinors

There is no completely standard set of spacetime and spinor conventions in the SUSY literature, but the choices that make the most sense to us are those by Dreiner et al. [117]; see their appendix for a thorough discussion of how to passing between metric conventions[47]. See also Problem 1 of Appendix C in Binetruy's supersymmetry textbook [118] which identifies all possible sources of sign ambiguities and writes relevant formulae with all choices made explicit. Pedagogical introductions to Weyl and Majorana spinors can be found in Aitchison [119] and the article by Pal [120].

4D Minkowski indices are written with lower-case Greek letters from the middle of the alphabet, $\mu, \nu, \cdots$. We use the particle physics ('West Coast,' mostly-minus) metric for Minkowski space, $(+,-,-,-)$. Our convention for $\sigma^0$ and the three Pauli matrices $\vec{\sigma}$ is

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.2}$$

The un-barred Pauli matrices have indices $\sigma^{\mu}_{\alpha\dot{\beta}}$ while the barred Pauli matrices, $\bar{\sigma}^{\mu} = (\sigma^0, -\vec{\sigma})$, have indices $\bar{\sigma}^{\mu\dot{\alpha}\beta}$. The two types of Pauli matrices are related by

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta} \sigma^{\mu}{}_{\beta\dot{\beta}}, \tag{A.3}$$

[47] To see this in action, see their source file at `http://zippy.physics.niu.edu/spinors.html`, which includes a macro to allow one to change metric conventions. The implementation is an excellent example of where the metric choice is (and isn't) relevant.

where our convention for the sign of $\epsilon$ is given below. The Weyl representation for the Dirac $\gamma$ matrices is

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar\sigma^\mu & \end{pmatrix} \qquad\qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & \\ & \mathbb{1} \end{pmatrix}. \tag{A.4}$$

Note that the definition of $\gamma^5$ is the usual 4D Weyl basis convention, whereas the sensible 5D convention is $\Gamma^5 = \text{diag}(i, -i)$ so that the 5D Clifford algebra is satisfied. The antisymmetric products of Pauli matrices are

$$\sigma^{\mu\nu} = \frac{i}{4}\sigma^{[\mu}\bar\sigma^{\nu]} \qquad\qquad \bar\sigma^{\mu\nu} = \frac{i}{4}\bar\sigma^{[\mu}\sigma^{\nu]}. \tag{A.5}$$

I don't like the factor of $i$, but this is the price of sticking with the conventions in [117].

The totally antisymmetric tensor [densities] are chosen to have

$$\epsilon^{12} = \epsilon_{21} = 1 \qquad\qquad \epsilon^{0123} = -\epsilon_{0123} = 1. \tag{A.6}$$

This convention agrees with Wess & Bagger [121], Terning [5], and Dreiner et al. [117] but has a relative sign from Bailin and Love [122]. The significance of this choice is described in footnotes 4–6 of Dreiner et al. [117], but the point is that Weyl spinor indices are raised and lowered via matrix multiplication from the left,

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \qquad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \qquad \bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta} \qquad \bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}, \tag{A.7}$$

where we've introduced the notation $\bar\psi_{\dot\alpha} = (\psi_\alpha)^*$ and $\chi^\alpha = (\bar\chi^{\dot\alpha})^*$. Note the use of $*$ here rather than $\dagger$, though the distinction is mostly poetic. If one is perturbed by this, an excellent reference is the relevant chapter in Aitchison's elementary text [119]. The relative sign between $\epsilon^{12}$ and $\epsilon_{12}$ sets $\epsilon_{\alpha\rho}\epsilon^{\rho\beta} = \delta_\alpha^\beta$ so that no signs appear when an index is raised and then lowered again. Alternately, this relative sign appears when relating the $\epsilon$ tensor to charge conjugation as we will see below. With this convention, special care is required to keep track of minus signs when raising and lowering indices of $\epsilon$ tensors (see [117]), but this is usually a silly thing to do to begin with. Using Lorentz invariance, one can write relations like $\theta^\alpha\theta^\beta \propto \epsilon^{\alpha\beta}\theta\theta$. The overall constant of proportionality can be found by contracting the indices of both sides. One finds

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta \qquad\qquad \theta_\alpha\theta_\beta = +\frac{1}{2}\epsilon_{\alpha\beta}\theta\theta \tag{A.8}$$

$$\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = +\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta \qquad\qquad \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta. \tag{A.9}$$

Similarly,

$$\theta\sigma^\mu\bar\theta\,\theta\sigma^\nu\bar\theta = +\frac{1}{2}\theta^2\bar\theta^2\eta^{\mu\nu} \tag{A.10}$$

$$(\theta\psi)(\theta\chi) = -\frac{1}{2}(\psi\chi)(\theta\theta) \tag{A.11}$$

$$(\bar\theta\bar\psi)(\bar\theta\bar\chi) = -\frac{1}{2}(\bar\psi\bar\chi)(\bar\theta\bar\theta). \tag{A.12}$$

The placement of Weyl spinors (with their natural index placement) within a Dirac spinor is

$$\Psi_{\mathrm{D}} = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{A.13}$$

Spinor contractions are descending for undotted indices and ascending for dotted indices:

$$\psi\chi \equiv \psi^\alpha \chi_\alpha \qquad\qquad \bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}. \tag{A.14}$$

With this convention, contractions are independent of the order of the spinors: $\psi\chi = \chi\psi$ and similarly for the barred spinors $\bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}$. The Dirac conjugate spinor is given by

$$\bar{\Psi}_D = \Psi^\dagger \gamma^0 = \begin{pmatrix} \psi^{\dagger\alpha} & \bar{\chi}^\dagger_{\dot{\alpha}} \end{pmatrix} \begin{pmatrix} & {\sigma^0}_{\alpha\dot{\beta}} \\ \bar{\sigma}^{0\dot{\alpha}\beta} & \end{pmatrix} = \begin{pmatrix} \psi^{\dagger\alpha} & \bar{\chi}^\dagger_{\dot{\alpha}} \end{pmatrix} \begin{pmatrix} & \mathbb{1}_{\alpha\dot{\beta}} \\ \mathbb{1}^{\dot{\alpha}\beta} & \end{pmatrix} \equiv \begin{pmatrix} \chi^\alpha & \bar{\psi}_{\dot{\beta}} \end{pmatrix}. \tag{A.15}$$

One may take this as a definition of $\chi$ and $\bar{\psi}$ in terms of $\psi$ and $\bar{\chi}$ in $\Psi_D$. It shows how $\gamma^0$ is used to convert the dotted index of $\bar{\chi}^\dagger$ into the undotted index of $\chi$ (and vice versa for $\psi^\dagger$ and $\bar{\psi}$).

The charge conjugate of a Dirac fermion $\Psi^c$ is given by

$$\Psi^c = C\bar{\Psi}^T \qquad\qquad C = \begin{pmatrix} i\bar{\sigma}^2 & \\ & i\sigma^2 \end{pmatrix} = \begin{pmatrix} \epsilon_{\alpha\beta} & \\ & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \tag{A.16}$$

This comes from taking the Hermitian conjugate of the Dirac equation

$$i(\partial\!\!\!/ - ieA\!\!\!/)\Psi = 0 \quad \Rightarrow \quad -i\bar{\Psi}\gamma^0\gamma^{\mu\dagger}(\overleftarrow{\partial}_\mu + ieA_\mu) = 0 \quad \Rightarrow \quad -i\gamma^{\mu\mathrm{T}}(\partial_\mu + ieA)\bar{\Psi}^{\mathrm{T}} = 0, \tag{A.17}$$

where we've made use of the identities $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$ and $(\gamma^0)^2 = \mathbb{1}$. Because $-\gamma^{\mu\mathrm{T}}$ satisfies the 4D Clifford algebra, there exists a charge conjugation matrix $C$ such that $C^{-1}\gamma^\mu C = -\gamma^{\mu\mathrm{T}}$. In particular, $C\bar{\Psi}^{\mathrm{T}}$ is a solution to the Dirac equation with opposite charge,

$$i\gamma^\mu(\partial_\mu + ieA_\mu)C\bar{\Psi}^{\mathrm{T}} = 0. \tag{A.18}$$

The above property of $C$ implies that $C \sim \gamma^0\gamma^2$. The constant of proportionality must be a pure phase so that $(\Psi^c)^c = \Psi$. We choose this proportionality so that

$$C = i\gamma^0\gamma^2, \tag{A.19}$$

which matches (A.16). This can be understood as the reason why the $\epsilon$ tensor density appears with a different overall sign when written with upper versus lower indices; the sign comes from $\sigma^2$ versus $\bar{\sigma}^2$. Writing out indices slightly more carefully,

$$\Psi^c = C\bar{\Psi}^T = \begin{pmatrix} i\bar{\sigma}^2 & \\ & i\sigma^2 \end{pmatrix} \begin{pmatrix} \chi^\alpha \\ \bar{\psi}_{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\alpha\beta} & \\ & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \begin{pmatrix} \chi^\alpha \\ \bar{\psi}_{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}. \tag{A.20}$$

A Majorana fermion obeys $\Psi_{\mathrm{M}} = \Psi^c_{\mathrm{M}}$ so that

$$\begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \tag{A.21}$$

that is $\psi_\alpha = \chi_\alpha$ and $\bar{\chi}^{\dot{\alpha}} = \bar{\psi}^{\dot{\alpha}}$. In other words,

$$\Psi_{\mathrm{M}} = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}. \tag{A.22}$$

Sometimes the right-hand side is written somewhat impressionistically as $(\psi, i\bar{\sigma}^2\psi^*)^{\mathrm{T}}$; the intended meaning is identical to the above expression.

## A.3 Superfields and superspace

The superspace measure is

$$d^2\theta = -\frac{1}{4}d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} = -\frac{1}{4}d\theta^\alpha d\theta_\alpha \qquad\qquad d^2\bar\theta = -\frac{1}{4}d\bar\theta_{\dot\alpha} d\theta_{\dot\beta}\epsilon^{\dot\alpha\dot\beta} = -\frac{1}{4}d\bar\theta_{\dot\alpha} d\bar\theta^{\dot\alpha}. \tag{A.23}$$

A [left] chiral superfield is given by

$$\Phi(x) = \phi(y) + \sqrt{2}\theta\psi(y) + (\theta\theta)F(y), \tag{A.24}$$

where the shifted coordinate is $y^\mu = x - i\theta\sigma^\mu\bar\theta$. The minus sign here is important for, among other things, obtaining the correct sign on the fermion kinetic term. Expanding in terms of fields evaluated at $x$, we have

$$\Phi(x) = \phi - i(\theta\sigma^\mu\bar\theta)\partial_\mu\phi - \frac{1}{2}(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)\partial_\mu\partial_\nu\phi + \sqrt{2}\theta\psi - i\sqrt{2}\theta(\theta\sigma^\mu\bar\theta)\partial_\mu\psi + (\theta\theta)F. \tag{A.25}$$

Using the relations (A.10) and (A.11) we may simplify this to

$$\Phi(x) = \phi + \sqrt{2}\theta\psi + (\theta\theta)F - i(\theta\sigma^\mu\bar\theta)\partial_\mu\phi + \frac{i}{\sqrt{2}}(\theta\theta)(\partial_\mu\psi\sigma^\mu\bar\theta) - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\partial^2\phi \tag{A.26}$$

$$\Phi^\dagger(x) = \phi^* + \sqrt{2}\bar\theta\bar\psi + (\bar\theta\bar\theta)F^* + i(\theta\sigma^\mu\bar\theta)\partial_\mu\phi^* - \frac{i}{\sqrt{2}}(\bar\theta\bar\theta)(\theta\sigma^\mu\partial_\mu\bar\psi) - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\partial^2\phi^*. \tag{A.27}$$

The field strength superfield is

$$\mathcal{W} = -i\lambda + [D - \sigma^{\mu\nu}F_{\mu\nu}]\,\theta - \theta\theta\sigma\partial\bar\lambda, \tag{A.28}$$

so that the SYM Lagrangian is $\mathcal{L} = \int d^2\theta\frac{1}{4}\mathcal{W}\mathcal{W} + \text{h.c.}$; occasionally I may write $\mathbb{W}$ instead of $\mathcal{W}$. I've chosen the definition $\sigma^{\mu\nu} = \frac{i}{4}\sigma^{[\mu}\bar\sigma^{\nu]}$, c.f. (A.1).

## A.4 SUSY NLΣM

The SUSY nonlinear sigma model (NLΣM) provides a nice application of these conventions. We provide a somewhat detailed derivation as a sanity check, but the reader may skip to the final result below. We expand $\Phi$ and $\Phi^\dagger$ about the Grassmann directions so that the expansion parameters are

$$\Delta = \sqrt{2}\theta\psi + \theta\theta F + i\theta\sigma^\mu\bar\theta\,\partial_\mu\phi - \frac{i}{\sqrt{2}}(\theta\theta)(\partial_\mu\psi)\sigma^\mu\bar\theta - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\partial^2\phi \tag{A.29}$$

$$\bar\Delta = \sqrt{2}\bar\theta\bar\psi + \bar\theta\bar\theta F^* - i\theta\sigma^\mu\bar\theta\,\partial_\mu\phi^* + \frac{i}{\sqrt{2}}(\bar\theta\bar\theta)\theta\sigma^\mu(\partial_\mu\bar\psi) - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\partial^2\phi^*. \tag{A.30}$$

We can thus write the Kähler potential as

$$K \to K + K_a\Delta^a + K_{\bar a}\bar\Delta^{\bar a} + \frac{1}{2}K_{ab}\Delta^a\Delta^b + \frac{1}{2}K_{\bar a\bar b}\bar\Delta^{\bar a}\bar\Delta^{\bar b} + K_{a\bar b}\Delta^a\bar\Delta^{\bar b} \tag{A.31}$$

$$+ \frac{1}{2}K_{\bar a bc}\bar\Delta^{\bar a}\Delta^b\Delta^c + \frac{1}{2}K_{a\bar b\bar c}\Delta^a\bar\Delta^{\bar b}\bar\Delta^{\bar c} + \frac{1}{4}K_{ab\bar c\bar d}\Delta^a\Delta^b\bar\Delta^{\bar c}\bar\Delta^{\bar d}. \tag{A.32}$$

where we've written barred indices to denote derivatives with respect to conjugate fields. The $K$ term on the right-hand side carries no Grassmann directions and can so can be dropped under the $d^4\theta$. We may now write out the relevant products of $\Delta$ and $\bar{\Delta}$s.

$$
\begin{aligned}
\Delta\Delta =& 2(\theta\psi)(\theta\psi) - 2\sqrt{2}i(\theta\psi)(\theta\sigma^\mu\bar{\theta})\partial_\mu\phi - (\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta})(\partial_\mu\phi)(\partial_\nu\phi)\\
\Delta\bar{\Delta} =& i(\bar{\theta}\bar{\psi})(\theta\theta)(\partial_\mu\psi\sigma^\mu\bar{\theta}) - i(\theta\psi)(\bar{\theta}\bar{\theta})(\theta\sigma^\mu\partial_\mu\bar{\psi}) + (\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta})(\partial_\mu\phi)(\partial_\nu\phi^*) + (\theta\theta)(\bar{\theta}\bar{\theta})|F|^2 + \cdots\\
\Delta\Delta\bar{\Delta} =& 2\sqrt{2}(\theta\psi)(\theta\psi)(\bar{\theta}\bar{\psi}) - 4i(\theta\psi)(\bar{\theta}\bar{\psi})(\theta\sigma^\mu\bar{\theta})\,\partial_\mu\phi + 2(\theta\psi)(\theta\psi)(\bar{\theta}\bar{\theta})F^* + \cdots\\
\Delta^2\bar{\Delta}^2 =& 4(\theta\psi)(\theta\psi)(\bar{\theta}\bar{\psi})(\bar{\theta}\bar{\psi}).
\end{aligned}
$$

The combinations with more $\bar{\Delta}$s than $\Delta$s are given by the replacement $\theta \leftrightarrow \bar{\theta}$, $\psi \to \psi^*$, and $F \to F^*$. We've dropped indices for simplicity, i.e. a term like $\theta\psi^a\phi^b + \theta\psi^b\phi^a$ is written as $2\theta\psi\phi$. Since all of the indices are summed over in the expansion for $K$, this is a reasonable simplification. (We'll restore indices as necessary below.) To simplify we use some of the handy expressions in the Appendix.

$$
\Delta\Delta|_{\theta^4} = -\frac{1}{2}(\partial\phi)^2 \tag{A.33}
$$

$$
\Delta\bar{\Delta}\big|_{\theta^4} = \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\psi} + \frac{1}{2}|\partial\phi|^2 + |F|^2 \tag{A.34}
$$

$$
\Delta\Delta\bar{\Delta}\big|_{\theta^4} = -i(\psi\sigma^\mu\bar{\psi})(\partial_\mu\phi) + (\psi\psi)F^* \tag{A.35}
$$

$$
\Delta\Delta\bar{\Delta}\bar{\Delta}\big|_{\theta^4} = (\psi\psi)(\bar{\psi}\bar{\psi}). \tag{A.36}
$$

Plugging this in to the expression for $K$ (and ignoring the constant with no support over $d^4\theta$),

$$
\begin{aligned}
K\,d^4\theta = & -\frac{1}{4}K_a\partial^2\phi^a - \frac{1}{4}K_{\bar{a}}\partial^2\phi^{*a} - \frac{1}{4}K_{ab}(\partial\phi^a)(\partial\phi^b) - \frac{1}{4}K_{\bar{a}\bar{b}}(\partial\phi^{*\bar{a}})(\partial\phi^{*\bar{b}})\\
& + K_{a\bar{b}}\left[\frac{i}{2}\psi^a\sigma^\mu\partial_\mu\bar{\psi}^{\bar{a}} - \frac{i}{2}\partial_\mu\psi^a\sigma^\mu\bar{\psi}^{\bar{a}} + \frac{1}{2}(\partial_\mu\phi^a)(\partial^\mu\phi^{*\bar{b}}) + F^aF^{*\bar{b}}\right]\\
& + \frac{1}{2}K_{\bar{a}bc}\left[-i(\psi^b\sigma^\mu\bar{\psi}^{\bar{a}})(\partial_\mu\phi^c) + (\psi^b\psi^c)F^{*\bar{a}}\right] + \frac{1}{2}K_{a\bar{b}\bar{c}}\left[i(\psi^a\sigma^\mu\bar{\psi}^{\bar{b}})(\partial_\mu\phi^{*\bar{c}}) + (\bar{\psi}^{\bar{b}}\bar{\psi}^{\bar{c}})F^a\right]\\
& + \frac{1}{4}K_{ab\bar{c}\bar{d}}(\psi^a\psi^b)(\bar{\psi}^{\bar{c}}\bar{\psi}^{\bar{d}}).
\end{aligned} \tag{A.37}
$$

We now invoke a bit of a trick. (There are other, equivalent, ways to do this, for example by taking supercovariant derivatives or by writing the Lagrangian in terms of Kähler geometric quantities.) Consider the total [spacetime] derivative term,

$$
\partial^2 K = \partial\left(K_a\partial\phi^a + K_{\bar{a}}\partial\phi^{*\bar{a}}\right) \tag{A.38}
$$

$$
\begin{aligned}
= \;& K_{ab}\partial\phi^a\partial\phi^b + K_a\partial^2\phi^a + K_{a\bar{b}}\partial\phi^a\partial\phi^{*\bar{b}}\\
& + K_{\bar{a}\bar{b}}\partial\phi^{*\bar{a}}\partial\phi^{*\bar{b}} + K_{\bar{a}}\partial^2\phi^{*\bar{a}} + K_{a\bar{b}}\partial\phi^a\partial\phi^{*\bar{b}}.
\end{aligned} \tag{A.39}
$$

We can thus add to the Kahler potential a total derivative $\partial^2K/4$. This cancels the first line and adds a factor of 2 to the usual complex scalar kinetic term in the second line so that

$$
\begin{aligned}
K\,d^4\theta = \quad & K_{a\bar{b}}\left[\frac{i}{2}\psi^a\sigma^\mu\partial_\mu\bar{\psi}^{\bar{a}} - \frac{i}{2}\partial_\mu\psi^a\sigma^\mu\bar{\psi}^{\bar{a}} + (\partial_\mu\phi^a)(\partial^\mu\phi^{*\bar{b}}) + F^aF^{*\bar{b}}\right] \\
& + \frac{1}{2}K_{\bar{a}bc}\left[-i(\psi^b\sigma^\mu\bar{\psi}^{\bar{a}})(\partial_\mu\phi^c) + (\psi^b\psi^c)F^{*\bar{a}}\right] \\
& + \frac{1}{2}K_{a\bar{b}\bar{c}}\left[i(\psi^a\sigma^\mu\bar{\psi}^{\bar{b}})(\partial_\mu\phi^{*\bar{c}}) + (\bar{\psi}^{\bar{b}}\bar{\psi}^{\bar{c}})F^a\right] \\
& + \frac{1}{4}K_{ab\bar{c}\bar{d}}(\psi^a\psi^b)(\bar{\psi}^{\bar{c}}\bar{\psi}^{\bar{d}}).
\end{aligned} \tag{A.40}
$$

We can solve the equation of motion for the auxiliary fields,

$$
\frac{\delta\mathcal{L}}{\delta F^{*\bar{a}}} = K_{b\bar{a}}F^b + \frac{1}{2}K_{\bar{a}bc}\psi^b\psi^c = 0. \tag{A.41}
$$

We thus find

$$
F^a = -\frac{1}{2}K^{a\bar{a}}K_{\bar{a}bc}\psi^b\psi^c \equiv -\frac{1}{2}\Gamma^a{}_{bc}\psi^b\psi^c \tag{A.42}
$$

$$
F^{*\bar{a}} = -\frac{1}{2}K^{a\bar{a}}K_{a\bar{b}\bar{c}}\bar{\psi}^{\bar{b}}\bar{\psi}^{\bar{c}} \equiv -\frac{1}{2}\Gamma^{\bar{a}}{}_{\bar{b}\bar{c}}\bar{\psi}^{\bar{b}}\bar{\psi}^{\bar{c}}, \tag{A.43}
$$

where we use upper indices to denote the inverse Kähler metric and have defined the Christoffel symbols. Plugging this back into the Kähler potential,

$$
\begin{aligned}
K\,d^4\theta = \quad & K_{a\bar{b}}\left[\frac{i}{2}\psi^a\sigma^\mu\partial_\mu\bar{\psi}^{\bar{a}} - \frac{i}{2}\partial_\mu\psi^a\sigma^\mu\bar{\psi}^{\bar{a}} + (\partial_\mu\phi^a)(\partial^\mu\phi^{*\bar{b}})\right] \\
& + \frac{1}{2}K_{\bar{a}bc}\left[-i(\psi^b\sigma^\mu\bar{\psi}^{\bar{a}})(\partial_\mu\phi^c)\right] + \frac{1}{2}K_{a\bar{b}\bar{c}}\left[i(\psi^a\sigma^\mu\bar{\psi}^{\bar{b}})(\partial_\mu\phi^{*\bar{c}})\right] \\
& + \frac{1}{4}R_{a\bar{b}c\bar{d}}(\psi^a\psi^c)(\bar{\psi}^{\bar{b}}\bar{\psi}^{\bar{d}}),
\end{aligned} \tag{A.44}
$$

where we've written the Riemann tensor

$$
R_{a\bar{b}c\bar{d}} = K_{a\bar{b}c\bar{d}} - K_{\bar{d}ac}\Gamma^{\bar{d}}{}_{\bar{b}\bar{d}} = K_{a\bar{b}c\bar{d}} - K_{d\bar{b}\bar{d}}\Gamma^d{}_{ac}. \tag{A.45}
$$

We can further simplify by defining the Kähler covariant derivatives

$$
D_\mu\psi^a = \left(\partial_\mu\delta^a_c + \Gamma^a{}_{bc}\partial_\mu\phi^b\right)\psi^c \tag{A.46}
$$

$$
D_\mu\bar{\psi}^{\bar{a}} = \left(\partial_\mu\delta^{\bar{a}}_{\bar{c}} + \Gamma^{\bar{a}}{}_{\bar{b}\bar{c}}\partial_\mu\phi^{*\bar{b}}\right)\bar{\psi}^{\bar{c}}. \tag{A.47}
$$

This allows us to group the $K^(3)$ terms with the $K^(2)$ terms to obtain,

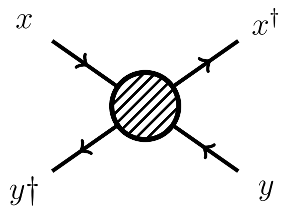
$$
K\,d^4\theta = \quad K_{a\bar{b}}\left[i\psi^a\sigma^\mu\overleftrightarrow{D}_\mu\bar{\psi}^{\bar{a}} + (\partial_\mu\phi^a)(\partial^\mu\phi^{*\bar{b}})\right] + \frac{1}{4}R_{a\bar{b}c\bar{d}}(\psi^a\psi^c)(\bar{\psi}^{\bar{b}}\bar{\psi}^{\bar{d}}), \tag{A.48}
$$

where we use the notation

$$
\overleftrightarrow{D}_\mu = \frac{\overrightarrow{D_\mu} - \overleftarrow{D_\mu}}{2}. \tag{A.49}
$$

## A.5 2-component plane waves

See [117] for details.




## A.6 OLD Notation and Conventions

[**Flip:** please check these for consistency and include above.]

4D Minkowski indices are written with lower-case Greek letters from the middle of the alphabet, $\mu, \nu, \cdots$. We use the particle physics ('West Coast,' mostly-minus) metric for Minkowski space, $ds^2 = (+, -, -, -)$. Typically (but not always) we will write superfields using capital Greek or Roman characters, e.g. $\Phi$ or $Z$. If we stray from this notation we will use the lowest component of the superfield to also refer to the entire superfield, e.g. $\varphi$. The components of a chiral superfield will be written as $\Phi = \phi + \theta\psi + \theta\theta F$. It is a terrible practice, but we will follow the standard convention that we will also refer to the vacuum expectation value of a field by the same symbol when there is no ambiguity, i.e.

$$\langle \Phi \rangle = \Phi,$$

when it is clear from context that the object being considered is the vacuum value, not the dynamical field itself. The Kähler potential and superpotential are denoted by $K$ and $W$ respectively. We will frequently take derivatives of these potentials in field space. For simplicity of notation we will frequently write these field derivatives (technically variations of functionals) compactly as

$$\partial_i W(\Phi_i) \equiv \frac{\delta}{\delta \Phi_i} W(\Phi_i).$$

We will further truncate this by writing

$$W_i \equiv \partial_i \left. W(\Phi_i) \right|_{\Phi = \langle \Phi \rangle}, \tag{A.50}$$

this is the standard notation in the literature.

We will make use of a notation originally inspired by Fernando Quevedo and write the field strength superfield as a 'sophisticated W,' $\mathbb{W}_a$, in order to avoid confusion with the superpotential $W$ or its derivatives $W_a = \partial_a W$. For whatever reason there are many silly normalizations for SUSY gauge theories. We shall employ what we feel is the least silliest (and what is used by [5]),

$$\mathscr{L}_{\text{SYM}} = \frac{1}{16\pi i}\int d^2\theta\ \tau \mathbb{W}^a_\alpha \mathbb{W}^{a\alpha} + \text{h.c.} \tag{A.51}$$

$$= -\frac{1}{4g^2}F^2 - \frac{\Theta_{\text{YM}}}{32\pi^2}F\widetilde{F} + \frac{i}{g^2}\lambda^\dagger\sigma^\mu D_\mu\lambda + \frac{1}{2g^2}D^2 \tag{A.52}$$

$$\mathbb{W}^a_\alpha = -i\lambda^a_\alpha + \theta_\alpha D^a(y) - (\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu}(y) - (\theta\theta)\sigma^\mu D_\mu \lambda^{a\dagger}(y) \tag{A.53}$$

$$\tau = \frac{4\pi i}{g^2} + \frac{\Theta_{\text{YM}}}{2\pi} \tag{A.54}$$

$$\widetilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F^a_{\alpha\beta}, \tag{A.55}$$

where $y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}$.

For SUSY QCD we denote the number of colors by $N$ and the number of flavors by $F$. This is a slight deviation from the canonical review literature which refers to these quantities as $N_c$ and $N_F$ respectively.

We make use of several abbreviations: ISS (Intriligator, Seiberg, Shih; metastable vacua according to [67]), GKK (Giveon, Katz, Komargodski; uplifted metastable vacua according to [123]), SUSY (supersymmetry), ~~SUSY~~ (supersymmetry breaking), MSSM (minimal supersymmetric Standard Model), FCNC (flavor-changing neutral current), WZ (Wess Zumino), LSP (lightest supersymmetric particle), $\chi$SF (chiral superfield), SYM (super Yang-Mills), SQCD (super QCD), DSB (dynamical SUSY breaking), MGM (minimal gauge mediation), OGM (ordinary gauge mediation), EOGM (extraordinary gauge mediation), DGM (direct gauge mediation), sDGM (semi-direct gauge mediation)...

# B The Coleman-Weinberg Effective Potential

The Coleman-Weinberg potential, also known as the [quantum] effective potential, is the potential term in the effective action after taking into account quantum corrections (say, to a given loop order). In other words, it is the potential that determines the vacuum expectation value of fields. Typically in field theory the Coleman-Weinberg potential is a small correction on top of the tree-level potential and it's not usually worth the trouble to calculate. In the case where the tree-level potential is flat, however, the Coleman-Weinberg potential determines the vacuum of the theory and whether or not spontaneous symmetry breaking occurs [124]. This is precisely what we have shown to occur in generalized O'Raifeartaigh models, c.f Theorem 18.4.

There are three roads to deriving the Coleman-Weinberg potential. We shall review each them for pedagogical value.

1. **Quantum Mechanically**: vacuum energy as a harmonic oscillator problem
2. **Diagrammatically**: calculate the vacuum bubble diagrams with vev insertions
3. **Functionally**: identify the momentum-independent part of the quantum effective action

## B.1 Quantum Mechanical Derivation

We begin with the most straightforward procedure[48]. This is the method presented in Dine's Cargese lectures [7]. We want to determine the vacuum energy of the theory. If you remember quantum mechanics from back when you were in kindergarden, you'll remember that it's really easy to calculate vac– I mean, *zero-point* energies. At least it's easy in the case of a harmonic oscillator. Fortunately, quantum fields are nothing but harmonic oscillators. The zero-point energy is

$$V_0 = \frac{1}{2}\hbar\omega = \frac{\omega}{2} \tag{B.1}$$

This is precisely the object that we want to promote to the Coleman Weinberg potential, $V_{\text{CW}} = V_0$. In particular,

$$\begin{aligned} V_{\text{CW}} = \frac{1}{2}\omega &= \frac{1}{2}\int đ^3k\,\sqrt{k^2+m^2} \\ &= \frac{1}{2}\cdot 4\pi \cdot \frac{1}{(2\pi)^3}\int dk\,k^2\sqrt{k^2+m^2} \\ &= \frac{1}{(2\pi)^2}\int dk\,k^3\left(1+\frac{1}{2}\frac{m^2}{k^2}-\frac{1}{8}\frac{m^4}{k^4}+\cdots\right), \end{aligned}$$

where we've written $k$ to mean 3-momentum. In the SUSY gauge theories that we'll be interested in, the mass is generally a function of the pseudomoduli, $m = m(X)$. Some comments are in order. First of all, it's not necessarily obvious why there's an integral over $k$, especially if you're trying to connect to formulae from quantum mechanics. It can sometimes be subtle going form QM $\to$ QFT. Recall that $\omega$ is the frequency (energy) of a *single* quantum mechanical oscillator. In quantum field theory these oscillators are tied together to form fields (see, e.g. Zee chapter 1 [125]). The $\omega$ that appears in the quantum mechanical expression is the term which appears in the potential *for that oscillator*. The potential for a given oscillator depends on the wave mode of the *quantum field*[49]. Thus the integral over the quantum field's momentum $k$ is interpreted by the single quantum mechanical oscillator as a sum over a continuum of different potentials (i.e. different oscillator systems). Next you might be concerned that we are only considering one such quantum oscilator. Indeed, the vacuum energy is given by the contribution from each quantum oscillator. This would just multiply the above result by the (infinite) volume of space. We must recall, however, that the Coleman-Weinberg potential is given by peeling this factor off of the vacuum energy, so our expression above is correct.

There is an additional source of infinities: the $dk$ integral over the first two terms in the sum. These infinities give a dependence on the UV cutoff $\Lambda$ that diverges as $\Lambda \to \infty$. Fortunately, in a theory of supersymmetry these contributions cancel within supermultiplets: for each boson

[48]We thank Felix Yu for sharing this derivation based on notes from Yuri Shirman's lectures.

[49]Here's a handy example if you're confused: consider a string (i.e. a field) with some stationary sinusoidal oscillation. Consider a single point on that string an quantize it. That is, imagine that it oscillates in some other sense (e.g. perpendicular to the string's plane) and that the classical Hooke's coefficient $k_H$ for this oscillation depends on the potential energy relative to the string oscillation. The $\omega$ ($\propto \sqrt{k_H}$) for the quantum mechanical system depends on the displacement of the point relative to the string and hence really depends on the wave number/momentum, $k$, of the string oscillation.

contribution there is a corresponding fermion contribution with opposite sign. This may seem strange, but one must recall that all of these terms are really an expansion in vacuum bubble diagrams with (pseudomoduli-dependent) mass insertions so that fermions really do pick up a minus sign. The cancellation of these terms (particularly the $m^2$ term) still carries over when SUSY is spontaneously broken due to the supertrace rule. Thus for supersymmetric theories the leading contribution comes from the $m^4$ piece,

$$V_{\mathrm{CW}} = \sum_i \frac{1}{(2\pi)^2}(-)^{F_i+1}\int_{m_i}^{\Lambda}\frac{m_i^4}{8}\frac{1}{k} \tag{B.2}$$

$$= \sum_i (-)^{F_i+1}\frac{1}{64\pi^2}m_i^4 \ln\frac{m_i^2}{\Lambda^2} \tag{B.3}$$

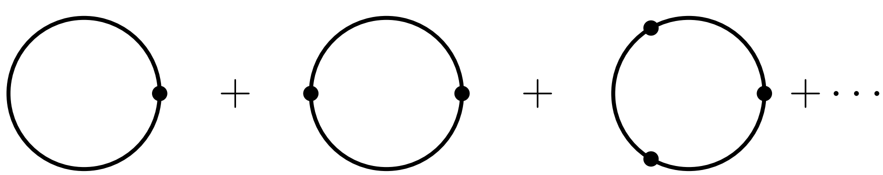
$$= \frac{1}{64\pi^2}\mathrm{STr}M^4 \ln\frac{M^2}{\Lambda^2}, \tag{B.4}$$

where $F_i$ is the fermion number of the state, STr is the supertrace, and $M$ represents the mass matrix for the supermultiplet with eigenvalues $m_i$.

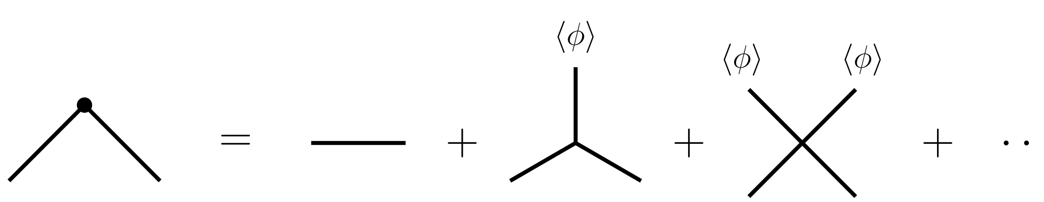
## B.2 Diagrammatic Derivation

While the quantum mechanical derivation is simple, it's perhaps unpalatable to those who are field theorists at heart. When particle physicists calculate things, they want to see an expansion in Feynman Diagrams. For this we turn to Coleman[50], whose pedagogy on this subject are highlighted in his Erice lectures in *Aspects of Symmetry* [25], chapter 5.3 and its appendix. Coleman also does the functional derivation in his section 5.3.4, but we'll get to this in the next section. Coleman also wrote the original paper on this subject [124].

Our strategy is to calculate the vacuum energy via bubble diagrams.

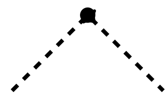



Where we've written the black dot to mean the two-point function including all tree-level $n$-point vertices with $(n-2)$ vev insertions.




[50]We are also grateful to Johannes Heinonen and Jay Hubisz for their insights on this derivation. We made use of Jay's solution set for Csaba Csáki's Physics 662: Quantum Field Theory II course at Cornell University.

We have to sum over all such diagrams, where the solid line can be a scalar, vector, or fermion. We just need to write down the appropriate two-point function for each case. Let's start with the scalar. The two-point function is



$$= -iU''(\phi_{\mathrm{cl}}),$$

where we've written $\phi_{\mathrm{cl}} = \langle \phi \rangle$ for readibility. For different flavors, we make the replacement $U'' \to \partial_i \partial_j U$. The loop diagrams are just a momentum integral with alternating two-point insertions and propagators. Thus the $n^{\mathrm{th}}$ two-point insertion diagram is

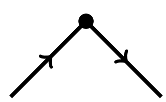
$$\mathcal{M}_n = \frac{1}{2n} \mathrm{Tr} \int \text{đ}^4 k \left( \frac{U''(\phi_{\mathrm{cl}})}{k^2} \right)^n , \tag{B.5}$$

where we've written the symmetry factor $1/2n$ coming from rotations and reflections. We can explicitly do the sum over each diagram,

$$\sum_{n=1}^{\infty} \mathcal{M}_n^{(\mathrm{scalar})} = \frac{1}{2} \mathrm{Tr} \int \text{đ}^4 k \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{U''(\phi_{\mathrm{cl}})}{k^2} \right)^n = -\frac{1}{2} \mathrm{Tr} \int \text{đ}^4 k \, \ln \left( 1 - \frac{U''(\phi_{\mathrm{cl}})}{k^2} \right) . \tag{B.6}$$

where we've used $\sum a^n/n = -\ln(1-a)$. We'll do the integrals in a moment.

Next we can do the same summation for the fermions. The two-point functions are the usual Dirac masses,



$$= im = i(H + A\gamma^5),$$

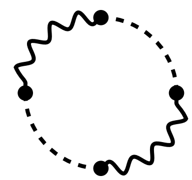
We've explicitly written the mass as a Hermitian plus and anti-Hermitian part as necessary for the Lagrangian to be real. If there are multiple flavors which mix under the mass term, we can add indices as appropriate. Now the $n^{\mathrm{th}}$ two-point contribution is

$$\mathcal{M}_n^{(\mathrm{fermion})} = -\frac{1}{2n} \mathrm{Tr} \int \text{đ}^4 k \left( \frac{1}{\not{k}} m \frac{1}{\not{k}} m^\dagger \right)^n = -\frac{1}{2n} \int \text{đ}^4 k \left( \frac{mm^\dagger}{k^2} \right)^n . \tag{B.7}$$

Once again, we sum this using the same identity,

$$\sum_i \mathcal{M}_n^{(\mathrm{fermion})} = \frac{1}{2} \mathrm{Tr} \int \text{đ}^4 k \, \ln \left( 1 - \frac{mm^\dagger}{k^2} \right) . \tag{B.8}$$

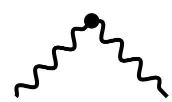
We'd like to move on to the gauge contribution. There are two things that we worry about. First, we have to choose a gauge. Not a big deal. Next, we worry about diagrams with both gauge bosons and scalars running in the loop, like



which come from the coupling of the scalar to a gauge boson. Fortunately, this problem goes away in Landau gauge these graphs vanish. This is because the gauge boson propagator goes like

$$\Delta_{\mu\nu}^{\text{Landau}} = \frac{i}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right). \tag{B.9}$$

The scalar coupling to the gauge boson is proportional to the scalar momentum, as one can check by dimensional arguments. Thus since the mixed scalar-gauge boson diagrams contain terms like $k^\mu \Delta_{\mu\nu} k^\nu = 0$, we don't have to worry about them. Instead, all we need to include are the vector mass insertions,



$$= iM.$$

As before we can include indices if a gauge group is broken so that gauge bosons mix (e.g. $B_\mu$ and $W^3_\mu$ mixing). The $n^{\text{th}}$ diagram is

$$\mathcal{M}_n^{(\text{gauge})} = \frac{1}{2n}\text{Tr}\int đ^4k \left[\frac{1}{k^2}\left(\delta^\mu{}_\nu - \frac{k^\mu k_\nu}{k^2}\right)M^2\right]^n \tag{B.10}$$

$$= \frac{1}{2n}\text{Tr}\int đ^4k \left(\delta^\mu{}_\mu - \frac{k^\mu k_\mu}{k^2}\right)\left(\frac{M^2}{k^2}\right)^n \tag{B.11}$$

$$= \frac{3}{2n}\text{Tr}\int đ^4k \left(\frac{M^2}{k^2}\right)^n, \tag{B.12}$$

where we were a little sloppy with indices in the first line, but made it clear that the indices all contract. It is not hard to see that

$$\left(\delta^\mu{}_\nu - \frac{k^\mu k_\nu}{k^2}\right)^n = \left(\delta^\mu{}_\mu - \frac{k^\mu k_\mu}{k^2}\right), \tag{B.13}$$

where on the left-hand side we assume that the indices are contracted. You can check the case $n = 2$ and prove inductively. The sum gives

$$\sum_i \mathcal{M}_n^{(\text{gauge})} = -\frac{3}{2}\text{Tr}\int đ^4k \,\ln\left(1 - \frac{M}{k^2}\right). \tag{B.14}$$

Great! All of the integrals take the same form, so we can just do them all in one fell stroke.

$$\int đ^4k \,\ln\left(1 - \frac{a}{k^2}\right) = \frac{i}{16\pi^2}\int_0^{\Lambda^2} dk_E^2 \,\ln\left(1 + \frac{a}{k_E^2}\right). \tag{B.15}$$

One can explicitly do the integral on the right-hand side using the usual tricks, e.g. invoking the appendix in Peskin and Schroeder [41]. Or, more practically, one can plug it into *Mathematica*,

$$= \frac{i}{32\pi^2}\left[\Lambda^2 a + \Lambda^4 \ln\left(1+\frac{a}{\Lambda^2}\right) - a^2\ln(\Lambda^2+a) - a^2\left(1+\frac{a}{\Lambda^2}\right) + a^2\ln a\right] \tag{B.16}$$

$$= \frac{i}{32\pi^2}\left[2\Lambda^2 a - \frac{1}{2}a^2 - a^2\ln\Lambda^2 + a^2\ln a + \mathcal{O}(\Lambda^{-2})\right] \tag{B.17}$$

Great. Plugging this in we get a nasty general formula

$$\begin{aligned} V_{\text{CW}} = \;& \frac{\Lambda^2}{32\pi^2}\text{Tr}\left[U''(\phi_{\text{cl}}) - mm^\dagger + 3M^2\right] \\ &+ \frac{1}{128\pi^2}\text{Tr}\left[(U''(\phi_{\text{cl}}))^2 - \left(mm^\dagger\right)^2 + 3(M^2)^2\right] \\ &+ \frac{1}{64\pi^2}\text{Tr}\left[(U''(\phi_{\text{cl}}))^2\ln\frac{U''(\phi_{\text{cl}})}{\Lambda^2} - \left(mm^\dagger\right)^2\ln\frac{mm^\dagger}{\Lambda^2} + 3(M^2)^2\ln\frac{M^2}{\Lambda^2}\right]. \end{aligned} \tag{B.18}$$

What a mess! This is the formula that you'd want to scribble down on your 'handy general formulae' page. For this current document, all of our Lagrangians are supersymmetric, so several cancellations occur. Let us ignore the gauge bosons (practically set $M = 0$), then the sums between $U''(\phi_{\text{cl}})$ and $mm^\dagger$ are really supertraces. Thus the first two lines of the above formula all cancel, and we're left with the usual formula, Eq. (B.4).

Before we move on, let's address a point about the loop expansion. One might wonder in which sense the loop expansion is valid, i.e. how do we explain the loop expansion in terms of some expansion parameter? Coleman (see also Srednicki chapter 21 [126]) shows us how to do this by parameterizing the loop expansion by a dimensionless parameter that we will suggestively call $\hbar$ [127]. We will set $\hbar = 1$ after we've proved what we wanted. Let us write the Lagrangian in terms of $\hbar$ as

$$\mathscr{L}(\hbar) = \frac{1}{\hbar}\mathscr{L}. \tag{B.19}$$

For a given Feynman diagram, we now define $P$ to be the power of $\hbar$ appearing in the expression for that graph. Each propagator carries a power of $a$ since it is the inverse of the kinetic term. Each interaction gives a power of $a^{-1}$. Thus, if we write $I$ be the number of internal lines and $V$ be the number of vertices, we have

$$P = I - V. \tag{B.20}$$

From the usual graph-ology, we know that the number of loops $L$ is given by

$$L = I - V + 1. \tag{B.21}$$

You can prove this by counting $\delta$ functions over momentum, appealing to fancy-schmancy graph theory, or just drawing a few diagrams and convincing yourself. Combining these equations, we get

$$P = L - 1, \tag{B.22}$$

so that indeed, $\hbar$ counts the number of loops. Great. Now what? Alright, so a loop expansion corresponds to an expansion in $\hbar$. We still want to understand why this expansion is meaningful. When we draw Feynman diagrams, we are expanding in small couplings. But we certainly aren't claiming that $\hbar$ is a small parameter: we set it to one. Instead, (quoting Coleman)

> The point is, rather, since the loop expansion corresponds to expansion in a parameter that multiplies the total Lagrange density, it is unaffected by shifts of fields, and by the redefinition of the division of the Lagrangian into free and interacting parts associated with such shifts.

## B.3 Functional Derivation

We've now given two derivations for the Coleman-Weinberg potential. The quantum mechanical derivation was quick and easy. The diagrammatic derivation was intuitive. Now we review a third derivation which has the benefit of field theoretic elegance. It is based on the functional integral. This method is a little "old school" and is what you would find in books that refer to things like "skeleton diagrams." Modern textbooks with useful presentations include those by Greiner [128] (whose notation is a bit odd in that $W[J] \leftrightarrow Z[J]$), Srednicki [126], and Banks [14]. A nice treatment can be found in the lecture by Hugh Osborn [129]. An explicit calculation can be found in Peskin (who writes $W[J]$ as $-E[J]$) chapter 11.4 [41], which in turn follows the surprisingly readable paper by Jackiw from 1974 [130]. We will roughly follow these last two references.

The general strategy is to calculate (practically to some low order in a loop expansion) the full 1PI quantum effective action $\Gamma$ and then read off the momentum-independent term, i.e. the term which survives when we specialize to a constant background field $\phi_{\text{cl}}$. We discuss some nuances about the effective action in Section 2.1, but for now it is sufficient to identify it as the action whose tree level matrix elements represent a summation over all loop diagrams contributing to the process. In other words, the vertices are actually 'blob' vertices which include a sum over 1PI contributions.

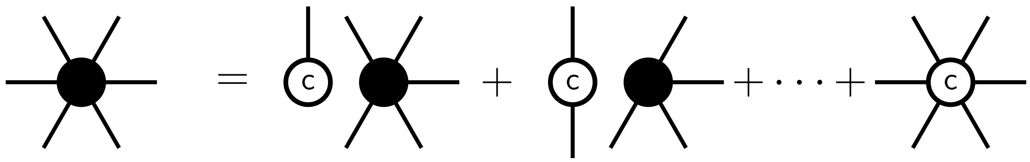
Let us remind ourselves about the usual objects in the path integral formalism. The generating functional of Green's functions is a function of the source $J(x)$ via

$$Z[J] = \int d\phi \, e^{iS[\phi]+iJ(x)\phi(x)}, \tag{B.23}$$

which allows us to calculate $n$-point Greeen's functions by taking $n$ functional derivatives of the source at the point in function space $J(x) = 0$. This generically is a sum over connected and disconnected diagrams. It turns out that $W[J]$, the generator of only connected diagrams, has a simple relation to $Z[J]$,

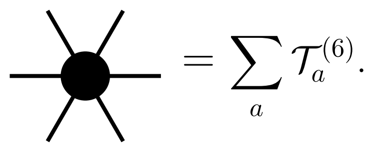
$$Z[J] = e^{iW[J]}. \tag{B.24}$$

*Proof.* For posterity, let's discuss why this is true. This is easiest to see diagrammatically.

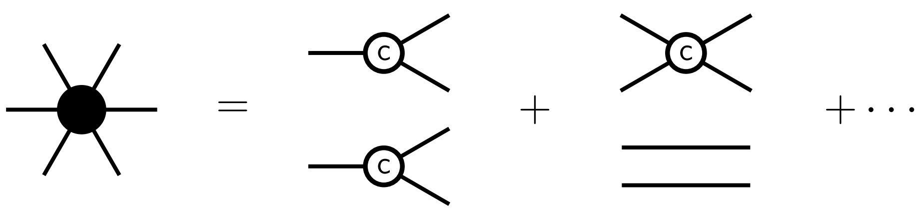


(B.25)

The black blobs represent the Green's function (connected and disconnected contributions) while the white blobs are connected Green's functions. Each external line represents a functional derivative with respect to $J(x_i)$, where $x_i$ is the endpoint of the external line. Each black blob on the right-hand side also has an expansion in products of lower order blobs. Each term in the sum, we'll call it $\mathcal{T}_a$ for transition matrix element[51],



Each $\mathcal{T}_a$ is a diagram which is generically disconnected. Let us write connected diagrams as $\mathcal{M}_i$. As an explicit diagrammatic example,



Each term on the right-hand side is one $\mathcal{T}_a$, while each connected diagram contributing to a given $\mathcal{T}_a$ is a $\mathcal{M}_i$. This is of course just a heuristic rewriting of Eq. (B.25) where we've fully expanded each Green's function (black blobs) in terms of connected Green's functions. The first term on the right-hand side is just $(\mathcal{M}_3)^2$. This contribution implicitly contains a symmetry factor between each identical connected piece[52]. We can write this out explicitly as

$$\mathcal{T}_a = \frac{1}{S_a} \prod_i (\mathcal{M}_i)^{n_i} . \tag{B.26}$$

The symmetry factor only counts the interchange of identical connected diagrams so is given by

$$S_a = \prod_i n_i!. \tag{B.27}$$

[51]Recall that the matrix element, $\mathcal{M}$, is a component of the scattering matrix, $S = T - 1$.
[52]This is not the same as the symmetry factor for a given connected diagram, which we will keep implicit.

Let us now write out the generating function of Green's functions, $Z[J]$. By definition, this is just the sum of all diagrams (up to a normalization which we ignore)

$$\begin{aligned} Z[J] &= \sum_n \sum_a \mathcal{T}_a{}^{(n)} \\ &= \sum_{\{n_i\}} \prod_i \frac{1}{n_i!} \left(\mathcal{M}_i\right)^{n_i} \\ &= \prod_i e^{\mathcal{M}_i} = e^{\sum_i \mathcal{M}_i}, \end{aligned} \tag{B.28}$$

where $\sum_i \mathcal{M}_i \equiv W[J]$ by definition. This then gives the desired result.

Let us prove this in a slightly more rigorous way. We can write out Eq. (B.25) more technically as

$$\begin{aligned} \left(\frac{\delta}{\delta J}\right)^n Z[J]_{J=0} &= i \sum_{r=1}^{n} \sum_{\text{comb.}} \left(\frac{\delta}{\delta J}\right)^r W[J]_{J=0} \cdot \left(\frac{\delta}{\delta J}\right)^{(n-r)} Z[J]_{J=0} \\ &= i \sum_{r=0}^{n-1} \sum_{\text{comb.}} \frac{\delta}{\delta J(x_1)} \left(\frac{\delta}{\delta J}\right)^r W[J]_{J=0} \cdot \left(\frac{\delta}{\delta J}\right)^{(n-r-1)} Z[J]_{J=0}, \end{aligned} \tag{B.29}$$

where we've explicitly written a sum over combinations of the external points $\{x_i\}$ but for simplicity of notation suppressed the position of each functional derivative. In the second line we pulled out an explicit factor of $\delta/\delta J(x_1)$ for future convenience. We can write the sum over combinations more explicitly as

$$\sum_{\text{comb.}} = \sum_{\{i_1,\cdots,i_r\}\subset\{1,\cdots n\}} = \frac{1}{n!} \sum_{\text{perm.}} \binom{n}{r}, \tag{B.30}$$

where 'perm.' means a sum over permutations of $\{1, \cdots, n\}$. Then we may invoke the generalized Leibniz rule,

$$\frac{d^n}{dx^n}\left(f(x)g(x)\right) = \sum_{r=0}^{n} \binom{n}{r} \frac{d^r}{dx^r} f(x) \cdot \frac{d^{n-r}}{dx^{n-r}} g(x). \tag{B.31}$$

Plugging this into Eq. (B.29), we get

$$\frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J]_{J=0} = i \frac{\delta}{\delta J(x_2)} \cdots \frac{\delta}{\delta J(x_n)} \left(\frac{\delta W[J]}{\delta J(x_1)} Z[J]\right)_{J=0}. \tag{B.32}$$

This result is manifestly symmetric in the $x_i$ and so the sum over permutations gives a factor of $n!$ which just cancels the $1/n!$ above. Since this equation holds for each value of $n$, we can reduce it to a simple [functional] differential equation, (i.e. the differential equation holds at each order in the Taylor expansion)

$$\frac{\delta}{\delta J(x_1)} Z[J]_{J=0} = i \frac{\delta W[J]_{J=0}}{\delta J(x_1)} Z[J]_{J=0}, \tag{B.33}$$

whose solution is simply Eq. (B.24). $\square$

Ok, that was a bit of a long aside. Let's move on to the 1PI quantum effective action, $\Gamma[\phi_{\rm cl}]$. We *define* $\Gamma$ to be the generator of 1PI diagrams. This means that if we treated $\Gamma$ to be the action of the theory, the tree-level diagrams would be exact (quantum mechanically) and there would be no loop corrections to those diagrams. In other words $\Gamma$ generates diagrams that already include loop effects. In practice, of course, this can only be calculated to a given order in a loop expansion. Let's see how we can formalize this. Let's define a generating functional $Z_\Gamma$ and a generating functional of connected graphs $W_\Gamma$ associated with this effective action,

$$Z_\Gamma[J] = \int d\phi \, e^{i\Gamma[\phi]+i\int d^dx\, J(x)\phi(x)} = e^{iW_\Gamma[J]}. \tag{B.34}$$

$W_\Gamma$ is a sum of connected diagrams whose internal lines are exact propagators and whose vertices are 1PI. By definition the restriction of $W_\Gamma$ to tree-level diagrams is equivalent to the usual unrestricted $W$. We can use this to get a handle on $\Gamma$, but first we need to figure out how to restrict $W_\Gamma$ to one-loop diagrams.

Fortunately we already discussed how to do this at length in the previous section when we did the diagrammatic derivation of the Coleman-Weinberg potential. We found that the natural parameter that counted the powers of loops is $\hbar$. Restoring this dependence, we have the

$$Z_{\Gamma,\hbar}[J] = \int d\phi \, e^{\frac{i}{\hbar}\left(\Gamma[\phi]+\int d^4x\, J(x)\phi(x)\right)} = e^{iW_{\Gamma,\hbar}[J]}, \tag{B.35}$$

where we know how to write the expansion in $\hbar$ in terms of number of loops,

$$W_{\Gamma,\hbar}[J] = \sum_L^\infty \hbar^{L-1} W_{\Gamma,L}[J]. \tag{B.36}$$

So our first step in connecting $\Gamma$ to our usual objects, $Z[J]$ and $W[J]$ is the relation

$$W[J] = W_{\Gamma,L=0}[J]. \tag{B.37}$$

We can go on and bring $\Gamma$ into the mix by evaluating $Z_{\Gamma,\hbar}$ in Eq. (B.35) using the stationary phase approximation,

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = -J(x). \tag{B.38}$$

This is sometimes called the quantum equation of motion. Define the **classical field** to be the field configuration $\phi(x) = \phi_{\rm cl}(x)$ that satisfies this equation. Then the generating functional associated with $\Gamma$ can be written as

$$Z_{\Gamma,\hbar} = \exp\left[\frac{i}{\hbar}\left(\Gamma[\phi_{\rm cl}] + \int d^4x\, J(x)\phi_{\rm cl}(x)\right) + \mathcal{O}(\hbar^0)\right]. \tag{B.39}$$

Putting this together with our loop expansion in $\hbar$ we get the important relation

$$\Gamma[\phi_{\rm cl}] = -W[J] + \int d^4x\, J(x)\phi_{\rm cl}(x). \tag{B.40}$$

In other words, $\Gamma[\phi_{\text{cl}}]$ is the Legendre transform of $W[J]$. Some treatments take this as the definition of the effective action and from there derive the more intuitive definition above, though we find it is more instructive to do things in this order.

We can better motivate the name 'classical field' by remembering that in the background $|\Omega\rangle$ of some general source $J(x)$, the 'background' value of the field is

$$\begin{aligned}
\langle\Omega|\phi(x)|\Omega\rangle_J &= \frac{\delta W[J]}{\delta J(x)} \\
&= \frac{\delta\Gamma[\phi_{\text{cl}}]}{\delta J(x)} + \phi_{\text{cl}}(x) + \int d^4y\, J(y)\frac{\delta\phi_{\text{cl}}(y)}{\delta J(x)} \\
&= \int d^4y\, \frac{\delta\Gamma[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(y)}\frac{\delta\phi_{\text{cl}}(y)}{\delta J(x)} + \phi_{\text{cl}}(x) + \int d^4y\, J(y)\frac{\delta\phi_{\text{cl}}(y)}{\delta J(x)} \\
&= \int d^4y\, \frac{\delta\phi_{\text{cl}}(y)}{\delta J(x)}\left(\frac{\delta\Gamma[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(y)} + J(y)\right) + \phi_{\text{cl}}(x) \\
&= \phi_{\text{cl}}(x).
\end{aligned} \tag{B.41}$$

Good. Now that we've thoroughly reviewed the basics, let's calculate the Coleman Weinberg potential. Our method will not be direct, but I promise it will be elegant. First let's expand about the classical field,

$$\phi(x) = \phi_{\text{cl}}(x) + \varphi(x). \tag{B.42}$$

Now consider the generator of connected diagrams,

$$\begin{aligned}
W[J] = &\int d^4x\, \left(\mathscr{L}[\phi_{\text{cl}}] + J(x)\phi_{\text{cl}}(x)\right) \\
&+ \int d^4x\, \varphi(x)\left(\frac{\delta\mathscr{L}}{\delta\phi} + J(x)\right) \\
&+ \frac{1}{2}\int d^4x\, d^4y\, \varphi(x)\varphi(y)\frac{\delta^2\mathscr{L}}{\delta\phi(x)\delta\phi(y)} \\
&+ \frac{1}{3!}\int d^4x\, d^4y\, d^4z\, \varphi(x)\varphi(y)\varphi(z)\frac{\delta^3\mathscr{L}}{\delta\phi(x)\delta\phi(y)\delta\phi(z)} + \cdots .
\end{aligned} \tag{B.43}$$

Let's drop the terms of $\mathcal{O}(\varphi^3)$ and perform the quadratic integral. The Gaussian integral is our bread-and-butter tool for path integrals, so you knew this was coming. Using the usual manipulations, we can write the generating functional $Z[J]$ as

$$\begin{aligned}
&\int d\varphi\, \exp\left[i\int d^4x\, \left(\mathscr{L}[\phi_{\text{cl}}] + J(x)\phi_{\text{cl}}(x)\right) + \frac{i}{2}\int d^4x\, d^4y\, \varphi(x)\frac{\delta^2\mathscr{L}}{\delta\phi(x)\delta\phi(y)}\varphi(y)\right] \\
&\qquad = \exp\left[i\int d^4x\, \left(\mathscr{L}[\phi_{\text{cl}}] + J(x)\phi_{\text{cl}}(x)\right)\right]\left(\det\left[-\frac{\delta^2\mathscr{L}}{\delta\phi\delta\phi}\right]\right)^{-1/2},
\end{aligned} \tag{B.44}$$

We can see explicitly the classical contribution and the first order contribution from quantum corrections. If we included higher order terms in $\varphi$ we would get a Feynman diagram expansion with respect to the classical background field. We can take the logarithm of $Z$ to get

$$iW[J] = i\int d^4x\,(\mathscr{L}[\phi_{\rm cl}] + J(x)\phi_{\rm cl}) - \frac{1}{2}\det\left[-\frac{\delta^2\mathscr{L}}{\delta\phi\delta\phi}\right] + \cdots\,, \tag{B.45}$$

where the "$\cdots$" represents connected diagrams and counter terms. We're not going to worry too much about the counter terms since we know from that in the supersymmetric limit there are no UV divergences that we have to regulate. But if you wanted to be precise, you would need to replace $\mathscr{L} \to \mathscr{L} + \mathscr{L}_{\text{c.t.}}$, where the first term (what we've written explicitly in our derivation here) is the renormalized Lagrangian and the second term contains counter terms. We would then need to identify $J(x)$ as the *renormalized* source which satisfies

$$\frac{\delta\mathscr{L}[\phi_{\rm cl}]}{\delta\phi(x)} + J(x) = 0, \tag{B.46}$$

and a counter term source $\delta J(x)$ which acts to enforce $\langle\phi(x)\rangle_J = \phi_{\rm cl}$. Upon expanding about $\phi_{\rm cl}$, the counter term Lagrangian just provides the usual counter term vertices and an overall constant that can be used to satisfy renormalization conditions for any divergences in the functional determinant. Those who really want to be careful with counter terms can follow the exposition in Peskin's chapter 11.4 [41].

We learned above that to get the effective action (finally!) we just take a Legendre transform of this object. We obtain

$$\Gamma[\phi_{\rm cl}] = \int d^4x\,\mathscr{L}[\phi_{\rm cl}] + \frac{i}{2}\ln\det\left[-\frac{\delta^2\mathscr{L}}{\delta\phi\delta\phi}\right] - i(\cdots). \tag{B.47}$$

As a sanity check, note that there is no $J(x)$ dependence. $\Gamma$ is only a function of $\phi_{\rm cl}$. Alright. Now we're getting somewhere. The effective potential is the momentum-independent part of the effective action, i.e. the part that isn't kinetic. It's easy to identify this: we just have to specialize to the case of a constant background classical field. Then $V_{\rm CW} = -\Gamma[\phi_{\rm cl}]/(\text{vol})$, with $\phi_{\rm cl} = \text{const}$ and the volume of spacetime being factored out,

$$V_{\rm CW} = V(\phi_{\rm cl}) - \frac{i}{2(\text{volume})}\ln\det\left[-\frac{\delta^2\mathscr{L}}{\delta\phi\delta\phi}\right] + \cdots \tag{B.48}$$

To calculate these functional determinants we use the handy relation

$$\ln\det\Delta = \operatorname{Tr}\ln\Delta, \tag{B.49}$$

where the trace is over eigenvalues of the operator $\Delta$. For our purposes,

$$\frac{\delta^2\mathscr{L}}{\delta\phi\delta\phi} = \partial^2 - U''(\phi_{\rm cl}) = \partial^2 + m^2 + \cdots\,. \tag{B.50}$$

Let's assume that $U'' = -m^2$. Since $U''$ is constant (because $\phi_{\rm cl}$ is constant), the eigenfunctions are plane waves whose eigenvalues are $-k^2+m^2$. The trace over the logarithm of these eigenvalues can be defined rigorously by taking the continuum limit of a discrete system (e.g. a large box),

$$\sum_k \ln(-k^2+m^2) \to (\text{volume})\int \text{đ}^4k\,\ln(-k^2+m^2). \tag{B.51}$$

Plugging this in and doing an implicit Wick rotation, we get

$$V_{\text{CW}} = V(\phi_{\text{cl}}) + \frac{1}{2}\int d\!\!{}^-\!{}^4k \, \ln(k^2 + m^2). \tag{B.52}$$

This now is now of the same form as the integrals of logarithms that we did in the previous section. Just to show off a little, we'll pull out a few more tricks to do these integrals explicitly. First we'll use a handy representation of the natural logarithm,

$$-\ln\frac{a}{b} = \int_0^\infty \frac{dz}{z}\left(e^{-az} - e^{-bz}\right). \tag{B.53}$$

We can use this with $a = k^2 + m^2$ and $b = 1$ to let us write the quantum correction as

$$-\frac{1}{2}\int d\!\!{}^-\!{}^4k \, \ln(k^2 + m^2) = \frac{1}{2}\int d\!\!{}^-\!{}^4k \int_0^\infty \frac{dz}{z}\left(e^{-(k^2+m^2)z} - e^{-z}\right). \tag{B.54}$$

The second term on the right-hand side is divergent and will ultimately be eaten by counter terms, so we'll just drop it like it's hot. The next trick that we'll do is to perform the $d\!\!{}^-\!{}^4k$ integral, which is now Gaussian.

$$\begin{aligned}
\frac{1}{2}\int d\!\!{}^-\!{}^4k \, \ln(k^2 + m^2) &= -\frac{1}{2}\int_0^\infty \frac{dz}{z} e^{-m^2 z} \int d\!\!{}^-\!{}^4k \, e^{-k^2 z} \\
&= -\frac{1}{2}\int_0^\infty \frac{dz}{z} e^{-m^2 z} \frac{1}{(4\pi z)^{d/2}} \\
&= -\frac{1}{2(4\pi)^{d/2}}\int_0^\infty dz \, z^{-1-d/2} e^{-m^2 z} \\
&= \frac{1}{(4\pi)^{d/2}}\frac{1}{d}\int_0^\infty dz \left(\frac{d}{dz} z^{-d/2}\right) e^{-m^2 z} \\
&= \frac{m^2}{(4\pi)^{d/2}}\frac{1}{d}\int_0^\infty dz \, z^{-d/2} e^{-m^2 z} \\
&= \frac{m^d}{(4\pi)^{d/2}}\Gamma\left(1 - \frac{d}{2}\right) = -\frac{m^d}{2(4\pi)^{d/2}}\Gamma\left(-\frac{d}{2}\right)
\end{aligned}$$

Using the $\overline{\text{MS}}$ renormalization prescription with a renormalization scale $\mu$ to remove divergences as $d = 4 - \epsilon \to 4$, this gives us

$$-\frac{m^d}{2(4\pi)^{d/2}}\Gamma\left(-\frac{d}{2}\right) = \frac{m^4}{64\pi^2}\left(\ln\left(\frac{m^2}{\mu^2}\right) - \frac{3}{2}\right). \tag{B.55}$$

This gives the leading quantum correction to the classical potential. One will note that the term with the logarithm matches that of Eq. (B.18), where we use $m^2 \to -U''(\phi_{\text{cl}})$. However, the term proportional to $(U'')^2$ does not appear to match. Osborn notes that this is due to the arbitrariness of our renormalization prescription, in particular the freedom to add a finite counter term that is a quartic polynomial in $\phi$ [129]. This would affect the coefficient of $(U'')^2$ so that this coefficient cannot be physical. One will note, on the other hand, that the coefficient of the $(U'')^2 \ln U''$ term *is* physical. These ambiguities can be removed by specifying the derivatives of the effective potential.

## B.4 Integrating out fields

As we know from our study of supersymmetric QCD, of the knobs that we have to play with[53] is to integrate out massive fields. Intriligator and Seiberg make some important pedagogical notes about what this means for our cherished arguments of holomorphy and the effective potential [2, 1]. Consider a superpotential which (say, at some point on the pseudomoduli space) takes the form

$$W = \frac{1}{2}\Phi^a M_{ab}\Phi^b + \cdots . \tag{B.56}$$

Integrating out $\Phi$ will give us an effective *Kähler* potential of the form

$$K_{\text{eff}} = -\frac{1}{32\pi^2}\text{Tr}\left[MM^\dagger \log\left(\frac{MM^\dagger}{\Lambda^2}\right)\right]. \tag{B.57}$$

[**Check:** Check this formula, e.g. see `hep-th/9605149` or Kuzenko. I probably need to do some supergraph calculations.] Now, in the limit of small SUSY-breaking, we can use this effecive Kähler potential as a trick to approximate the Coleman-Weinberg potential. Suppose that the mass matrix $M$ depends on the pseudomodulus $X$. Then the approximate CW potential, which Intriligator and Seiberg call the 'truncated' potential, is

$$V_{\text{trunc}} = (K_{\text{eff}\,X,\overline{X}})^{-1}|\partial_X W|^2. \tag{B.58}$$

This is just the tree-level scalar potential that one would get with a non-trivial Kähler potential. $V_{\text{trunc}}$ approximates $V_{\text{CW}}$ to leading order in

$$F_X = -(K_{\text{eff}\,X,\overline{X}})^{-1}\overline{\partial_X W}. \tag{B.59}$$

This is verified in the ISS paper [67].

One ought to be careful at the origin of a theory where fields have been integrated out. At the points on the pseudomoduli space (usually the origin) where the integrated-out fields become massless, the effective theory becomes singular. This non-analyticity is the way the theory is telling us that another degree of freedom becomes operative, i.e. our effective theory is breaking down. This is of course what we would expect since it makes no sense to integrate out a massless (or very light relative to the scale) field.

## B.5 An illustrative example

In their SUSY-breaking lecture notes, Intriligator and Seiberg make some important notes about the Coleman-Weinberg potential [1]. We will follow their analysis of a simple metastable SUSY-breaking model (from Section 2.3 of their lectures) to do a sample calculation and explore some further technical details.

The simple model is composed of two chiral superfields, $X$ and $q$ with a canonical Kähler potential and a superpotential

$$W = \frac{1}{2}hXq^2 + fX. \tag{B.60}$$

[53] Giggle if you're British.

This already has a form that is similar to the ISS model. We note that if the first term were absent this would simply be the Polonyi model $W = fX$, which is a simple (the simplest?) SUSY-beaking model. A simple analysis of the potential for this model, however, yields

$$V = |hXq|^2 + \left|\frac{1}{2}hq^2 + f\right|^2$$

so that supersymmetric vacua exist at $\langle X\rangle = 0$ and $\langle q\rangle = \pm\sqrt{-2f/h}$. Now some SUSY intuition should kick in: when restricted to the submanifold $\langle q\rangle = 0$, there is a pseudoflat direction parameterized by $\langle X\rangle$. We can then move to a region of large $\langle X\rangle$, where the $q$s thus obtain large mass terms and can be integrated out. In this regime we return to the SUSY-breaking Polonyi model. Thus we can start thinking about constructing a metastable vacuum along this pseudomoduli. (One can pause to briefly reflect on how this is a simple case of the ISS 'macroscopic model I' in Section 20.2.)

Let's start by considering the spectrum along this pseudoflat direction. We can be optimistic and hope that the Coleman-Weinberg potential stabilizes a SUSY-breaking vacuum. (It will not.) The quarks obtain masses

$$m_0^2 = |hX|^2 \pm |hf| \qquad\qquad m_{1/2} = hX,$$

where we've been lazy and have written $X = \langle X\rangle$. We note immediately that the squarks are tachyonic if $|X|^2 < |f/h|$. This means the potential slopes downward along the $\langle q\rangle$ direction down to the supersymmetric vacuum described above. That's fine, we should have expected this to happen at tree-level since we already knew the lower-energy SUSY vacuum existed. Let's work in the non-tachyonic regime $|X|^2 > |f/h|$ so that we may expand in the parameter

$$z \equiv \left|\frac{f}{X^2 h}\right|.$$

Let's work out how the Coleman-Weinberg potential lifts the pseudomodulus. Recall that

$$V_{\mathrm{CW}} = \frac{1}{64\pi^2}\mathrm{STr}\left(\mathcal{M}^4 \log\frac{\mathcal{M}^2}{\Lambda^2}\right), \tag{B.61}$$

where $\mathcal{M}$ is the classical (i.e. tree-level) mass matrix. Thus the Coleman-Weinberg potential for the pseudomodulus $X$ is

$$V_{\mathrm{CW}}^{(1)} = -\frac{1}{64\pi^2}2|hX|^4 \log\frac{|hX|^2}{\Lambda^2} + \sum_{\pm}\frac{1}{64\pi^2}\left(|hX|^2 \pm |hf|\right)^2 \log\frac{|hX|^2 \pm |hf|}{\Lambda^2} \tag{B.62}$$

$$= \frac{|hf|^2}{32\pi^2}\left[\log\left|\frac{hX}{\Lambda}\right|^2 + \frac{3}{2} - \frac{z^2}{12} + \mathcal{O}(z^4)\right], \tag{B.63}$$

where we've expanded in our 'small parameter' $z$. This potential lifts the degeneracy of the pseudomoduli $\langle X\rangle$ (recall that for brevity we've been habitually droping the angle brackets) in such a way that the potential increases with $|X|$. Thus we see that the Coleman-Weinberg potential is indeed pushing us back into the tachyonic region that we were hoping to avoid.

So we've now worked through a very simple example of not-quite metastale SUSY-breaking. It's nice to see an example where the effective potential does *not* stabilize our pseudomoduli where we want, since most papers only present successful cases. In practice when dealing with larger global symmetries (e.g. super QCD with some number of flavors) it can become very tedious to calculate pseudomoduli by hand. One can usually get away with tricks to determine the stability of the pseudomoduli space (e.g. in the ISS macro model discussed in Section 20.2.4), but to compute the entire one-loop Coleman-Weinberg effective potential one generally has to diagonalize mass matrices via some computer algebra system like *Mathematica*.

One interesting development on this front is a computational tool by Korneel van den Broek called *Vscape* [131]. It is a software package that calculates the effective potential for the pseudomoduli space of an ungauged theory of chiral superfields, such as the ISS macroscopic model I.

## B.6 Cutoff dependence

Now it's somewhat important to discuss the cutoff dependence of the Coleman-Weinberg formula that we've derived. Let's focus on the case of a supersymmetric theory, where we do not have any divergent terms in the CW effective potential, but we still have the explicit appearance of $\Lambda$. One might be somewhat perturbed by the $\Lambda$ in the formula for the effective potential: what does it mean and how do we pick it to get meaningful results?

It turns out that the cutoff dependence can be removed explicitly if we work with running couplings [1]. Now things are starting to sound familiar from the standard theory of renormalization. Consider the simple illustrative example in the previous section, where we are again living on the pseudomoduli $\langle q \rangle = 0$. One can see that Eq. (B.63) has explicit terms with $|\Lambda|$ appearing. Let us define the running coupling

$$f(\mu) = f_0 \left[ 1 + \frac{|h|^2}{64\pi^2} \left( \frac{3}{2} + \log \frac{\mu^2}{\Lambda^2} + \mathcal{O}(h^4) \right) . \right] \tag{B.64}$$

We will motivate this in a moment. Let us first behold a 'miracle': with respect to this running coupling, the Coleman-Weinberg potential is independent of the cutoff $\Lambda$:

$$V_{\text{CW}} = |f(|hX|)|^2 \left[ 1 - \frac{|h|^2}{32\pi^2} \left( -\frac{z^2}{12} + \mathcal{O}(z^4) \right) + \mathcal{O}(h^4) \right] , \tag{B.65}$$

where we've evaluated $f(\mu)$ at the scale of the massive fields $q$: $\mu = |hX|$.

We review super QCD below, but you might wonder why we're talking about a running coupling when we know from Seiberg-ology that the holomorphic couplings in the superpotential do not run, i.e. they are not renormalized. On the other hand, we do know that there is still wavefunction renormalization and indeed, we can understand the above running in terms of the wavefunction renormalization $Z_X$ of the field $X$.

The tree-level potential above comes from $F_X$, so that at leading order only $Z_X$ can affect $V$.

$$V_{\text{eff}} = Z_X^{-1} |W_X|^2 + \text{finite} = Z_X^{-1} |f|^2 + \text{finite}. \tag{B.66}$$

This gives us

$$-\frac{\partial V_{\text{eff}}}{\partial \log \Lambda^2} = \gamma_X |f|^2 = \frac{1}{64\pi^2} \text{STr}\,\mathcal{M}^4 + \mathcal{O}(h^2), \tag{B.67}$$

where $\gamma_X$ is the anomalous dimension of $X$.

[**Work:** Flesh this out a little bit, it's kind of important.]

# C Phases of Gauge Theories

See: Srednicki chapter 82, Preskill notes. Larsen/Terning notes. Banks. Fradkin[54].

| Phase | $V(r)$ |
|---:|:---|
| Coulomb | $\sim \frac{1}{r}$ |
| Free electric | $\sim \frac{1}{r\log(\Lambda r)}$ |
| Free magnetic | $\sim \frac{\log(\Lambda r)}{r}$ |
| Higgs | $\sim$ constant |
| Confining | $\sim r$. |

Under electromagnetic duality,

| | | |
|---:|:---:|:---|
| Free electric | $\leftrightarrow$ | Free magnetic |
| Coulomb | $\leftrightarrow$ | Coulomb |
| Higgs | $\leftrightarrow$ | Confining |

Can check for confinement using Wilson loops. In $s$-confined theories the Higgs and Confined phases are identical. (Why?)

*** Oblique confined phase. `http://www.springerlink.com/content/l563v000661j125r/`

# D Review of Anomaly Cancellation

Here we collect the basic technical machinery for calculating the cancellation of gauge anomalies. This is not meant to be a comprehensive review, please refer to other resources.

## D.1 Overview and background

An anomaly is a classical symmetry which is broken quantum mechanically. For example, the chiral (ABJ, abelian) anomaly is often the first example of an anomaly where a global symmetry is broken in a gauge background. Classically, the axial current is conserved $\partial_\mu J^\mu_A = 0$, where as quantum mechanically it is proportional to a non-perturbative term, $\partial_\mu J^\mu_A \sim F^{\mu\nu}\widetilde{F}_{\mu\nu}$. Since this can be calculated from the triangle diagram with an axial current and two photons, we call this the $\text{U}(1)_A\ \text{U}(1)^2_{\text{EM}}$ anomaly. A few other remarks about the chiral anomaly:

[54] `http://webusers.physics.illinois.edu/~efradkin/phys583/physics583.html`

- The anomaly is generated by chiral 'zero mode' (with respect to the Dirac operator) fermions and are independent of the fermion mass.
- The anomaly is one-loop exact; higher order corrections are lower superficial degree of divergence.
- In dimensional regularization, the anomaly appears in the definition of $\gamma^5$ in $d \neq 4$.
- Fujikawa showed that the anomaly comes from the non-invariance of the path integral measure.
- In non-abelian theories, Green's functions with odd numbers of axial couplings up to 5-point functions contribute anomalous terms. However, if the triangle diagram vanishes then so do all other anomalous diagrams.

Other examples of anomalies include gravitational anomalies and the conformal anomaly; the latter famously manifested through the renormalization group.

The main anomaly we'll consider here are non-Abelian gauge symmetries[55]. Since gauge symmetries are really redundancies of how we describe a theory, an anomaly in this symmetry would be manifestly non-sensical. We thus require theories to be gauge anomaly-free. Non-Abelian anomalies are intimately related to instantons[56].

Anomalies can be calculated perturbatively through triangle diagrams with chiral fermions, or alternately non-perturbatively using the path integral methods pioneered by Fujikawa. As this is standard fare in quantum field theory, we will not dwell on the technical calculation. Fore more details about anomalies in the spirit of this document, see Preskill's review[57] [132] or one's favorite textbooks (Terning, Banks, Nakahara, and Weinberg are especially good).

## D.2 The Anomaly Coefficient

The anomaly is quantified by the non-vanishing divergence of the Noether current associated with the anomalous symmetry. In the divergence is non-zero, but is proportional to a total derivative. More importantly, for gauge anomalies, it is proportional to the **anomaly coefficient**,

$$A^{abc} = \mathrm{Tr}\left[T^a\{T^b, T^c\}\right], \tag{D.1}$$

where the $T$s are the generators of the appropriate symmetry and $A^{abc}$ depends on the fermion representation. The trace here refers to a sum over all fermions running in the loop, as Burgess and Moore say, "every color of every flavor of quark and every lepton in each generation with $T$ denoting the action of the symmetry on that particular particle type." You can see that this is precisely the group theoretic factor that appears wen you draw a triangle diagram with gauge currents at each

[55]In fact, it is worth pointing out the very elegant differential geometry that rigorously unifies many of the heuristic manifestations of anomalies in quantum field theory, e.g. the relation of the Abelian and non-Abelian anomalies in different dimensions via the Stora descent equations. For more on this see `http://www.lepp.cornell.edu/~pt267/files/BSMclub/Flip_11April11_notes.pdf`.

[56]`http://www.lepp.cornell.edu/~pt267/files/documents/A_instanton.pdf`

[57]`http://www.theory.caltech.edu/~preskill/pubs/preskill-1991-anomalies.pdf`

corner. To simplify gauge anomaly calculations, it is convenient to define an anomaly coefficient $A(\mathbf{r})$ for fermions in representation $\mathbf{r}$ relative to the fundamental representation,

$$A^{abc}(\mathbf{r}) = A(\mathbf{r})\mathrm{Tr}\left[T_F^a\{T_F^b, T_F^c\}\right]. \tag{D.2}$$

Now some useful properties of the anomaly coefficient:

- If $\mathbf{r}$ is a complex representation, then $A(\bar{\mathbf{r}}) = -A(\mathbf{r})$.
- $A(\mathbf{r}_1 \oplus \mathbf{r}_2) = A(\mathbf{r}_1) + A(\mathbf{r}_2)$
- $A(\mathbf{r}_1 \otimes \mathbf{r}_2) = A(\mathbf{r}_1)\dim(\mathbf{r}_2) + A(\mathbf{r}_2)\dim(\mathbf{r}_1)$.

This tells us, for example, that chiral fermions in vector-like (left-right symmetric) representations (e.g. Dirac fermions) also do not contribute to anomalies. Further, chiral fermions in *real* ($\bar{\mathbf{r}} = \mathbf{r}$) or pseudo-real ($\bar{\mathbf{r}} = U^\dagger \mathbf{r} U$) representations do not contribute to anomalies. The value of $A(\mathbf{r})$ for various representations of SU($N$) is given below, copied from [5].

| Irrep | $\dim(\mathbf{r})$ | $2T(\mathbf{r})$ | $A(\mathbf{r})$ |
|---|---|---|---|
| $\square$ | $N$ | $1$ | $1$ |
| **Ad** | $N^2-1$ | $2N$ | $0$ |
| $\ydiagram{1,1}$ | $\frac{N(N-1)}{2}$ | $N-2$ | $N-4$ |
| $\ydiagram{2}$ | $\frac{N(N+1)}{2}$ | $N+2$ | $N+4$ |
| $\ydiagram{1,1,1}$ | $\frac{N(N-1)(N-2)}{6}$ | $\frac{(N-3)(N-2)}{2}$ | $\frac{(N-3)(N-6)}{2}$ |
| $\ydiagram{3}$ | $\frac{N(N+1)(N+2)}{6}$ | $\frac{(N+2)(N+3)}{2}$ | $\frac{(N+3)(N+6)}{2}$ |
| $\ydiagram{2,1}$ | $\frac{N(N-1)(N+1)}{3}$ | $N^2-3$ | $N^2-9$ |
| $\ydiagram{2,2}$ | $\frac{N^2(N+1)(N-1)}{12}$ | $\frac{N(N-2)(N+2)}{3}$ | $\frac{N(N-4)(N+4)}{3}$ |
| $\ydiagram{4}$ | $\frac{N(N+1)(N+2)(N+3)}{24}$ | $\frac{(N+2)(N+3)(N+4)}{6}$ | $\frac{(N+3)(N+4)(N+8)}{6}$ |
| $\ydiagram{2,1,1}$ | $\frac{N(N+1)(N-1)(N-2)}{8}$ | $\frac{(N-2)(N^2-N-4)}{2}$ | $\frac{(N-4)(N^2-N-8)}{2}$ |

It is conventional to work with only left handed fields, e.g. $L$ and $\bar{e}_R$.

## D.3 Cancellation of gauge anomalies

The key point is that for a sensible gauge symmetry, the anomaly must vanish. In other words,

$$\sum_i A(\mathbf{r_i}) = 0. \tag{D.3}$$

Before getting to the nitty-gritty of checking such an expression, let's remark that the easy way for anomalies to cancel it to work within theories where there is no anomaly. This condition condition of having only (pseudo-)real gives representations us a list of groups which, in four dimensions, have vanishing anomaly coefficients: SU(2), SO($2n+1$), SO($4n$), SO($4n+2$), Sp($2N$), $\mathrm{G}_2$, $\mathrm{F}_4$, $\mathrm{E}_6$, $\mathrm{E}_7$ and $\mathrm{E}_8$. Note that SO(10) and $\mathrm{E}_6$ are potential GUT candidates because they can fit the Standard Model as a subgroup. Alternately, for an arbitrary gauge group, one can enforce anomaly-freedom by only including fermions in vector-like representations. A useful fact is that if anomalies cancel in a group, then anomalies will cancel in any subgroup. Thus if you construct a unified theory without anomalies, e.g. SU(5) with the $\mathbf{10} \oplus \bar{\mathbf{5}}$ representation, then you know that the anomalies of the Standard Model must also cancel.

Next let's note that the importance of anomaly cancellation only holds for *gauge* symmetries. There is nothing 'wrong' with a theory whose *global* symmetries are anomalous.

Let us now confirm that the Standard Model is anomaly-free. This list is from Burgess and Moore [13].

- $A(3,3,3)$: Since the quarks are left-right symmetric (vector-like) with respect to $\mathrm{SU}(3)_c$, the anomalies cancel.

- $A(3,3,2)$: This one is also easy. We can ignore the SU(3) parts and just focus on the $\mathrm{SU}(2)_L$ piece. The trace will include a trace over the Pauli matrices for each doublet. Since the pauli matrices are traceless, this anomaly vanishes.

- $A(3,3,1)$: Now we consider the case when there are U(1) generators. To do this it is useful to note that $\lambda_a, \lambda_b = \frac{4}{3}\delta_{ab} + 2f_{abc}\lambda_c$, where the $\lambda$s are Gell-Mann matrices. The color trace gives a factor of three in the first term and causes the second term to vanish. Thus the anomaly is given by the sum of the hypercharges of each quark: $A(3,3,1) = 3(2Y[Q_L] + Y[\bar{U}_R] + Y[\bar{D}_R]) = 2[2(1/6) + (-2/3) + (1/3)] = 0$.

- $A(3,X,Y)$: For $X, Y \neq 3$ this will be proportional to the trace of a Gell-Mann matrix and so vanishes (just like $A(3,3,2)$).

- $A(2,2,2)$: Unlike color, the the electroweak group is not left-right symmetric. However, we noted above that SU(2) is anomaly free. This is because it is pseudo-real: $\bar{\sigma}^i = -\sigma^2\tau^i\sigma^2$.

- $A(2,2,1)$: Here we have $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$. Counting the generation and color multiplicities, we thus have a sum over the hypercharge of each doublet, $A(2,2,1) = 3(Y[L] + 3Y[Q]) = 3[(-1/2) + 3(1/6)] = 0$.

- $A(2,2,1)$: This is proportional to the trace of a single generator and vanishes.

- $A(1,1,1)$: This is the sum over all fermions with respect to the cube of their hypercharges,

$$A(1,1,1) = 3(2Y[L]^3 + Y[\bar{E}_R]^3 + 6Y[Q]^3 + 3Y[\bar{U}_R]^3 + 3Y[\bar{D}_R]^3) \tag{D.4}$$

$$= 3\left[2\left(-\frac{1}{2}\right)^3 + (1)^3 + 6\left(\frac{1}{6}\right)^3 + 3\left(-\frac{2}{3}\right)^3 + 3\,(13)^3\right] = 0. \tag{D.5}$$

Note that anomaly cancellation sets a rigorous, non-trivial condition on the hypercharges and cubes of hypercharges of particles. This prevents giving an arbitrarily small, but finite, charge to the neutrino by shifting its hypercharge by an small amount. Finally, Witten an Alvarez-Gaumé showed that gravitational anomalies impose an additional constraint on U(1) gauge group factors: in order for consistent gravitational coupling, the U(1) generators must be traceless over fermions, $\sum Y = 0$.

## D.4 Comments on global anomalies

These are mainly form Burgess and Moore.

- One can also calculate the anomalies for global symmetries. We already met the chiral anomaly, $A(A,1,1)$. We can also consider baryon number, for which $A(3,3,B) = 0$ but $A(2,2,B) = 3$. Similarly, lepton number, e.g. $A(2,2,L) = 1$. Note that by 'lepton number' here we mean a particular flavor of lepton.
- Note that $A(B,B,B) = 0$ while $A(L,L,L) - 2$. Further, $A(G,G,B) = 0$ while $A(G,G,L) = 1$, where $G$ represents gravity.
- Global anomalies needn't vanish. The effect of the anomalies on low-energy physics can be interpreted as topological objects, instantons and sphaelerons. These effects are proportional to $e^{-8\pi/g^2}$ so that anomalous global symmetries with respect to weakly coupled gauge groups are good approximate symmetries, whereas anomalous global symmetries with respect to strongly coupled gauge groups are strongly broken.
- The anomaly-free global symmetries of the Standard Model are given by linear combinations of the anomalous symmetries above. Including gravitational anomalies, these are $L_e - L_\mu$ and $L_e - L_\tau$, where $L_\mu - L_\tau$ is linearly dependent on the other two.
- Notice that all of the SM anomalies are the same for baryon number as they are for total lepton number ($3L$). The gravitational, $B^3$, and $L^3$ anomalies agree if we include right-handed neutrinos. Thus the combination $B - L$ is anomaly free in the theory with right-handed neutrinos.
- **The $\eta'$ problem**: see my A-exam for more details[58]. The chiral U(3) symmetries of QCD are generally anomalous. The anomalies with $\mathrm{SU}(3)_c$ are all proportional to the trace of the generator's $3 \times 3$ representation. Thus the traceless symmetries are non-anomalous in the limit where the electroweak interactions are negligible. Since, as an equation of Lie algebras, $U(3) = SU(3) \times U(1)$, only the $U(1)$ generator carries a trace and is thus strongly violated by $\mathrm{SU}(3)_c$. Thus QCD anomalies break $U(3)_L \times U(3)_R \to SU(3)_L \times SU(3)_R \times U(1)_B$ where $U(1)_B$ is the non-anomalous $U(1)$ that is vectorlike with respect to the quarks. The 'broken' $U(1)_A$ symmetry is the reason why there is no ninth pseudo-Goldstone pion, i.e. why the $\eta'$ is so heavy.

# E Analytic Continuation into Superspace

As we saw in Section 16.2.3, unless you are the Rambo of loop integrals, the gauge-mediated SUSY-breaking masses in the visible sector (i.e. the soft terms in the MSSM) can be very tedious to calculate since these terms appear at one- and two-loop order. The exact formulae were rather involved, even though the limit of small $(F/M^2)$ simplified the expressions dramatically to forms that one could have guessed from pure 'dimensional analysis[59].'. Fortunately we can do better.

[58] `http://www.lepp.cornell.edu/~pt267/files/documents/A_instanton.pdf`

[59] By 'dimensional analysis' we mean: $\alpha$ carries units of 'gauge-ness,' $(4\pi)$ carries units of inverse loop, $F$ carries units of SUSY-breaking, and $M^{-1}$ carries units of mediation.

We now present a very handy trick for easily calculating the soft SUSY-breaking terms in that limit based on holomorphy. This so-called **analytic continuation into superspace** was first developed by Giudice and Rattazzi [133] and was later expanded to include higher-loop corrections in collaboration with Arkani-Hamed and Luty [134]. Further references are Patrick Meade's TASI09 lectures[60] and John Terning's textbook [5].

## E.1 Overview

In gauge mediated supersymmetry breaking, a chiral superfield (or set of superfields) $X$ in the hidden sector spontaneously breaks SUSY by obtaining a vacuum expectation value

$$\langle X \rangle = M + \theta^2 F.$$

In minimal gauge mediation, the lowest-component (SUSY-preserving) vev $M$ gives a mass to the messenger fields $\phi, \varphi$ which transmit SUSY breaking to the MSSM. The higher component vev $\theta^2 F$ is the actual SUSY-breaking term and is transmitted to the MSSM only through the messengers. A sensible thing to consider is to use the power of effective field theory by integrating out the messenger fields and considering effective operators with MSSM fields coupled to the vevs of the SUSY-breaking hidden-sector fields $X$. In such a formalism we treat the $X$ as a SUSY-breaking spurion in the visible sector[61]. In such a set-up the effective operators would heuristically take the form

$$\mathscr{L}_{\text{eff}} = c_1 \int d^2\theta \, \frac{X}{M} \, \mathbb{W}_a \mathbb{W}^a + c_2 \int d^4\theta \, \frac{X^\dagger X}{M^2} Q^\dagger Q.$$

The problem with this approach is now staring us in the face: in order to go through the EFT procedure straightforwardly[62], one still has to compute one- and two-loop diagrams and do a matching to determine the $c_1$ and $c_2$ coefficients. Our usual approach has failed us[63].

Now we can be clever. Giudice and Rattazzi reminded us that the lowest-order vevs for these effective operators, i.e. the non-SUSY-breaking vevs, are just terms that are contributions to the usual kinetic Lagrangian in supersymmmetry. The coefficients of these terms are just the (holomorphic) gauge coupling $\tau \sim g^{-2}$ and the wavefunction renormalization $Z$ of the chiral superfields. Further, we already know the RG behavior of the gauge coupling and wavefunction renormalization from well-known one-loop calculations. It would be great if we could insert these physical quantities could serve as the lowest component of the spurion coefficients in $\mathcal{L}_{\text{eff}}$ and then

[60] Recordings available at `http://www.colorado.edu/physics/Web/tasi09_annc.html`.

[61] We can proceed as if the $X$ field is nothing more than a 'trick' in the visible sector to parameterize the *a priori* unknown physics of SUSY-breaking. In such a framework we would never have to consider whether or not $X$ is in any sense a physical field. In this case, however, by the assumption of gauge mediation we *know* that $X$ is actually a physical field that is just hidden from the visible sector through couplings via heavy messengers. In this sense we can interpret our spurion analysis 'literally.'

[62] The implementation of EFT in particle physics is an under-appreciated skill. Good introductions can be found in, e.g., Witek Skiba's lectures at TASI09 at `http://www.colorado.edu/physics/Web/tasi09_annc.html` or the lectures by Cliff Burgess [13] or James Wells [135]. The most immediate application of the effective field theory framework are electroweak precision observables; the main papers for phenomenologists are Barbieri, Pomarol, Rattazzi, and Strumia [136], Han and Skiba [137], and Cacciapaglia, Csáki, Marandella, and Strumia [138].

[63] `http://xkcd.com/55/`

'promote' their well-known RG dependence to a form for the SUSY-breaking higher-component spurion vevs. We can, in fact, do this. The running values of $\tau(\mu, \Lambda)$ and $Z(\mu, \Lambda)$ at some scale $\mu$ and for some UV cutoff $\Lambda$, are given by the solution of the RG equations. These solutions include terms that come from integrating out the messenger fields at the scale $M$. If we promote the $M$-dependence of these expressions to a dependence on the spurion superfield $X$, then we convert $\tau$ and $Z$ into superfields whose higher-component (SUSY-breaking) vevs are given straightforwardly in terms of the $X$ vevs. This is called **analytic continuation into superspace**. We will see that the miracle is that in the $F \ll M$ limit which is usually sufficient in most gauge mediation models, these higher-component vevs have precisely the coefficients that we would obtain via explicit calculation of two-loop results.

This result is at first magical and then, after some thought, tautological: such a result *had* to be true due to holomorphy and the constraints of supersymmetry. In a broader sense, this is an example of the use of supersymmetry to constrain the behavior of a quantum field theory that would otherwise be much more difficult to ascertain.

## E.2 Preliminary results

Before we do any heavy-lifting, let us remind ourselves of a few results and notation. Recall that the **soft SUSY-breaking terms** for a SUSY gauge theory (in particular, a Wess-Zumino model coupled to a gauge field) take the form

$$\mathscr{L}_{\text{soft}} = -\frac{1}{2}m_\lambda \lambda\lambda - \frac{1}{6}A\phi^3 - \frac{1}{2}B\phi^2 - \frac{1}{2}\phi^*\phi + \text{h.c.} \tag{E.1}$$

These are terms which are manifestly non-supersymmetric but that do not spoil the cancellation of quadratic divergences that solves the Hierarchy problem. This is identical to saying that these are the terms that can appear when supersymmetry is broken spontaneously. Heuristically this is sensible since at energies well above the scale of the vacuum energy (the SUSY-breaking order parameter) the theory should appear supersymmetric with all nasty divergences canceling. We can write this out more formally,

**Theorem E.1.** *The soft supersymmetry-breaking terms that otherwise respect all of the symmetries and constraints of the theory (e.g. gauge symmetries, renormalizability) are identical to terms which can be obtained by promoting the couplings of the manifestly supersymmetric theory to spurion superfields which obtain higher-component vacuum expectation values.*

*Proof.* If one sits down and thinks about this for a moment then it follows tautologically. More formally, one can construct an isomorphism between any soft breaking term to an appropriate SUSY-preserving term which would contribute to the soft breaking term if its coupling obtained a higher-component vev. Let us assume renormalizability so that we only have to look at the superpotential. The generalization to non-renormalizable theories is straightforward. First, consider any soft SUSY-breaking term and promote each of the fields to superfields. By the assumption that the soft-terms respect all of the symmetries and constraints of the theory, this term must exist in the superpotential. Next we want to map any term in the superpotential to a soft term when the the superpotential term's coupling is given an $F$-term vev. This is also straightforward since the $\theta^2$ in the higher-component coupling creates a term in the Lagrangian in which all of the non-spurion superfields are constrained to their lowest components. This term is manifestly a part of the soft breaking Lagrangian. $\square$

Now we are reassured that we can really describe all soft-SUSY breaking terms by discussing the higher-component vevs of the couplings, i.e. by promoting the couplings to spurion superfields. As mentioned above, these spurions will be defined via the usual running (non-superfield) couplings by promoting the dependence on the messenger threshold $M$ to a superfield $X$. It is now useful to discuss the usual notation employed to describe the higher component vevs of these objects. If $f(M)$ is a non-superfield analytic function of the scale $M$, we may promote $M \to X = M + \theta^2 F$ so that $f$ obtains a higher component vev given by Taylor expansion in $\theta^2 F/M$:

$$f(\langle X \rangle) = f\left(M\left(1+\theta^2 F/M\right)\right) = f(M) + \theta^2 \frac{\partial f(M)}{\partial X} F,$$

where we note that the expression on the right-hand side is exact since $\theta^4 = 0$. We can also write the $F$-term using partial derivatives with respect to logarithms of superfields,

$$f(\langle X \rangle)|_{\theta^2} = \frac{\partial f(M)}{\partial X} F = \frac{\partial \ln f(M)}{\partial X} f(M) F = \frac{\partial \ln f(M)}{\partial \ln X} f(M) \frac{F}{M}. \tag{E.2}$$

Finally, it is worth noting that the meaning of a logarithm of a superfield is given by its Taylor expansion,

$$\ln X = \ln(M + \theta^2 F) = \ln M + \theta^2 \frac{F}{M},$$

which again terminates and is thus exact.

Now we are ready to derive our main results from analytic continuation into superspace. The discussion in this section should prepare you to compare all of our derivations to the results in the original literature. Let us emphasize that the following results depend on the assumption of gauge mediation as the only source of SUSY-breaking. They are invalidated if there are other contributions to the soft terms of non-negligible strength. Further, the results that we obtain will assume the $F \ll M^2$ limit.

## E.3 Gaugino mass

Let's start by determining the gauge mediation prediction for the gaugino mass soft-term. As mentioned above, the real 'trick' is to use a result that we already know: the renormalization group equations for the physical gauge coupling. The running coupling depends on the messenger sector via the threshold at $\mu = M$ where the messenger fields are integrated out. The RG flow is shown in Fig. 3. In the limit $F/M^2 \ll 1$, we may neglect the threshold effects of supersymmetry breaking in our renormalization. In other words, when the SUSY-breaking scale is low (e.g. just above the TeV scale), we can (1) neglect the non-supersymmetric renormalization group flow between $\sqrt{F}$ and the electroweak scale and consider only the manifestly supersymmetric flow from the cutoff scale down to $\sqrt{F}$ and (2) we can neglect the non-supersymmetric effects when integrating over thresholds (when we integrate out the gauginos).

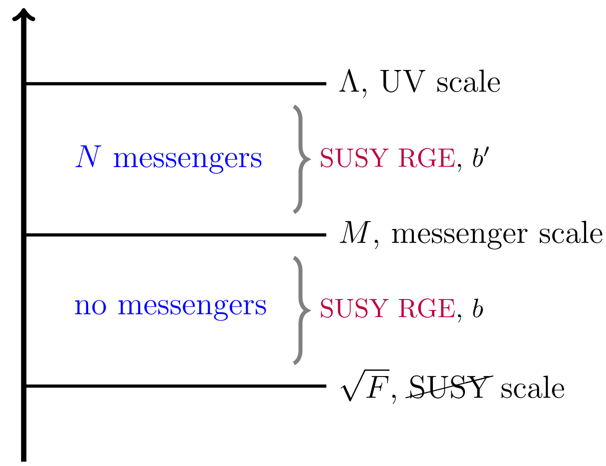



Figure 3: The renormalization group evolution of a gauge-mediated model with $F \ll M^2$.

The renormalization group equation for the coupling $g$ at a scale below $M$ can be integrated to yield

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\Lambda)} + \frac{b'}{8\pi^2}\ln\frac{M}{\Lambda} + \frac{b}{8\pi^2}\ln\frac{\mu}{M}. \tag{E.3}$$

In terms of the holomorphic coupling, this is written as

$$\tau(\mu) = \tau(\Lambda) + i\frac{b'}{2\pi}\ln\frac{M}{\Lambda} + i\frac{b}{2\pi}\ln\frac{\mu}{M}. \tag{E.4}$$

One of the nice results of $SU(N_c)$ super-Yang-Mills theories is that the beta function is written simply in terms of the number of superfields transforming in the fundamental[64],

$$b_0 = 3N_c - N_f.$$

Thus we know that the difference in the beta functions is precisely the number of messenger fields $n$ at the scale $M$,

$$b - b' = n.$$

The expression for $\tau(\mu)$ depends on the messenger scale $M$. We can ask ourselves where the scale $M$ comes from. In minimal gauge mediation, for example, we know that it is the vev of lowest component of the SUSY-breaking field $X$, e.g. Eq. (16.7),

$$\langle X\rangle = M + \theta^2 F.$$

The trick behind analytic continuation into superspace is to promote $M$ back to the superfield from whence it originated. This, in turn, promotes $\tau$ into a chiral superfield,

$$\begin{aligned}\tau(\mu) &= \tau(\Lambda) + i\frac{b'}{2\pi}\ln\frac{X}{\Lambda} + i\frac{b}{2\pi}\ln\frac{\mu}{X} \\ &= i\frac{b'-b}{2\pi}\ln X + \cdots .\end{aligned} \tag{E.5}$$

[64]The fancy way of writing this for a group $SU(N_c)$ and representation $r$ is $b_0 = 3C_2(N_c) - \sum_i C(r_i)$, where $C_2\mathbb{1} = (t^a t^a)_r$ and $C(r)\delta^{ab} = \mathrm{Tr}_r(t^a t^b)$. Am I the only one who forgets these things?

We know from the form of $\mathcal{L}_{\text{SYM}}$ in Eq. (A.51) that the soft term corresponding to the gaugino mass can be written as

$$\hat{m}_\lambda = -2 \left.\frac{\tau}{16\pi i}\right|_{\theta^2} . \tag{E.6}$$

where the factor of 2 comes from the 1/2 in front of the gaugino mass in the soft breaking Lagrangian, Eq. (E.1). [**Comment:** I'm not sure where the minus sign comes from since $\mathbb{W} \sim i\lambda$, thus $\mathbb{W}\mathbb{W} \sim = -\lambda^2$ already.] We've labelled $\hat{m}_\lambda$ with a hat to indicate that it is not yet canonically normalized. Recall that we've written our gauge Lagrangian with the 'natural' normalization in which the kinetic term has an overall factor of $g^{-2} = \tau/4\pi i$. Upon canonical normalization the gaugino mass takes the form

$$m_\lambda = -2\frac{g^2}{16\pi i}\ \tau(X)|_{\theta^2} = -\frac{1}{2\tau}\ \tau(X)|_{\theta^2} .$$

Note that canonically normalizing cancels any arbitrariness in how we defined $\tau$ relative to $g^{-2}$ so that this equation is correct no matter what prefactor multiplies $\tau \sim g^{-2}$. Let's now use the grown-up notation Eq. (E.2) and the expression Eq. (E.5) to write this more elegantly,

$$m_\lambda = -\frac{1}{2} \left.\frac{\partial \ln \tau}{\partial \ln X}\right|_{X=M} \frac{F}{M} = -\frac{i}{2\tau}\frac{b'-b}{2\pi}\frac{F}{M} = \frac{ng^2}{16\pi^2}\frac{F}{M} = n\frac{\alpha}{4\pi}\frac{F}{M}. \tag{E.7}$$

Lo and behold we get *exactly* Eq. (16.11), the leading order contribution in the SUSY-breaking parameter $F/M^2$. Take a moment and bask in the glory of what we've done: we've reproduced the leading order contribution to what would otherwise have been a two-loop calculation. Armed with the one-loop exact beta function for the gauge coupling, we didn't even have to calculate *any* loops.

Before moving on to the other soft terms, let us make the following emphatic caveat: this trick is only valid in the limit $F \ll M^2$. We relied on the assumption that SUSY-breaking effects were small as we went through renormalization group thresholds. For example, we did not pick up the logarithms in the full loop calculation for $m_\lambda$ in Eq. (16.10) nor would we pick up the dilogarithms in the full two-loop calculation for the scalar masses.

## E.4 Analytic Continuation: Wess-Zumino soft terms

Let's proceed the formulae for the Wess-Zumino (i.e. superpotential) soft terms. Thanks to Seiberg know that the parameters of the superpotential don't renormalize [139], but we do know that we have wavefunction renormalization and that this affects the physical couplings. We would like to identify the $M$ dependence of $Z(M)$ and then promote it to $X$ dependence. There is an immediate subtlety: $Z$ is not a holomorphic quantity, but a real function. Thus it has to be a function of both $X$ and $X^\dagger$, in particular, $Z = Z(M) = Z(\sqrt{XX^\dagger})$. The one-loop expansion for $Z$ coming from the Wess-Zumino model takes the form

$$Z(\mu) = 1 + |\lambda|^2 \ln\left|\frac{\Lambda}{\mu}\right| . \tag{E.8}$$

However, this is not the quantity that we want to calculate since to this order it doesn't involve the messengers which only couple via gauge interactions, and hence it is manifestly supersymmetric. What we want is the wavefunction renormalization from gauge interactions, which is succinctly written in the RGE

$$\frac{d\ln Z}{d\ln\mu} = \frac{C_2(r)}{\pi}\alpha(\mu).$$

We've already calculated $\alpha(\mu) = i\tau^{-1}$ in Eq. (E.5), so that

$$\alpha^{-1}(\mu) = \alpha^{-1}(\Lambda) + \frac{b_0}{4\pi}\ln\frac{M^2}{\Lambda^2}.$$

We can then integrated the RGE taking into account the threshold at $M$,

$$Z(\Lambda, M, \mu) = Z(\Lambda)\left[\frac{\alpha(\Lambda)}{\alpha(M)}\right]^{2c/b'}\left[\frac{\alpha(M)}{\alpha(\mu)}\right]^{2c/b}. \tag{E.9}$$

So we seem to be well on our way to performing analytic continuation, we just have to plop $\sqrt{X^\dagger X}$ everywhere we see $M$. Not so fast. We should not forget to canonically normalize our fields with respect to $Z$. We can go ahead and write

$$\begin{aligned}\mathscr{L}_{\text{kin}} &= \int d^4\theta\ \left(Z + F_Z\theta^2 + F_Z^*\overline{\theta}^2 + D_Z\theta^2\overline{\theta}^2\right)\Phi^\dagger\Phi \\ &= \int d^4\theta\ \left(Z + \frac{\partial Z}{\partial X}F\theta^2 + \frac{\partial Z}{\partial X^\dagger}F^*\overline{\theta}^2 + \frac{\partial^2 Z}{\partial X\partial X^\dagger}FF^*\theta^2\overline{\theta}^2\right)\Bigg|_{X=M}\Phi^\dagger\Phi.\end{aligned}$$

In the second line we just wrote $F_Z$ in terms of the $F$-terms of the spurion $X$. We can canonically normalize up to order $\mathcal{O}(\theta^2,\overline{\theta}^2)$ by redefining our fields

$$\Phi \to \Phi' = Z^{1/2}\left(1 + \frac{\partial\ln Z}{\partial X}F\theta^2\right)\Bigg|_{X=M}\Phi.$$

From now on we drop the prime on the field, $\Phi' \to \Phi$. When we need to we'll refer to the original, non-canonically normalized superfield as $\Phi_0$. I know, we're being excessively pedantic, but I'm easily confused. Our normalization doesn't get rid of the $D$-term, so that the kinetic term now looks like

$$\mathscr{L}_{\text{kin}} = \int d^4\theta\ \left[1 - \left(\frac{\partial\ln Z}{\partial X}\frac{\partial\ln Z}{\partial X^\dagger} - \frac{1}{Z}\frac{\partial^2 Z}{\partial X\partial X^\dagger}\right)FF^*\theta^2\overline{\theta}^2\right]\Bigg|_{X=M}\Phi^\dagger\Phi. \tag{E.10}$$

The $\theta^2\overline{\theta}^2$ is precisely a scalar mass term, $\widetilde{m}^2$,

$$\widetilde{m}^2 = -\left.\frac{\partial^2\ln Z}{\partial\ln X\partial\ln X^\dagger}\right|_{X=M}\frac{FF^*}{M^2}. \tag{E.11}$$

But wait, there's more! If we go back to the superpotential and plug in our rescaled field $\Phi$, we get $A$ and $B$ terms 'for free.' Of course, we expected this since we know that the only running of the terms in the physical superpotential terms comes from wavefunction renormalization. Let's see how this works. The superpotential was written in terms of the non-canonically normalized field,

$$W(\Phi_0) = W\left(Z^{-1/2}\left(1 - \frac{\partial \ln Z}{\partial \ln X}\frac{F}{M}\theta^2\right)\Phi\right).$$

We want to isolate the soft terms that appear when one of the non-canonically normalized fields picks up the $F\theta^2$. We will write this down by taking a derivative of $W$ with respect to the *non-canonically normalized field* and multiply by the $F\theta^2$ term,

$$\Delta\mathcal{L}_{\text{soft}} = \left.\frac{\partial W}{\partial \Phi_0}\right|_{\Phi_0=\phi_0} Z^{-1/2}\left(-\frac{\partial \ln Z}{\partial \ln X}\frac{F}{M}\right).$$

Thus to leading order ($Z = 1$) we obtain

$$A = 3\lambda \frac{\partial \ln Z}{\partial \ln X}\frac{F}{M}. \tag{E.12}$$

Thus we have the useful result that the $A$ terms will be suppressed by the Yukawa coupling times powers of $F/M$ and will thus be small.

We could proceed to plug in our simple Wess-Zumino superpotential to extract the exact form of the $B$ terms, but we'll stop here since we now that $B$ terms are a sensitive subject in gauge mediation since it needn't be generated by loops of the messenger fields. In other words, $B$ (or '$B_\mu$' in the Standard Model), is a **hard parameter**.

## E.5 Remarks

Now that we've established our main results and demonstrated our method, let's make a few important remarks.

First of all, we might ask what we can do to incorporate higher orders in the messenger loops? Before analytic continuation into superspace, one would have to calculate two loop diagrams for the gaugino masses and three loop diagrams for the scalar masses. Patrick Meade remarks, "Now I've never calculated a three loop diagram; maybe some of you have, but it sounds hard." Just as we were able to capture the one and two loop effects using well known RG equations at one loop order, we may calculate the two and three loop effects by using the RG equations at two loop order. There are subtleties when we go to higher loops due to the higher-loop evolution of $\tau$. In any practical renormalization scheme (e.g. DRED), $\tau$ loses its holomorphicity at two-loop order. This is precisely due to the dilogarithms (and $n$-logarithms at higher orders) that we saw in the full two-loop formula for the soft masses in Section 16.2.3. We thus can no longer simply promote $M \to X$ in our analytic continuation. Giudice and Rattazzi teamed up with Arkani-Hamed in a follow-up paper that shows how to tip-toe through these subtleties for to analytically continue in superspace to all orders in perturbation theory [134].

Note that we are always stuck in the $F \ll M^2$ limit, no matter how many messenger loops we include. The threshold effects that we throw away in this limit are functions of logarithms and dilogarithms presented in Section 16.2.3; we will never obtain such functions using analytic continuation. Just how small does $F$ have to be relative to $M^2$? Giudice and Rattazzi found that this approximation is still very good for $F/M^2 \sim 0.3$ [133]. They note that this is true because the actual expansion parameter is $F^2/M^4$.

Next let use make some general remarks about the wavefunction renormalization spurion superfield $Z(X, X^\dagger, \mu)$ following the discussion by Giudice and Rattazzi around their equation (16). They remark that this superfield is a power series in logarithms of the form

$$L_\Lambda = \ln\left(\mu^2/\Lambda^2\right) \qquad\qquad L_X = \ln\left(\mu^2/XX^\dagger\right).$$

Thus to $\ell$-loop order we may write

$$Z(X, X^\dagger, \mu) = \alpha^{\ell-1}(\Lambda)\, P_\ell(\alpha(\Lambda)L_X, \alpha(\Lambda)L_\Lambda), \tag{E.13}$$

where $P_\ell$ is a function that comes from integrating the $\ell$-loop RG equation. This means, for example, that the scalar mass in Eq. (E.11) takes the form

$$\widetilde{m}^2 = \alpha^{\ell+1}(\mu)\widetilde{P}_\ell\left(\alpha(\mu)L_X\right), \tag{E.14}$$

for $\widetilde{P}$ related to the second derivative of $P$. Thus we can see explicitly that it is sufficient to consider the $\ell = 1$ loop result to obtain $\mathcal{O}(\alpha^2)$ contributions to the soft scalar masses.

Moving on, we expressed our wavefunction renormalization in terms of $\alpha$, which we related to the renormalization of $\tau$ from Eq. (E.5). It is important to recognize, however, that the $Z$ spurion is a *real* superfield while the $\tau$ spurion is a *chiral* superfield. Thus the proper identification is

$$\alpha^{-1}(X) = \mathbb{I}\mathrm{m}\,(\tau) = \alpha^{-1}(\Lambda) + \frac{b'}{2\pi}\ln\frac{XX^\dagger}{\Lambda^2}. \tag{E.15}$$

[**Check:** There should also be a b' term.] With this we can write out more explicit forms of our scalar mass and $A$ term by plugging into Eq. (E.9),

$$\widetilde{m}^2(\mu) = 2C_2\frac{\alpha^2(\mu)}{16\pi^2}n\left[\xi^2 + \frac{n}{b}(1-\xi^2)\right]\left(\frac{F}{M}\right)^2 \tag{E.16}$$

$$A_i(\mu) = 2\frac{C_i}{b}\frac{\alpha(\mu)}{4\pi}n(\xi - 1)\frac{F}{M}, \tag{E.17}$$

where

$$\xi \equiv \frac{\alpha(M)}{\alpha(\mu)} = \left(1 + \frac{b}{2\pi}\alpha(\mu)\ln\frac{M}{\mu}\right)^{-1}. \tag{E.18}$$

If the superfield $\Phi$ is charged under multiple gauge groups, then the appropriate generalization is to sum over the contributions from the different gauge couplings. Note that we've even explicitly included the leading log effect from fthe renormalization from $M$ down to $\mu$: $A_i = 0$ at $\mu = M$, but at low energies acquires a renormalization proportional to the gaugino mass.

Now let us close by reminding ourselves of something to be happy about: we have been able to determine the leading-order (in $F/M^2$) effect of supersymmetry breaking in the hidden sector without having to calculate any loops and in a way that is by and large insensitive to the details of *how* supersymmetry is broken in the hidden sector. We should be very proud of ourselves.

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