\( \mathcal{L}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \left[ g_3 \bar{e}_L \gamma^5 e_R \nu_L \bar{\nu}_R + g_1 \bar{e}_L \gamma^5 H^\dagger \bar{\nu}_L e_R + g_9 \bar{e}_L \gamma^5 H^\dagger \bar{\nu}_L e_R + g_6 \bar{e}_L \gamma^5 H^\dagger \bar{\nu}_L e_R \right] \\
+ \frac{1}{12} G_F \bar{e}_L \tilde{\nu} \gamma^5 (V - q \tilde{q})^\dagger \gamma^5 \tilde{q} \frac{\sqrt{2}}{6} \bar{q} \gamma^5 (V - q \tilde{q})^\dagger \gamma^5 \tilde{q} \\
+ \text{terms that vanish for RS models} \\
\)

Once you know the effective couplings, you can plug into the \( Br \) formula (hep-ph/0501161)

\[
Br(\mu \rightarrow 3e) = \frac{2(g_3^2 + g_9^2)}{3} + \frac{g_9^2}{3} - \frac{g_6^2}{2}
\]

\[
Br(\mu \rightarrow e) = \frac{\frac{8G_F^2 f_\pi^4 m_\mu^2 D_{2 \pi \tau}}{2\pi^2 \tau} \frac{Q_\mu}{2} \cdot \frac{2(2\tau + q)}{(2\tau + q)^2} \frac{Q_\mu}{2}}{L_{\text{eff}}} 
\]

\[
\text{Fernberg-Weinberg approximation (1957)} \\
\begin{align*}
E_e & \approx \frac{P_e}{M_e} \\
F_\pi & \approx 0.55 \\
2\pi & \approx 6.61 \\
L_{\text{eff}} & \approx 2.6 \times 10^{-12} \text{ cm}
\end{align*}
\]

Discussion: Coupling to Nuclei

- In going from \( \nu_L \rightarrow Br(\mu \rightarrow e) \), we have to dress the \( \bar{e} \) CURRENT \( \rightarrow \) nuclear CURRENT

  This is done using QCD, which is partially-conserving \( \gamma \)-Axial current / pseudoscalar \( \gamma \)-Axial current vanishings: \( \langle N|\bar{q}^\gamma_{\mu} q|N \rangle = \langle N|\bar{q}^\gamma_{\mu} q|N \rangle = 0 \).

- Note the normalization of \( v^8 \); \( v^8 \) is the vector coupling to electrons: \( \text{lepton} + \text{hadrons} \).

  For example, consider the \( Z \) coupling to up-type quarks:

\[
\frac{g}{\sin^2 \theta_W} \left[ \bar{u} \gamma^\mu (1 - q\bar{q}) P_L u + \bar{d} \gamma^\mu (1 - q\bar{q}) P_L d \right] Z^\mu = \frac{g}{\sin^2 \theta_W} \left[ v^8 \bar{u} \gamma^\mu u + v^8 \bar{d} \gamma^\mu d \right] \\
= \frac{1}{2} Z_{\text{eff}} \bar{u} \gamma^\mu u + \bar{d} \gamma^\mu d
\]

Remark: It is more natural to write \( \mathcal{L}_{\text{eff}} \) \( \rightarrow \) in terms of chiral currents

\[
\mathcal{L}_{\text{eff}} \rightarrow \frac{1}{2} G_F \left[ \epsilon (v + a) \gamma^\mu P_L + \epsilon (v - a) \gamma^\mu P_R \right] \cdot \frac{\sqrt{2}}{6} \bar{q} \gamma^\mu q.
\]

Sample matching calculation: \( p \rightarrow e \), via \( SM \) \( Z \)

\[
e \rightarrow \frac{\sqrt{2}}{Z} \gamma \rightarrow \frac{9 g_3}{64 \sin^3 \theta_W} \bar{e} \gamma^\mu P_L e \frac{1}{\sqrt{2}} \bar{u} \gamma^\mu \gamma^5 v^8 q = \frac{9 G_F (v + a)}{64 \sin^3 \theta_W} e \gamma^\mu P_L e \frac{\sqrt{2}}{6} \bar{q} \gamma^\mu v^8 q \\
(\gamma) = 2g_L
\]

\[
\frac{Z_{\text{eff}}}{\alpha \gamma} = \frac{g_3^2}{8 m_Z^2 c_W^2}
\]

\[
\text{This also fixes connection for } Q_\gamma
\]

\[
\Rightarrow (v + a) = 2g_L
\]
Now let's review the properties of bulk fermions & bosons in RS.

The general solution for the $n^{th}$ KK mode gauge boson profile is:

$$ h^{(n)}(z) = \int_{-\infty}^{\infty} \left[ Y_0(M_{kk}^n R) J_n(M_{kk}^n z) - J_0(M_{kk}^n R) Y_n(M_{kk}^n z) \right] $$

(Revised) We know that the SD EOM has a general solution:

$$ h^{(n)}(z) = a J_n(M_{kk}^n z) + b Y_n(M_{kk}^n z) $$

The $M_{kk}^n$ factor comes from solving the $\psi^a$ wave EOM.

Since the $2$ boson has a zero mode, it must have Neumann BC by invoking the formulae for derivatives of Bessel functions. We find that the $Z = R'$ BC is:

$$ Y_0(M_{kk}^n R) J_0(M_{kk}^n R') = J_0(M_{kk}^n R) Y_0(M_{kk}^n R') $$

We know that $M_{kk}^n \sim n/R'$ & $R' \ll R' \Rightarrow M_{kk}^n R \approx 0$.

Now recall two important properties of the $J_0 \neq Y_0$ Bessel functions:

1. $J_0(0) = 1$ & $J_0(x) \approx 1$ for $x < 1$.
2. $J_0(x) \approx \infty$ & $Y_0(x) \approx 0$ for $x > 1$.

Thus the RHS of $\psi^a(0)$ is very large in the limit $M_{kk}^n R' \approx 0$ while the LHS is a product of "under control" ($\ll 1$) or zero numbers.

$\Rightarrow J_0(M_{kk}^n R') \approx 0 \Rightarrow M_{kk}^n R' \approx 0$ is a zero of $J_0$.

The first KK mode thus satisfies $M_{kk}^1 R' = x_1 \approx 2.405$.

More generally, the spacing of the KK tower follows the zeros of $J_0$.


The Zero Mode $Z$:

1. Write down SM $Z$ coupling in terms of SD parameters.
2. Identify the non-universal (SU(2)) coupling of the SM $Z$.

(These $\Rightarrow$ SD zero mode becomes slightly non-universal.

$\Rightarrow$ A new flavor violating coupling to fermions)

We approximate the zero-mode $Z$ boson wavefunction profile by expanding the Bessel functions for small argument ($M^2 \ll M_{kk}^n$ or $M_{kk}^n R' \ll 1$):

$$ h^{(0)}(z) \approx \sqrt{1 + \frac{M_{kk}^n R'^2}{2} \log \frac{z^2}{R'} + \cdots} $$
To fix the normalization $|\omega|$, we canonically normalize the 4D kinetic term

$$\int d^4x \int' d\bar{z} \left( \frac{e}{2} \right)^2 F_{\mu\nu}F^{\mu\nu} \rightarrow \int d^4x \int' d\bar{z} \frac{e}{2} \left( \frac{e}{2} \right)^2 F_{\mu\nu}F^{\mu\nu} \left( \ln' (\bar{z}) \right)^2 + \ldots$$

requiring the $\left( \frac{e}{2} \right)^2$ to be canonically normalized after the $1/2$ integral.

$$\ln' (\bar{z}) = \frac{1}{\sqrt{R \log R/R}} \left[ 1 - \frac{Me^2}{4} \left( e^2 - 2z^2 \log \frac{2}{R} \right) \right]$$

This term vanishes for a massless zero mode (eq A(0))

$$\Rightarrow \text{profile for such gauge bosons is flat}$$

Non-degenerate couplings to fermions. Recall the 4D fermion zero-mode profile,

$$\Psi_{\bar{c}}^{(0)}(x,z) = \frac{1}{R^2} \left( \frac{2}{R} \right)^2 \left( \frac{2}{R} \right)^c \bar{c}_{\bar{c}} \frac{\Psi_{\bar{c}}^{(0)}(x)}{\sqrt{2} c_c}$$

$$\Rightarrow \text{cannonically normalized 4D field}$$

$$\bar{c} = \sqrt{\frac{1 - z_c}{1 - (z/R)^{1 - z_c}}}$$

Thus in the c-basis (5D canonical basis) the fermion coupling to (performing $d(5D)$)

$$g_{5D}^{(0)} \bar{c} \int d\bar{z} \left( \frac{2}{R} \right)^2 \left( \frac{2}{R} \right)^c (x) \bar{c}_c \frac{\Psi_{\bar{c}}^{(0)}(x)}{\sqrt{2} c_c}$$

$$g_{5D}^{(0)} = g_{5D}^{(0)} T_{\bar{c}} - g_{5D}^{(0)} \delta_{\bar{c}}$$

$$\Rightarrow \text{universal part} \Rightarrow \text{non-universal}$$

Thus in the $M$-basis, the fermion coupling comes from the universal term

$$g_{5D}^{(0)} \int d\bar{z} \left( \frac{2}{R} \right)^2 \left( \frac{2}{R} \right)^c (x) \bar{c}_c \frac{\Psi_{\bar{c}}^{(0)}(x)}{\sqrt{2} c_c}$$

$$\Rightarrow \text{universal part}$$

The sum coupling: comes from the universal term

$$g_{5D}^{(0)} = \frac{g_{5D}^{(0)}}{R^2} \int d\bar{z} \frac{1}{\sqrt{R \log R/R}} \left( \frac{2}{R} \right)^2 \left( \frac{2}{R} \right)^c \bar{c}_c \frac{\Psi_{\bar{c}}^{(0)}(x)}{\sqrt{2} c_c}$$

$$\Rightarrow \text{universal part}$$

Note: still in $c$-basis!

The FNC non-universal part

$$\text{change vars: } \gamma = \frac{2}{R}$$

$$g_{5D}^{(0)} \frac{g_{5D}^{(0)}}{R^2} \int \frac{d\bar{z}}{\sqrt{R \log R/R}} \left( \frac{2}{R} \right)^2 \left( \frac{2}{R} \right)^c \bar{c}_c \frac{\Psi_{\bar{c}}^{(0)}(x)}{\sqrt{2} c_c} \gamma^{2 - 2 \gamma} (1 - 2 \log \frac{2}{R})$$

$$\Rightarrow \text{evaluate by careful integration by parts (see appendix)}$$

$$= \frac{g_{5D}^{(0)}}{R^2} \int \frac{d\bar{z}}{\sqrt{R \log R/R}} \left( \frac{2}{R} \right)^2 \left( \frac{2}{R} \right)^c \bar{c}_c \frac{\Psi_{\bar{c}}^{(0)}(x)}{\sqrt{2} c_c} \gamma^{2 - 2 \gamma} (1 - 2 \log \frac{2}{R})$$

$$\Rightarrow \text{non-leading, can drop}$$
\[ q_{4D}^2 = \frac{\alpha_{\text{em}}^2}{R \log R'/R} \frac{M_e^2}{2(3 - \alpha)} (R')^2 \log \frac{R'}{R} = -\frac{2\pi}{\alpha_{\text{em}}} \frac{(M_e R)^2}{2(3 - \alpha)} \log \frac{R'}{R} \]

The full coupling is \( q_{4D} = q_{4D}^2 + q_{4D}^\text{NC} \). Note we're still in c-basis.

**RS Gauge Mechanism:** In this 5D c-basis, the nonuniversal couplings are diagonal, but not proportional to \( 1 \). When we rotate into the physical (ke) basis, we get \( q_{4D}^\text{NC} \).

**Fact:** The rotation from c-basis flavor \( i \to j \) is in the basis flavor; go like \( f_i / f_j \).

For \( j \) see 14 dec. Notes.

\[ q_{4D}^2 = \left( \frac{\alpha_{\text{em}}}{f_i} q_{4D} \right) \frac{M_e^2}{f_i} \left( \frac{f_j^2}{2(3 - \alpha)} - \frac{f_j^2}{2(3 - \alpha)} \right) (M_e R)^2 \cdot \frac{1}{2} \log \frac{R'}{R} \frac{2\pi}{\alpha_{\text{em}}} \]

**Fundamental Universal**

The KK Z: for now we will write \( Z' = Z^{(0)} \) (will be more careful in historical model).

Recall: \( h_\perp^2 z^2 \times Z^{(0)} (\mathbf{M}_R) \mathbf{Y}, (\mathbf{M}_L) - \mathbf{Y}_0 (\mathbf{M}_R) \mathbf{Y}, (\mathbf{M}_L) \)

**Remarks:** The second term is much smaller than the first, over most of the range of \( z \).

\[ \begin{array}{cc}
\sim 1 & R' \\
-2z & -1 \mathbf{Y}_0 (\mathbf{M}_R) \mathbf{Y}, (\mathbf{M}_L) \\
\sim 0 & 2 \mathbf{Y}_0 (\mathbf{M}_R) \mathbf{Y}, (\mathbf{M}_L) 
\end{array} \]

\[ Y_1 (z) = -\frac{z}{2} \mathbf{Y}_0 + \Theta (\log z) z \]

(Taylor-like expansion)

Thus \( 2Y_1 (z) \) is flat to leading order in \( z \). This certainly isn't a valid approximation at large \( z \), but the point isn't that \( 2Y_1 (z) \) is constant \( 1 \). The point is that in the expansion of \( 2Y_1 (z) \) is a universal part. This gives a flavor-conserving coupling as we saw for the zero mode \( Z \) since the orthogonality (normality) of the fermion wavefunctions removes any \( c \)-dependence. There are flavor-violating terms in the rest of the expansion for \( Y_1 (z) \), but as seen in the plot, these are negligible compared to the flavor-violating profile of the \( Z^{(0)} \) term.

**Sanity Check:** \( Y_1 (z) = \frac{1}{2} z + \Theta (z^2) \), i.e \( Z^{(0)} (z) \) does not contain a flat piece in its expansion. Thus the universal part of \( Z^{(0)} (z) \) is indeed the only source of flavor-conserving couplings. [\( - \) the non-flat term] and give a flavor-conserving piece, but we will shortly see that this is suppressed by the fermion \( f_i \) functions.\]
Another heuristic way to understand the contribution of the $\hat{h}_2^{(1)}(z)$ term is to appeal to the AdS/CFT dictionary. In the CFT the UV brane $\sim$ elementary states while IR brane $\sim$ composite states. Naively we expect our light ($\sim 260$ GeV) fields to be elementary. However, the flat (1,1) gauge boson zero mode probes both branes $\hat{y}$ is thus a mixture of elementary with some composite. More precisely, the zero mode is a rotation of elementary with some composite. This means that the KK modes, which are naively composite, must also contain some elementary state. It is this "elementary state component" of the KK $\hat{y}$ that we are considering in the leading flavor-universal term coming from the $\hat{h}_2^{(1)}(z)$ term.

First we need the normalization of $\hat{h}_2^{(1)}(z)$. Recall that this comes from requiring the 4D kinetic term (ie KK decompose from $d\phi_2$) to be canonically normalized; cf. p. 2 for the zero mode.

$$\hat{h}_2^{(1)}(z) = \hat{N}_2 \left[ \hat{Y}_+(\hat{N}^2) \hat{y}_0, (M_2^*) \right] - 3 \hat{N}_2 \hat{Y}_+(\hat{N}^2) \hat{y}_0 (M_2^*)$$

Let us redefine $\hat{N}_2$ to absorb a factor of $\hat{Y}_+(\hat{N}^2)$

$$\hat{N}_2 = \left[ \hat{J}_+(M_2) - A \hat{Y}_+(M_2) \right]$$

Where: $$A = \frac{\hat{J}_+(\hat{N}^2)}{\hat{Y}_+(\hat{N}^2)} \quad M = M_2^{(1)} = \frac{x_1}{R}$$

We know that the $A\hat{y}_+(M_2)$ term is small compared to the first term. Thus let us simplify our job by neglecting it in our determination of $\hat{N}_2$. The error will be small since the $A\hat{y}_+(M_2)$ term is roughly a few % of the leading term. Most of the $d\phi_2$ integral.

Our normalization condition is $\int_0^\pi d\phi_2 \frac{R^2}{2} (\hat{h}_2^{(1)}(z))^2 = 1$.

This integral is straightforward if one uses the orthogonality of Bessel functions of the first kind, namely:

$$\int_0^\pi J_v \left( \frac{p}{a} \right) J_v \left( \frac{p}{a} \right) \rho p d \rho = \frac{1}{2} a^2 \left[ J_{v+1} \left( \frac{p}{a} \right) \right]^2$$

Alternatively, one may use

$$J_\nu (z) = \frac{e^{i\nu(z)}}{2^\nu \Gamma(\nu)} \left( J_{\nu-1}(z) + J_{\nu+1}(z) \right) \quad J_\nu (z) = \frac{1}{2} (J_{\nu+1}(z) - J_{\nu-1}(z))$$

One finds that the correctly-normalized approximation for $\hat{h}_2^{(1)}(z)$ (neglecting the $A\hat{y}_+(M_2)$ term) is:

$$\hat{h}_2^{(1)}(z) \approx \sqrt{\frac{2}{R}} \frac{1}{J_1(x_1 R)} \cdot \hat{Y}_+(x_1 R)$$
We already made the case that the $\Delta y_1(M^2)$ term gives the leading universal contribution, so we cannot completely neglect it. We will assume the normalization \( N \) from the previous $J_1(M^2)$ term approximation:

\[
h_2^{(1)}(z) \approx \frac{\frac{1}{z}}{J_1(x_i R)} \left( J_1(x_i \frac{R}{R'}) \right) \frac{d_0(x_i, y_i)}{y_o(x_i, R/R')} \approx 2 y_1(x_i, \frac{R}{R'})
\]

Where we will only consider the second term for the universal coupling. We now proceed analogously to what we did for $h_2^{(0)}$ on p. 3.

**Universal $k R^2$ Coupling**

For this we only need to consider the second term.

Let's make some approximations:

\[
J_0(x_i, \frac{R}{R'}) \approx J_0(0) = 1
\]

\[
y_0(x_i, \frac{R}{R'}) \approx \frac{2(Y + \log(\frac{x_i}{2}) + \log(\frac{R}{R'}))}{\pi} \approx \frac{2}{\pi} \frac{1}{\log(\frac{R}{R'})}
\]

\[
Y_1(x_i, \frac{R}{R'}) \approx -\left( \frac{R'}{x_i \frac{R}{R'}} \right) + 0(2)
\]

Next we pull out the universal part of $2 y_1(x_i, \frac{R}{R'})$:

\[
y_1(x_i, \frac{R}{R'}) \approx -\left( \frac{R'}{x_i \frac{R}{R'}} \right) + 0(2)
\]

So we have $2 y_1(x_i, \frac{R}{R'})$ gives universal term (0(2)).

**Recall:** We are not approximating $y_1(x_i, \frac{R}{R'})$, this would be a bad approx! This is identifying and isolating the universal part of $h_2^{(1)}$. It is easy to see that $2 y_1(x_i, \frac{R}{R'})$ does not have a universal part.

Now we follow exactly the same procedure as on page 3.

\[
h_2^{(1)}(z) \left|_{\text{universal}} \approx \frac{\frac{1}{z}}{J_1(x_i R)} \left( - \left[ \frac{-\frac{2}{\pi}}{2 \log(\frac{R}{R'})} \right] \right) \approx \frac{1}{\log(\frac{R}{R'})} \approx 1.13 \rightarrow 1
\]

Then following the analysis of $g_{2\phi}^{4\phi}$ on p. 3, we obtain

\[
\frac{g_{2\phi}^{4\phi}}{4 \pi} = \frac{g_{2\phi}^{4\phi}}{4 \pi} \frac{1}{\log(\frac{R}{R'})} = \frac{g_{2\phi}^{4\phi}}{4 \pi \sqrt{\log(\frac{R}{R'})}}
\]

Dimensionless
NON-UNIVERSAL (FVNC) COUPLING

OK, NOW THAT WE'RE DONE WITH THE UNIVERSAL PART, WE CAN FORGET THE $2v_1(h^2)$
TERM ALTOGETHER. ITS CONTRIBUTION TO THE FVNC PART IS NEGligible SINCE
ITS INTEGRAL IS SO SMALL. THUS WE'RE BACK TO

$$h^{(0)}_2(z) = \frac{2}{\sqrt{R}} \frac{2}{J_1(\sqrt{x}) R} J_1(\sqrt{x}, \frac{2}{R}).$$

NOW WE PERFORM THE OVERLAP INTEGRAL WITH FERMIONS TO GET THE
4D EFFECTIVE NON-UNIVERSAL COUPLING

$$g_{4D,FVNC}^{2\mu} = g_{SM}^{2\mu} \frac{1}{R} \frac{1}{J_1(\sqrt{x}) R^2} \left[ \frac{1}{J_1(\sqrt{x}) R^2} \right] \frac{1}{1 - c} \int \frac{2}{\sqrt{R}} \frac{2}{J_1(\sqrt{x}) R} J_1(\sqrt{x}, \frac{2}{R})$$

$$= g_{SM}^{2\mu} \frac{1}{R} \int_0^1 dx \frac{1}{R} \int_0^1 dx \frac{1}{J_1(\sqrt{x}) R^2} \sqrt{x} \int_0^1 dx \frac{1}{J_1(\sqrt{x}) R^2}$$

$$= g_{SM}^{2\mu} \frac{1}{R} \int_0^1 dx \frac{1}{J_1(\sqrt{x}) R^2} \sqrt{x} \int_0^1 dx \frac{1}{J_1(\sqrt{x}) R^2}$$

$$= \sqrt{x} \int_0^1 dx \frac{1}{J_1(\sqrt{x}) R^2} \sqrt{x} \int_0^1 dx \frac{1}{J_1(\sqrt{x}) R^2}$$

$$= \sqrt{x} \int_0^1 dx \frac{1}{J_1(\sqrt{x}) R^2} \sqrt{x} \int_0^1 dx \frac{1}{J_1(\sqrt{x}) R^2}$$

NOW ROTATE TO THE KK MASS BASIS

$$g_{4D}^{2\mu} = g_{SM}^{2\mu} \sqrt{\frac{R}{R'}} \gamma_\mu \gamma_5 f^5$$

REMARKS: REMEMBER THAT THE WHOLE POINT OF THE UNIVERSAL PIECE WAS THAT
THE UNIVERSALITY PREVENTS ANY FERM. EFFECTS EVEN AFTER ROTATING
INTO THE KK BASIS.

HOWEVER, THE NON-UNIVERSAL PART DOES CONTRIBUTE TO THE FLAVOR-
CONSERVING COUPLING,

$$g_{4D, \text{non-universal}}^{2\mu} = g_{SM}^{2\mu} \sqrt{\frac{R}{R'}} \gamma_\mu \gamma_5 f^5.$$

WE CAN SEE, HOWEVER, THAT FOR ZERO MODE PRECISIONS $f_i \ll 1$
(ESPECIALLY FOR LIGHT FERMIONS IN THE AVERAGING SCENARIO)
SO THAT THIS IS SUPPRESSED RELATIVE TO $g_{4D}^{2\mu}$ ON P.6.
MATCHING TO THE EFFECTIVE LFV $\lambda_3$ (See p. 1)

Let us remind ourselves of our notation (cf. Peskin p. 7-9)

\[ \Delta \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}} - \mathcal{L}_{\text{SM}} = \frac{g}{c_w} Z_F \left[ \bar{e}_L Y^e \left( s_w^2 - \frac{1}{3} \right) e_L + \bar{e}_R Y^e (s_w^2) e_R + \ldots \right] \]

Important def. of SM couplings

\[ \Delta \mathcal{L}_{\text{eff}} = \frac{4 G_F}{\sqrt{2}} \left( \bar{e}_L Y^e (T^3 - s_w^2 Q) e_R \right)^2 \]

\[ \frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W^2} = \frac{3}{8 G_F M_W^2} \]

Now consider \[ \Delta \mathcal{L}_{\text{eff}} = \frac{4}{3} G_F g_3 \left( \bar{e}_R Y^e (e_L Y^e) \right) \]

Let us ignore the KK contributions for now. Let us match this effective operator to the $Z$-exchange diagram.

\[ M_R \quad e_2 \quad e_3 \quad \bar{e}_R \quad e_3 \]

\[ = g_{4 \bar{q}} \quad \frac{1}{M_Z^2} \quad g_{2 \bar{q} \bar{q}} \quad \frac{1}{M_Z^2} \quad \left( \bar{e}_R Y^e \right) \left( \bar{e}_R Y^e \right) \]

\[ \text{Flavor-conserving} \]

\[ \text{Flavor-violating:} \quad g_{2 \bar{q} \bar{q}} = g_{2 \bar{q} \bar{q}} \Delta_{\text{eff}} = \frac{1}{2 (1 - 2 \zeta)} \log \frac{R}{R} \quad \Delta_{\text{eff}} \]

\[ \Delta_{\text{eff}} = \frac{(M_R^2)^2}{2 (1 - 2 \zeta)} \log \frac{R}{R} \Delta_{\text{eff}} \]

\[ \text{Minimal model approximation:} \quad \Delta_{\text{eff}} \approx \frac{\lambda^n}{\sqrt{\lambda^n}} \]

\[ f_c = \sqrt{1 - \frac{1 - 2 \zeta}{(2 \zeta - 1)^2}} \]

\[ f_{\text{inv}} = \sqrt{\frac{1}{\sqrt{\lambda^n}}} \]

Now doing the matching:

\[ \frac{1}{2} G_F g_3 = \frac{g_{2 \bar{q} \bar{q}}}{M_Z^2} = \frac{1}{2} \frac{1}{M_Z^2} \Delta_{\text{eff}} = \left[ \frac{3}{4} g_{2 \bar{q} \bar{q}} \right]^2 \frac{1}{M_Z^2} \Delta_{\text{eff}} \]

From which we derive: \[ g_3 = 2 g_{2 \bar{q} \bar{q}} \Delta_{\text{eff}} \]

Similarly:

\[ \frac{g_{2 \bar{q} \bar{q}}}{2 G_F M_Z^2} g_4 = \left[ \frac{3}{4} g_{2 \bar{q} \bar{q}} \right]^2 \frac{1}{M_Z^2} \Delta_{\text{eff}} \Rightarrow g_4 = 2 g_{2 \bar{q} \bar{q}} \Delta_{\text{eff}} \]

\[ \frac{1}{2} \frac{1}{M_Z^2} \Delta_{\text{eff}} = \left[ \frac{3}{4} g_{2 \bar{q} \bar{q}} \right]^2 \frac{1}{M_Z^2} \Delta_{\text{eff}} \Rightarrow g_5 = 2 g_{2 \bar{q} \bar{q}} \Delta_{\text{eff}} \]
Now consider the $\mu \rightarrow e$ effective \( Z \)

\[
\sum_{\text{f}} \bar{\nu} \gamma_{\text{f}} (\gamma_{\text{f}}) P_{\nu_{\text{f}}} \nu \rightarrow \bar{\nu} \gamma_{\text{f}} P_{\nu_{\text{f}}} \nu \rightarrow \frac{\Delta M}{\Delta M} \frac{\Delta M}{\Delta M} \frac{\Delta M}{\Delta M} \frac{\Delta M}{\Delta M} \frac{\Delta M}{\Delta M} \frac{\Delta M}{\Delta M} \\
\Rightarrow \nu \pm q = 2 g_{\nu q} \Delta_{e e}^{(0)}
\]

Now include a KK $Z$ to the minimal model.

We introduce a handy notation

\[
g_{\nu q} = \frac{g_{\nu q}}{g_{\nu q}} \left( g_{\nu q} \Delta_{e e}^{(0)} + \Delta_{e e}^{(0)} \right)
\]

\[
\Delta_{e e}^{(0)} = \left( \frac{6}{2} \right) \Delta_{e e}^{(0)}
\]

Now we can write the modified effective couplings

Heuristically: (effective coupling) = \( (g_{\nu q})^2 \) \( \text{[zero mode]} + g_{\nu q} \frac{M^2}{M^2} \Delta_{e e}^{(0)} \)

Thus:

\[
\begin{align*}
3_{3,4} &= 2(3,4)^2 \left[ \Delta_{e e}^{(0)} + \frac{M^2}{M^2} \Delta_{e e}^{(0)} \right] \\
5_{5,6} &= 2(5,6)^2 \left[ \Delta_{e e}^{(0)} + \frac{M^2}{M^2} \Delta_{e e}^{(0)} \right] \\
\nu \pm q &= 2 g_{\nu q} \left[ \Delta_{e e}^{(0)} + \frac{M^2}{M^2} \Delta_{e e}^{(0)} \right]
\end{align*}
\]

Now include the KK photon

\[
e_{\text{SM}} = \frac{g_{\nu q}}{g_{\nu q}} \rightarrow g_{\nu q} = g_{\nu q} \left( g_{\nu q} \Delta_{e e}^{(0)} \right)
\]

\[
g_{3,4} = 2(3,4)^2 \left[ \Delta_{e e}^{(0)} + \frac{M^2}{M^2} \Delta_{e e}^{(0)} \right] - 2(3,4)^2 \left[ \Delta_{e e}^{(0)} + \frac{M^2}{M^2} \Delta_{e e}^{(0)} \right]
\]

\[
g_{5,6} = 2(5,6)^2 \left[ \Delta_{e e}^{(0)} + \frac{M^2}{M^2} \Delta_{e e}^{(0)} \right] - 2(5,6)^2 \left[ \Delta_{e e}^{(0)} + \frac{M^2}{M^2} \Delta_{e e}^{(0)} \right]
\]

For the $\nu \rightarrow e$ amplitude we will write the coupling as:

\[
e_{\text{SM}} Q_{N} \gamma \left[ \frac{\Delta_{e e}^{(0)}}{\cos \theta_{W}} \frac{Q_{N}}{Q_{N}} \right] \Rightarrow \left[ \frac{\Delta_{e e}^{(0)}}{\cos \theta_{W}} \frac{Q_{N}}{Q_{N}} \right]
\]

\[
\tilde{Q}_{N} = Q_{N} \left( 22 + N \right) + Q_{N} \left( 22 + 2 \right), \text{ electric charge of nucleus}
\]

Similarly:

\[
e_{\text{SM}} Q_{L} = \left[ \frac{\Delta_{e e}^{(0)}}{\cos \theta_{W}} \frac{Q_{N}}{Q_{N}} \right] \Rightarrow \left[ \frac{\Delta_{e e}^{(0)}}{\cos \theta_{W}} \frac{Q_{N}}{Q_{N}} \right]
\]

Note: convention for lepton charge

Now it is easy to match coefficients:

\[
(\nu \pm q) = 2 g_{\nu q} \left[ \Delta_{e e}^{(0)} + \frac{M^2}{M^2} \Delta_{e e}^{(0)} \right] - 2 g_{\nu q} \frac{M^2}{M^2} \frac{Q_{N}}{Q_{N}} \Delta_{e e}^{(0)}
\]
THE CUSTODIALLY PROTECTED MODEL

Details of the custodially-protected RSI model can be found in Monika's thesis [0903.2415]. We will only summarize the relevant results.

* RS models with bulk fields suffer from a large $T$-parameter. One way to solve this is to expand the bulk gauge symmetry to $SU(3)_C \times SU(2)_L \times SU(2)_R \rightarrow U(1)_X$ ($\text{hep-ph/0308056, 0308058}$)

\[ \rightarrow \text{breaks to $U(1)_Y$ on UV brane (UVs nec. to get correct $U(1)_Y$ charges)} \]

* One can impose a discrete $\mathbb{Z}_2$ symmetry: $SU(2)_L \rightarrow SU(2)_R$. This is equivalent to gauging custodial symmetry. This protects the experimentally-constraining $Zb\bar{b}$ coupling ($\text{hep-ph/0605341}$) and can be used to protect against tree-level FCNCs.

How this works: $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V \rightarrow U(1)_V$, generated by $T_3^e \otimes T_3^e$. $\mathbb{Z}_2$ imposes $T_3^e = T_3^e$ so that the effect of new physics must obey $Q_3 = Q_3^R$. On the other hand, $Q_4 = Q_4^L + Q_4^R$ is conserved. This $\rightarrow S_3^L = S_3^R$. \[ \Rightarrow S_3^L = 0 \] from the BSM sector. Recall the $Z$ coupling is $Q_3^L - Q_3^R S_3^L$. Since BSM terms are conserved, new physics cannot give an anomalous $Zb\bar{b}$ coupling.

* The low energy gauge eigenstate spectrum includes a $Z'$, $Z''$, $Z_\gamma$, which mix into mass eigenstates $Z, Z', Z_\gamma$. (Note: previously we wrote $Z' = Z''$.)

* The $Z, Z'$ FCNC coupling to LH fermions is protected ($\mathcal{O}$), but the RH coupling is unconstrained. The leading LH FCNC comes from the $Z'$ component of the $Z'_{\gamma}$:

\[ Z_{\gamma} \approx \frac{Q_3^L - Q_3^R}{\sqrt{2}} + \frac{\sin \phi_{\gamma} Z_{\gamma}^{\mu \nu}}{\sqrt{2}} + \frac{\beta Z_{\gamma}}{\sqrt{2}} \]

\[ \Rightarrow \text{no coupling to leptons} \text{ (no x change)} \rightarrow \text{no FCNC} \]

* Our strategy: instead of the minimal model ($L = L_A$), we will try to push all the FCNC into the LH sector, where custodial protection takes care of most of it. This means pushing the LH fermions toward the IR brane and the RH fermions to the UV brane.

* $\mathbb{Z}_2$ is broken on the UV brane, but we will ignore this small effect.
WE WILL HAVE TO TREAT THE LH & RH SECTORS SEPARATELY. THE LH SECTOR WILL HAVE FCNC ONLY FROM THE 2H \rightarrow Y(0). (IN PARTICULAR, ONLY THE 20^3 \subset Z_4 \ GIVES LEPTON FLAVOR VIOLATION.) THE RH SECTOR WILL HAVE THE SAME FCNC STRUCTURE AS IN THE MINIMAL MODEL. WE WILL WANT TO MINIMIZE Br(\mu \rightarrow e) OVER \text{PMNS} \text{ Values Subject to the SM Mass Spectrum.}

A NICE SHORTCUT: \( Br(\mu \rightarrow e) \sim \left[ \frac{A}{\sin \theta_{\mu e}} \right] \left[ \frac{B}{\csc \theta_{\mu e}} \right] \frac{1}{m_{\mu} m_{e}} \)

Then use: \((a-b)^2 = a^2 - 2ab + b^2 \Rightarrow A + B \geq 2 \sqrt{AB} \)

\[
Br(\mu \rightarrow e) \geq 2 \sqrt{AB} \frac{A_{\mu e} f_{\mu e} f_{\tau e} f_{\mu \tau}}{Y_{\nu} m_{\mu} m_{e}} = 2 \sqrt{AB} \frac{m_{\mu} m_{e}}{Y_{\nu} m_{\mu} m_{e}}
\]

ANTHROPIAN ASSUMPTION

\( \gamma' = \frac{m}{Y_{\nu} m_{\mu} m_{e}} \)

SINCE Br(\mu \rightarrow e) IS THE STRONGEST BOUND, WE WILL ONLY FOCUS ON THIS.

SOME USEFUL CONVERSIONS: LIMIT OF UNBROKEN P_{\mu} \Rightarrow \left\{ \begin{array}{c} \cos \frac{\theta}{2} = \frac{1}{2} \cos \phi \\ \sin \frac{\theta}{2} = \frac{1}{2} \sin \phi \end{array} \right.

\Rightarrow \frac{\sin \theta}{2} = \frac{\tan \theta_{\mu e}}{\cos \phi} = \frac{\tan \theta_{\mu e}}{\cos \theta_{\mu e}} = \frac{1}{2} \frac{\sin \theta}{2}

2 \cos \frac{\theta}{2} \approx 0.60

\( g_{\chi} = \frac{g'}{\cos \phi} = \frac{\tan \theta_{\mu e}}{\cos \phi} \frac{g}{\frac{1}{2} \cos \frac{\theta}{2}} \)

CUSTOMER EFFECTIVE COUPLINGS FOR \( \mu \rightarrow 3e \)

\[
\begin{align*}
g_3 &= 2g_{\mu}^3 \left[ \frac{1}{\Delta_{\mu e}^2} + \frac{M_{\psi}/M_{\chi}^2}{M_{\psi}/M_{\chi}^2} \Delta_{\mu e} \right] - 2 \left( q_{\mu}^2 - q_{\mu}^2 \right) \frac{M_{\psi}/M_{\chi}^2}{M_{\psi}/M_{\chi}^2} \frac{M_{\psi}/M_{\chi}^2}{M_{\psi}/M_{\chi}^2} \\
g_4 &= 2g_{\mu}^2 g_{\mu}^2 \frac{M_{\psi}/M_{\chi}^2}{M_{\psi}/M_{\chi}^2} \Delta_{\mu e} \csc \theta_{\mu e} - 2 \left( q_{\mu}^2 - q_{\mu}^2 \right) \frac{M_{\psi}/M_{\chi}^2}{M_{\psi}/M_{\chi}^2} \frac{M_{\psi}/M_{\chi}^2}{M_{\psi}/M_{\chi}^2} \\
g_5 &= 2g_{\mu} g_{\mu} g_{\mu} \left[ \frac{1}{\Delta_{\mu e}^2} + \frac{M_{\psi}/M_{\chi}^2}{M_{\psi}/M_{\chi}^2} \Delta_{\mu e} \right] - 2 \left( q_{\mu}^2 - q_{\mu}^2 \right) \frac{M_{\psi}/M_{\chi}^2}{M_{\psi}/M_{\chi}^2} \frac{M_{\psi}/M_{\chi}^2}{M_{\psi}/M_{\chi}^2} \\
g_6 &= 2g_{\mu} g_{\mu} g_{\mu} \frac{M_{\psi}/M_{\chi}^2}{M_{\psi}/M_{\chi}^2} \Delta_{\mu e} \csc \theta_{\mu e} - 2 \left( q_{\mu}^2 - q_{\mu}^2 \right) \frac{M_{\psi}/M_{\chi}^2}{M_{\psi}/M_{\chi}^2} \frac{M_{\psi}/M_{\chi}^2}{M_{\psi}/M_{\chi}^2}
\end{align*}
\]

WHERE \( \Delta_{\mu e}^2 \) IS WRITTEN WITH \( \sin^2 \theta_{\mu e} \) ONLY.
CUSTODIAL EFFECTIVE COUPLINGS FOR $L \to e$.

This requires more work.

- Leptons have $N_a \times$ charge, quark couples to $Z^0 \subset Z_a$.
- Quarks couple to both $Z^0 \neq Z_a$ (a component is negligible).

$$Z_H = \cos \frac{\phi}{2} Z^{\text{ee}} \sin \frac{\phi}{2} Z^{\text{ee}} + \beta Z^{\text{ee}}$$

The $\beta$ is a new gauge boson. Let's work out its couplings.

(See, e.g., Monica's thesis.)

Custodial model has:

\[ W_R^R \quad \sin \phi \quad X \quad W_R^L \quad \cos \phi \quad X \]

\[ \sigma_{L \times X} = \sin \phi \quad \sigma_{R \times X} = \cos \phi \]

Breaking to $U(1), Y$.

\[ Z = c_w W^L + s_w B \]

\[ A = s_w W^L + c_w B \]

\[ c_w = \frac{g}{\sqrt{g^2 + g^2}} \Rightarrow \cos \phi = \frac{3}{\sqrt{g^2 + y^2}} \]

\[ g_{\text{mix}} = g \cos \phi T^3_R - s_w \sin \phi T^3 \]

For the quarks:

\[ \text{SU}(2)_L \quad \text{SU}(2)_R \quad \text{U}(1)_Y \]

\[ Q_L \quad 0 \quad 0 \quad \frac{1}{3} \]

\[ u_R \quad 1 \quad 1 \quad \frac{1}{3} \]

\[ d_R \quad 3 \quad 3 \quad \frac{1}{3} \]

\[ \bar{d}_R \quad \bar{3} \quad \bar{3} \quad \frac{1}{3} \]

By choice of BC, the only fields with zero nodes are:

- $q^6, q^4, u_R, d_R$

\[ T^3_R = -\frac{1}{2} \quad -\frac{1}{2} \quad 0 \quad -1 \]

\[ T^3 = \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{3} \]

\[ q^2_{\text{NN}} = g \cos \phi \left[ (2N+1)(\frac{3}{2}) + (2N+2)(-\frac{3}{2}) \right] - g \sin \phi \left[ (3N+3)(\frac{3}{2}) \right] \]

\[ \theta = \frac{\phi}{4} \cos \theta \]

\[ Q^2_{\text{NN}} = \cos \theta \left[ (2N+1)(\frac{3}{2}) + (2N+2)(-\frac{3}{2}) \right] - \frac{3\sin \theta \cos \theta}{4} (2N) \]

\[ (V-q) = 2g \left[ \frac{\Delta^0_{\text{KK}}}{M^2_{\text{KK}}} + \frac{\Delta^0_{\text{KK}}}{M^2_{\text{KK}}} \Delta K \right] - 2g \Delta^0_{\text{KK}} \frac{M^2_{\text{KK}}}{M^2_{\text{KK}}} \frac{Q_N}{Q_N} \Delta K \]

\[ (V+q) = 2g \frac{M^2_{\text{KK}}}{M^2_{\text{KK}}} \Delta K \cos \frac{\phi}{2} + 2g \frac{M^2_{\text{KK}}}{M^2_{\text{KK}}} \frac{Q_N}{Q_N} \Delta K \cos \frac{\phi}{2} \sin \frac{\phi}{2} - 2g \frac{M^2_{\text{KK}}}{M^2_{\text{KK}}} \frac{Q_N}{Q_N} \Delta K \]

\[ \Delta K = 2g \left[ \frac{M^2_{\text{KK}}}{M^2_{\text{KK}}} \Delta K \cos \frac{\phi}{2} + 2g \frac{M^2_{\text{KK}}}{M^2_{\text{KK}}} \frac{Q_N}{Q_N} \Delta K \cos \frac{\phi}{2} \sin \frac{\phi}{2} - 2g \frac{M^2_{\text{KK}}}{M^2_{\text{KK}}} \frac{Q_N}{Q_N} \Delta K \right] \]
\[ \text{Non-universal part: change vars to } y = \frac{R}{r} \]
\[ \int \frac{\gamma^2}{\sqrt{1 - \frac{2\gamma^2}{R^2}}} \cdot \frac{M^2}{4} \int_{\gamma}^{1} dz \left( \frac{\gamma^2 z}{R^2} \right)^{2c} z^2 \left( 1 - 2 \log \frac{\gamma^2}{R^2} \right) \]
\[ = B \cdot \frac{M^2}{4} \left( \frac{R}{r} \right)^{2c} \int_{\gamma}^{1} dy \left( \frac{R}{r} \right)^{2c} \left( 1 - 2 \log \gamma \right) \]
\[ = B \frac{M^2}{4} \left( \frac{R}{r} \right)^{2c} \int_{\gamma}^{1} dy \left( 1 - 2 \log \gamma \right) \]
\[ = \frac{1}{3-2c} \left[ y^{3-2c} \right]^{\gamma/R} = (\gamma) \]
\[ (\gamma) = -2 \int_{1}^{\gamma/R} dy \left( 1 - 2 \log \gamma \right) \]

**Trick: Integrate by parts**
\[ \int dy \gamma^a \log y = \frac{a+1}{a+1} \gamma^{a+1} \log y - \int dy \frac{1}{a+1} \gamma^{a+1} \]
\[ (\gamma) = -2 \left[ \frac{1}{3-2c} \gamma^{3-2c} \log y \right]^{\gamma/R} + 2 \int dy \frac{1}{3-2c} \gamma^{2-2c} \]
\[ = \frac{2}{3-2c} \cdot \frac{1}{3-2c} \left[ \gamma^{3-2c} \right]^{\gamma/R} \]

No more integrals. Just algebra.
\[ \left[ \frac{1}{3-2c} \gamma^{3-2c} \right]^{\gamma/R} = \left( 1 - \frac{1}{3-2c} \right) \gamma^{3-2c} \left[ \gamma^{3-2c} \right]^{\gamma/R} - 2 \left[ \frac{1}{3-2c} \gamma^{3-2c} \log y \right]^{\gamma/R} \]
\[ = \frac{5-2c}{3-2c} \cdot \frac{1}{5-2c} \left( \frac{\gamma^{3-2c}}{R^{3-2c}} - 1 \right) - \frac{2}{3-2c} \left( \frac{R}{r} \right)^{3-2c} \log \frac{R}{r} \]

Subleading! Leading term in \( R/R' \)
APPENDIX (That's right, these notes have an appendix!)

Now we prove some useful facts about the anarchic Yukawa matrices.

In the $c$-basis, the SM Yukawas look like

\[
\begin{pmatrix}
 f_1 f_1 & f_1 f_2 & f_1 f_3 \\
 f_2 f_1 & f_2 f_2 & f_2 f_3 \\
 f_3 f_1 & f_3 f_2 & f_3 f_3
\end{pmatrix}
\]

Where all the $c_i$ are $O(1)$ (or $O(N)$) with no hierarchies. The $f$'s introduce the observed mass hierarchies.

To make this manifest, let us define

\[
\begin{align*}
 s_1^2 &= f_1 / f_3 \\
 s_2 &= f_2 / f_3
\end{align*}
\]

Then the Yukawas take the form

\[
\begin{pmatrix}
 s_1^4 c_{11} & s_1^2 s_2 c_{12} & s_1^2 c_{13} \\
 s_2 s_1^2 c_{21} & s_2^2 c_{22} & s_2 c_{23} \\
 s_1^2 c_{31} & s_2 c_{32} & c_{33}
\end{pmatrix}
\]

Claim: Upon diagonalization, $\lambda \sim O(1)$ (or $O(N)$) in we get a realistic hierarchy from generic $c_i$'s. This is important because we would then know that the rotation matrix will be something with $s_i$ on the off-diagonal elements.

Use perturbation theory for the hierarchies in the $s_i$'s. The eigenvalues are given by solutions to

\[
\det (\lambda - \lambda_i) = 0
\]

\[
= (1 - \lambda_i) (s_2^2 - \lambda) (s_4 - \lambda_i) + s_4 s_2^2 = 0
\]

Consider the largest eigenvalue, $\lambda_3$. We may write

\[
\begin{align*}
 (1 - \lambda_3) &= \frac{s_4 s_2^2}{(s_2^2 - \lambda_3) (s_4 - \lambda_3)} \\
 &\sim O(\delta)
\end{align*}
\]

Since $\lambda_3$ is largest $\sim (\lambda_3 + 1)$

The denominator is $O(1)$

Thus is $O(\delta)$, $\Rightarrow \lambda_3$ is indeed $\sim 1$.

Hence is $O(1)$. 
For the smaller eigenvalues, $\lambda_2$,

$$
(\delta_2^2 - \lambda_2) = \frac{-4 \delta_1 \delta_2}{(1 - \lambda_2) (\delta_1^2 - \delta_2^2)} \quad \sim \quad \Theta(\delta^2)
$$

Again gives $\lambda_2 \sim \Theta(\delta^2)$.

Thus:

$$
X \sim f_3^2 \left( \delta_1, \delta_2^2, 1 \right) \sim \left( \frac{f_1^2}{f_2^2}, f_2^2, f_3^2 \right)
$$

Corollary:

If $U^T X U = X$,

then the off-diagonal elements of $U$ go like these $g_i$.

$$
\Theta: \left( \begin{array}{cc}
\frac{f_1f_2}{f_2 f_4} & \frac{f_1f_2}{f_2 f_4} \\
\frac{f_2 f_4}{f_1 f_2} & \frac{f_2 f_4}{f_1 f_2}
\end{array} \right) = f_1^2 \left( \begin{array}{cc}
1 & \Theta \\
\Theta & \Theta
\end{array} \right)
$$

Diag. via

$$
\begin{pmatrix}
1 & \Theta \\
\Theta & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & \Theta \\
\Theta & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} + \Theta(\delta^2)
$$