P3318 HW #10 Solutions   due 6 May

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1. (H&F question 3, p.290) DISPLACED AXIS THM

(8.11) $I_{\alpha\beta} = \sum_i m_i \left(r_i^2 \delta_{\alpha\beta} - r_{i\alpha} r_{i\beta} \right)$

write $\vec{r}_i = \vec{s}_i + \vec{a}$

$\vec{s}_i$: cm coordinate $\vec{a}$: shift of origin from cm

$$I_{\alpha\beta} = \sum_i m_i \left[(\vec{s}_i + \vec{a})^2 \delta_{\alpha\beta} - (s_{i\alpha} + a_\alpha)(s_{i\beta} + a_\beta) \right]$$

$$= \sum_i m_i \left(s_i^2 \delta_{\alpha\beta} - s_{i\alpha} s_{i\beta} \right) \quad \leftarrow I_{cm}$$
$$+ \sum_i m_i \left(a^2 \delta_{\alpha\beta} - a_\alpha a_\beta \right) \quad \leftarrow = (\Sigma m_i)(a^2 \delta_{\alpha\beta} - a_\alpha a_\beta)$$
$$+ \sum_i m_i \left(2 \vec{s}_i \cdot \vec{a}\, \delta_{\alpha\beta} - s_{i\alpha} a_\beta - a_\alpha s_{i\beta} \right)$$

3rd line vanishes by cm coords: $\sum_i m_i s_{i\alpha} = 0$

$$= \boxed{I_{cm} + M(a^2 \delta_{\alpha\beta} - a_\alpha a_\beta)}$$

2. (H&F QUESTION 8, P.299) <u>A paradox?</u>

The question asks: if $\vec{L} = I\vec{\omega}$ and $\vec{L}$ & I are constants, why isn't $\vec{\omega}$ constant?

This is referenced below (8.32) where it was pointed out that $\vec{\omega}$ is <u>not</u> constant. This is because in the <u>BODY</u> FRAME

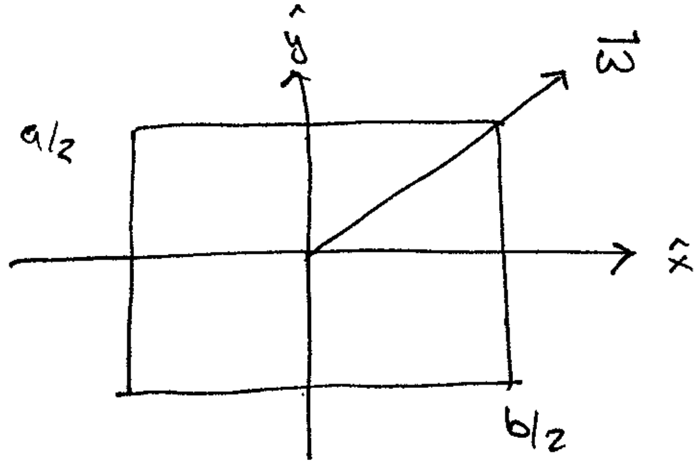
$$(8.33) \quad \dot{\vec{L}} = \overset{\circ}{\vec{L}} + \underbrace{\vec{\omega} \times \vec{L}}$$

this leads to the inhomog. terms in Euler's eqns that lead to $\dot{\vec{\omega}} \neq 0$ in general

IN OTHER WORDS, IN THE BODY FRAME ONE CAN SEE THAT $\overset{\circ}{\vec{L}} \neq 0$.

3. (H&F 8.15) <u>ROTATING RECTANGULAR PLATE</u>

a)



IN THE PRINCIPLE AXIS FRAME, WE KNOW THAT $\vec{\omega}$ IS PARALLEL TO (b, a) WITH MAGNITUDE $|\omega|$

$$\Rightarrow \quad \vec{\omega} = \frac{\omega}{\sqrt{a^2+b^2}}\left(b\,\hat{x} + a\,\hat{y}\right)$$

b) THIS COORDINATE SYSTEM IS THE PRINCIPLE AXIS FRAME WHERE THE MOMENT OF INERTIA TENSOR IS DIAGONAL.

c) IN THE PRESENCE OF TORQUE, EULER'S EQNS ARE (cf (8.35))

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = \tau_1$$

+ cyclic.

c) $\dot{\vec{\omega}} = 0 \Rightarrow$

$$
\begin{aligned}
-\omega_2\omega_3(I_2 - I_3) &= \tau_1 \\
-\omega_1\omega_3(I_3 - I_1) &= \tau_2 \\
-\omega_1\omega_2(I_1 - I_2) &= \tau_3
\end{aligned}
$$

SINCE $\vec{\omega}$ is in the xy plane, $\omega_3 = 0$

$$
\Rightarrow \tau_2 = 0
$$

$$
\tau_3 = 0
$$

WE'RE LEFT WITH: $\tau_3 = -\omega_1\omega_2(I_1 - I_2)$

$$
I_1 = \int d^3x\ \rho\,(y^2 + \cancel{z^2}^{\,0}) \quad \leftarrow \text{continuum limit of (8.12)}
$$

$$
\rho = \frac{M}{ab}\,\delta(z) \quad \leftarrow \text{IDEAL } \infty\text{'LY THIN PLATE}
$$

$$
= \frac{M}{ab}\int_{-a/2}^{a/2}\int_{b/2}^{b/2} dx\,dy\ y^2
$$

$$
= \frac{M}{a}\cdot\frac{1}{3}y^3\Big|_{-a/2}^{a/2}
$$

$$
= \frac{2}{3}\frac{M}{a}\left(\frac{a}{2}\right)^3
$$

$$
= \frac{Ma^2}{12}
$$

SIMILARLY

$$I_2 = \frac{M}{ab}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} dx\, dy\; x^2$$

$$= \frac{Mb^2}{12}$$

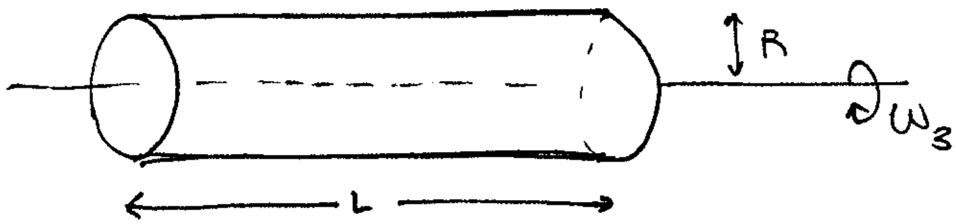
$$\Rightarrow \quad \tau_3 = -\underbrace{\omega_1}_{\frac{\omega b}{\sqrt{a^2+b^2}}}\underbrace{\omega_2}_{\frac{\omega a}{\sqrt{a^2+b^2}}}\left(\frac{Ma^2}{12} - \frac{Mb^2}{12}\right)$$

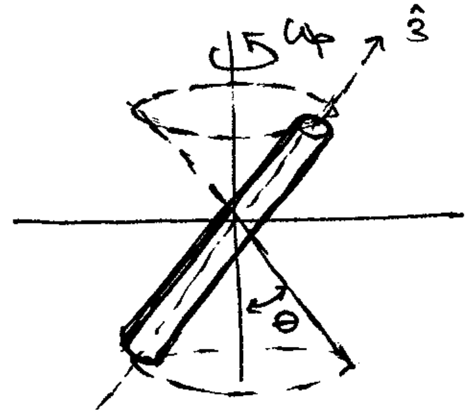
$$= \frac{-Mab}{12\,(a^2+b^2)}\,(a^2-b^2)$$

$$\boxed{\vec{\tau} = + \frac{Mab}{12(a^2+b^2)}\,(b^2-a^2)}$$

note sign differs from book.

4. TUBEWORLD





a) PRINCIPAL MOMENTS of INERTIA

$$I = I_1 = I_2 = \int d^3x \, \underbrace{\rho}_{\frac{M}{2\pi R L}\delta(r-R)} \, (\underbrace{y^2}_{(r\sin\theta)^2} + z^2)$$

$$= \frac{M}{2\pi R L} \int_{-L/2}^{L/2} dz \int_0^{2\pi} d\theta \int_0^R r\, dr \; \delta(r-R) \left(r^2 \sin^2\theta + z^2\right)$$

$$= \frac{M}{2\pi L} \int_{-L/2}^{L/2} dz \int_0^{2\pi} d\theta \left[R^2 \sin^2\theta + z^2\right]$$

$$= \frac{M}{2\pi L} \int_{-L/2}^{L/2} dz \left[\pi R^2 + 2\pi z^2\right] = \boxed{\frac{MR^2}{2} + \frac{ML^2}{12}}$$

$$I_3 = \int d^3x \left(\frac{M}{2\pi R L}\,\delta(r-R)\right)(x^2+y^2)$$

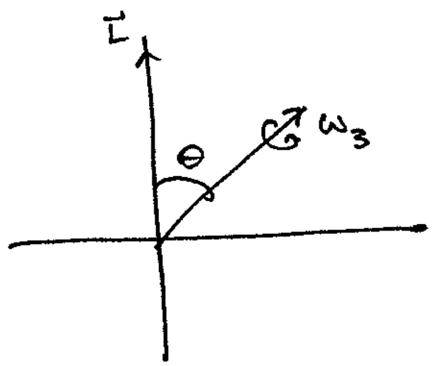
$$= \frac{M}{2\pi L}\int_{-L/2}^{L/2} dz \int_0^{2\pi} d\theta \left(R^2\cos^2\theta + R^2\sin^2\theta\right)$$

$$= \frac{MR^2}{2\pi}\int_0^{2\pi} d\theta \quad = \quad \boxed{MR^2}$$

b) THIS SYSTEM IS A <u>SYMMETRIC TOP</u>, SO WE MAY USE THE RESULTS of § 8.5 in H&F; note that Ω in H&F is ω_3 in this system.

$$(8.37) \qquad \vec{\omega}_{tot} = \underbrace{\vec{\omega}_3}_{\omega_3 \hat{3}} + \vec{\omega}_p$$

in H&F fig (8.5):



$$\cos\theta = \hat{L}\cdot\hat{3}$$

$$\hat{L} = \frac{1}{L}\vec{L}$$

$$L = I\omega_p \qquad (8.47)$$

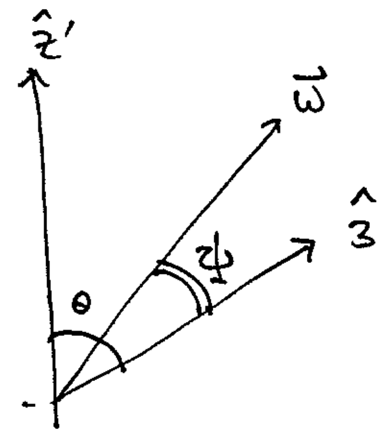
$$\vec{L}\cdot\hat{3} = I_3\omega_3 \qquad (\vec{L} = I\vec{\omega})$$

$$\Rightarrow \cos\theta = \frac{I_3\omega_3}{I\omega_p} = \frac{\omega_3}{\omega_p}\cdot\frac{12R^2}{6R^2+L^2}$$

$$\boxed{\Rightarrow \left(\frac{\omega_p}{\omega_3}\right)\cos\theta = \frac{12}{6+L^2/R^2}}$$

CORRECTION
10 MAY

c)



In the principle axis frame

$$L_1 = 0$$

$$L_2 = L \sin\theta = I_2 \omega_2 = I_2 \omega \sin\psi$$

$$L_3 = L \cos\theta = I_3 \omega_3 = I_3 \omega \cos\psi$$

$$\Rightarrow \frac{L_2}{L_3} = \boxed{\tan\theta = \frac{I_1}{I_3} \tan\psi}$$

d) [CORRECTION to may]

We ~~cannot~~ assume $T \to 0$ since $T = L^2/2I$, this would require $L \to 0$, but L is conserved.

$$\vec{L} = (I\omega_p)\hat{L} \quad \text{CONSERVED} \rightarrow \boxed{\omega_p' = \omega_p}$$

from 'FOOTBALL' LECTURE NOTES:

$$\vec{\omega} = \underbrace{\omega_p \hat{L}}_{\text{CONST.}} - \Omega \hat{3}$$

PROLATE: $I_3 < I \Rightarrow \Omega < 0$

SO TO MINIMIZE $T \sim \frac{1}{2}\vec{\omega}^T I \vec{\omega}$

WE WANT $\Omega \to 0$

$$\left.\begin{array}{l} \Omega \to 0 \\ \Omega = \omega_3 \underbrace{\left(\frac{I_3}{I} - 1\right)}_{\text{CONST}} \end{array}\right\} \Rightarrow \boxed{\omega_3 \to 0}$$

$$\cos\theta = \frac{I_3 \omega_3}{I \omega_p} \to 0 \Rightarrow \boxed{\theta' = \pi/2}$$

TUBEWORLD SETTLES TO A STATE OF FIXED AXIS ~~PRECESSION~~ ROTATION w/ AXIS $\perp$ SYM. AXIS.