Assignment 7

1. H&F 6.2
2. H&F 6.6
3. H&F 6.8
4. H&F 6.10
5. H&F 6.14
6. Consider the most general "point transformation" for one degree of freedom, $Q = Q(q,t)$. Show that the *form* of the Euler-Lagrange equation is unchanged by such a general transformation. Specifically, starting from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q},$$

and the definition of the transformed Lagrangian,

$$L(q, \dot{q}, t) = L'(Q, \dot{Q}, t),$$

show that

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{Q}}\right) = \frac{\partial L'}{\partial Q}.$$

Be sure to exercise great discipline in your application of the rules of calculus!

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b) Find the new Hamiltonian $\bar{H}$ and solve Hamilton's equations of motion.

c) Interpret the new canonically conjugate variables Q, P geometrically in q, p phase space.

d) Check whether the requirement $\delta F_1 = 0$ at the end points of the time integral for the action is equivalent to $p\,\delta q - P\,\delta Q = 0$ at the end points.

e) Find generating functions of the other three types that will generate this same canonical transformation.

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Problem 2: *(Generating function produces a canonical transformation)* You are given the generating function $F_3(p, Q) = -(e^Q - 1)^2 \tan p$. Prove that it generates the canonical transformation

$$Q = \log(1 + \sqrt{q}\cos p),$$
$$P = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p. \qquad (6.115)$$

(From Goldstein, 1980.)

Problem 3: *(Motion in an arbitrary reference frame solved with a canonical transformation)* Let z be the position of a particle (mass m) in an inertial frame, and Z be the position measured from the origin of a possibly noninertial frame displaced by the function $D(t)$ from the origin of the inertial frame. Then $Z = z - D(t)$. In the inertial frame, the Hamiltonian is

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(3)

$$H(z, p) = \frac{p^2}{2m} + V(z). \qquad (6.116)$$

Find an explicit form for the generating function $F_2(z, P, t)$ that generates the transformation from z, p to Z, P. What is the canonical transformation $Z(z, p)$, $P(z, p)$? Find $\bar{H}(Z, P, t)$ and Hamilton's equations of motion in terms of Z, P, t. Show that, in this case, there is a problem with converting F_2 into F_1 by the usual method of a Legendre transformation and explain why this occurs.

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(From Percival and Richards.)

roblem 4: *(Solve the freely falling body with a canonical transformation)* The [amiltonian of a freely falling body is, in one dimension (neglect x, y motion),

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$$H = \frac{p^2}{2m} + mgz. \qquad (6.117)$$

nd a time-independent generating function $F_4(p, P)$ such that $\bar{H}(Q, P) = P$. Demine the explicit form of the canonical transformation: $Q(z, p)$ and $P(z, p)$. Solve $z(p, P)$. Prove that Q is the time.

(From Percival and Richards.)

blem 5: *(Jacobians and canonical transformations)* For the case of 1 degree of dom, the determinant of the Jacobian of dynamical variables related by a canonical

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transformation is

$$\boxed{\det\left|\frac{\partial(P,Q)}{\partial(p,q)}\right| = 1.} \tag{6.118}$$

For any functions $F(x, y)$, $G(x, y)$, the Jacobian is defined by

$$\frac{\partial(F,G)}{\partial(x,y)} \equiv \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}. \tag{6.119}$$

Prove that Equation (6.118) holds for any canonical transformation generated by a generating function of the type $F_1(q, Q)$ with q, Q considered to be independent variables. Since we can always produce a generating function of this type by a Legendre transformation or series of transformations from F_2, F_3, F_4, proving Equation (6.118) for F_1-generated transformations is completely general.

One consequence of (6.118) is that a closed curve in p, q space encloses the same area that the image of that curve encloses in P, Q space. An application of (6.118) is used in the proof of the canonical invariance of Poisson brackets in Problem 8.

Hints: The *general chain rule* holds for Jacobian matrices* if we transform from the independent variables x, y to the new independent variables u, v, which are functions $u(x, y)$, $v(x, y)$:

$$\frac{\partial(F,G)}{\partial(x,y)} = \underbrace{\frac{\partial(F,G)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(x,y)}}_{\text{matrix multiplication}}. \tag{6.120}$$

As a special case of the general chain rule:

$$\frac{\partial(u,v)}{\partial(x,y)}\,\frac{\partial(x,y)}{\partial(u,v)} = \text{identity matrix} \equiv I. \tag{6.121}$$

Problem 6: *(Possible generating functions)* Use (6.118) to prove whether or not the two functions below can be used as generating functions:

$$F_1(q, Q) = qe^{Q}, \quad F_1(q, Q) = q^2 + Q^4. \tag{6.122}$$

If it is a possible generating function, determine the transformation $q, p \to Q, P$ explicitly.

(Adapted from Percival and Richards.)

* *Advanced Calculus*, 3rd ed., by Kaplan, pp. 106ff.

Problem 7*: *(Dynamics on a rotating turntable using polar coordinates)* This is a follow-up to Problem 14 in Chapter 5. Consider a generating function of the $F_2(q, P, t)$ type: $F_2 \equiv r_{\text{lab}} P_r + (\phi_{\text{lab}} - \theta(t))P_\phi$. The old coordinates p, q are the lab coordinates of the bug, and the new coordinates P, Q are the rotating coordinates of the bug. There are two degrees of freedom, so F_2 contains the sum over these. Using the relations in Table 6.1 and the rules for finding the new Hamiltonian $\bar{H}$, show that the relations between coordinates (r, ϕ) and $(r_{\text{lab}}, \phi_{\text{lab}})$ are correct and $\bar{H}$ is the Hamiltonian (5.102). This demonstrates a link between this type of canonical transformation and a change of coordinate systems.

General Properties of Poisson Brackets

Problem 8: *(Poisson brackets are invariant under canonical transformations)* Prove that any canonical transformation leaves the Poisson brackets $[F, G]$ invariant. That is, if P and Q are obtained from q and p by a canonical transformation, then

$$[F, G]_{Q,P} \equiv \frac{\partial F}{\partial Q}\frac{\partial G}{\partial P} - \frac{\partial F}{\partial P}\frac{\partial G}{\partial Q} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} \equiv [F, G]_{q,p}. \tag{6.123}$$

Hints: For one degree of freedom, the Poisson bracket can be regarded as the determinant of a Jacobian (6.119). The determinant of a product of two matrices is the product of their determinants.

Problem 9*: *(How to test when a transformation is canonical)*

a) First prove that Equation (6.39) is *necessary* for canonical transformations. Sufficiency means that (6.39) is a litmus test for the canonical equivalence of two sets of dynamical variables. To prove this, we have to show that (6.39) implies the existence of a generating function that defines the transformation. This problem also provides a method for finding generating functions, at least in principle.

b) Explain, using (6.123) for one degree of freedom, why the areas enclosed by corresponding closed curves in (q, p) or (Q, P) phase spaces are the same. Also explain why this implies

$$\oint p\, dq = \oint P\, dQ. \tag{6.124}$$

(The symbol $\oint$ stands for an integral around a closed curve.) Hint: Use Stokes' theorem.[†]

c) Now take the point of view that Q, q are the independent variables, and p, P are dependent variables, as with F_1-type generating functions. Then Equation (6.124)

[†] See Kaplan, *Advanced Calculus*, 3rd ed., p. 328.

implies that around any closed curve in (q, Q) space

$$\oint p(q, Q)\,dq - \oint P(q, Q)\,dQ = 0. \qquad (6.125)$$

Imagine a closed curve in (q, Q) space. Explain why (6.125) means that any path integral in this space between fixed end points is independent of the path taken, so can be considered a perfect differential of a function of q, Q. Stated as an equation, this is

$$\int p\,dq - P\,dQ = \int dF_1(q, Q). \qquad (6.126)$$

d) Using (6.126), prove that (6.39) implies the correct relationships for generating functions ($p = \frac{\partial F_1}{\partial q}$ and $P = -\frac{\partial F_1}{\partial Q}$).

Problem 10: *(Finding generating functions)* Suppose you try the contact transformation

$$Q = \log\left(\frac{\sin p}{q}\right), \qquad P = q \cot p. \qquad (6.127)$$

a) Find $[Q, P]_{q,p}$. Is (6.127) canonical?

b) Now show that

$$p\,dq - P\,dQ = d(pq + q\cot p). \qquad (6.128)$$

c) Find $F_1(q, Q)$ explicitly. Useful information: $\int \sin^{-1} x\,dx = \sqrt{1-x^2} + x\sin^{-1} x$.

Problem 11: *(Possible canonical transformation)* Is the transformation below canonical?

$$Q = \log(1 + \sqrt{q}\,\cos p), \qquad P = (1 + \sqrt{q}\cos p)\sqrt{q}\sin p \qquad (6.129)$$

Problem 12*: *(Poisson brackets for many degrees of freedom)*

a) Prove, using Hamilton's equations of motion and the definition of the Poisson brackets (6.36), that the total time derivative of any function $D(q_1, \ldots, p_1, \ldots)$ of the ps and qs obeys the equation

$$\frac{dD}{dt} = [D, H] + \frac{\partial D}{\partial t}, \qquad (6.130)$$

where H is the Hamiltonian of the system. Equation (6.130) can be regarded as the most general and canonically invariant way to state Hamilton's equations of motion. Explain.

b) For the I_j defined by (5.14), if the Lagrangian is invariant under transformations that generate I_j, show $[I_j, H] = 0$. Notice that I_j does not contain the time explicitly. This in known as the *Hamiltonian form of Noether's Theorem.*

Problem 13*: *(Landau's proof)* In their book *Mechanics*, Landau and Lifshitz give a proof of the important relation (6.123) as follows (our notation and our formula references have been used below):

> First of all, it may be noticed that the time appears as a parameter in the canonical transformation(s)... It is therefore sufficient to prove (6.123) for quantities which do not depend explicitly on time. Let us now formally regard G as the Hamiltonian of some fictitious system. Then, by formula (6.130), $[F, G]_{p,q} = \frac{dF}{dt}$. The derivative $\frac{dF}{dt}$ can depend only on the properties of the motion of the fictitious system, and not on the particular choice of variables. Hence the Poisson bracket $[F, G]$ is unaltered by the passage from one set of canonical variables to another.

Comment on whether or not you think this is a valid and completely general proof of (6.123).

Problem 14*: *(Poisson brackets of constants of the motion can generate new constants of the motion)* Consider the uniform motion of a free particle of mass m. The Hamiltonian is a constant of the motion and so is the quantity F defined as

$$F(x, p, t) \equiv x - \frac{pt}{m}. \tag{6.131}$$

a) Compare $[H, F]$ with $\frac{\partial F}{\partial t}$. Prove from (6.130) that F is also a constant.

b) Prove that the Poisson bracket of two constants of the motion is itself a constant of the motion, even if the constants $F(x, p, t)$ and $G(x, p, t)$ depend explicitly on the time. (Part a is one example of this.)

c) Show *in general* that if the Hamiltonian and a quantity F are constants of the motion then $\frac{\partial F}{\partial t}$ is a constant of the motion also.

Problem 15: *(Poisson brackets with angular momentum)*

a) Angular momentum is defined as $\vec{l} = \vec{r} \times \vec{p}$. Prove that $[l_x, l_y] = l_z$ for all cyclic permutations of l_x, l_y, l_z.

b) Calculate all the Poisson brackets of the components of $\vec{r}$ and $\vec{p}$ with the components of the angular momentum (for example, $[x, l_z]$, $[p_x, l_z]$, etc.).

Problem 16: *(Poisson brackets and spherical symmetry)* Let $\phi(\vec{r}, \vec{p})$ be any function that is spherically symmetric about the origin (invariant under rotations).

a) ϕ can depend only on the components of $\vec{r}$ and $\vec{p}$ through the combinations r^2, p^2, and $\vec{r} \cdot \vec{p}$. Why is this true?

b) Evaluate the Poisson bracket $[\phi, l_z]$ (l_z is the z component of the angular momentum) and show that it vanishes.

(Adapted from Landau and Lifshitz, 1986.)

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P3318 HW #7 SOLUTIONS 29 MARCH 2013

1. (H&F 6-2) Generating function → canonical transform

WE ARE GIVEN $F_3(p, Q, t) = -(e^Q - 1)^2 \tan p$

RECALL (table 6.1 in H&F, table 9.1 in Goldstein) THAT THE TRANSFORMATION IS GIVEN BY:

$$F = F_3(p, Q, t) + qp$$

THEN THE "PHYSICALLY EQUIVALENT LAGRANGIAN" CONDITION GIVES

$$\cancel{p\dot{q}} - H = P\dot{Q} - K + \frac{dF}{dt}$$

$$\frac{\partial F_3}{\partial t} + \left(\frac{\partial F_3}{\partial p} + q\right)\dot{p} + \frac{\partial F_3}{\partial Q}\dot{Q} + \cancel{p\dot{q}}$$

$$\Rightarrow \begin{cases} q = -\dfrac{\partial F_3}{\partial p} & \text{matching } \dot{p} \text{ coefficients} \\ P = -\dfrac{\partial F_3}{\partial Q} & \text{matching } \dot{Q} \text{ coefficients} \\ K = H + \dfrac{\partial F_3}{\partial t} & \end{cases}$$

PLUGGING IN THE FORM OF F_3 GIVES:

$$q = +(e^Q - 1)^2 \sec^2 p$$

$$P = +2(e^Q - 1)e^Q \tan p$$

from q eqn: $(e^Q - 1) = \sqrt{q}\, \cos p$

$$e^Q = \sqrt{q}\, \cos p + 1$$

PLUG INTO P EQN:

$$P = 2\sqrt{q} \cos p\, (\sqrt{q} \cos p + 1) \tan p$$

$$= \boxed{2(\sqrt{q}\cos p + 1)\sqrt{q} \sin p}$$

WORK ON INVERTING THE q EQN:

$$e^Q - 1 = \sqrt{q} \cos p$$

$$\Rightarrow \boxed{Q = \log(1 + \sqrt{q} \cos p)}$$

2. (H&F 6.6) Possible Generating functions

a) $F_1(q, Q) = q e^Q$

b) $F_1(q, Q) = q^2 + Q^4$

the expressions for the (Q, P) coordinates are

$$p = \frac{\partial F_1}{\partial q} \quad \leftarrow \text{implicit equation for } Q$$

$$P = \frac{-\partial F_1}{\partial Q}$$

a) $\frac{\partial F_1}{\partial q} = e^Q \Rightarrow \boxed{Q = \log p}$

$-\frac{\partial F_1}{\partial Q} = -q e^Q \Rightarrow \boxed{P = -q p}$

$\frac{\partial Q}{\partial q} = 0$

$\frac{\partial Q}{\partial p} = 1/p$

$\frac{\partial P}{\partial q} = -p$

$\frac{\partial P}{\partial p} = -q$

$$\begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{p} \\ -p & -q \end{vmatrix} = +1 \quad ✓$$

b) $\frac{\partial F_1}{\partial q} = 2q = p$ } hmm! doesn't relate

$-\frac{\partial F_1}{\partial Q} = -4Q^3 = P$ } Q to q & p!

suspect something is wrong

$\det\left|\frac{\partial(P,Q)}{\partial(p,q)}\right| = 1 \Leftrightarrow \{Q,P\}_{p,q} = 1 \Leftrightarrow$ canonical

EASIEST CHECK:

$$\{Q,P\}_{p,q} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$$

$$= \frac{\partial Q}{\partial q}(-12)\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}(-12)\frac{\partial Q}{\partial q}$$

$$= 0$$

$\neq 1 \Rightarrow$ $\boxed{\text{not canonical}}$

3 (H&F 6.8) <u>Poisson Brackets & Canonical transforms</u>

From the hint: observe that

$$\{F,G\}_{q,p} = \left| \frac{\partial(F,G)}{\partial(q,p)} \right|$$

Now consider $\{F,G\}_{Q,P}$ as a transformation:

$$\{F,G\}_{Q,P} = \left| \frac{\partial(F,G)}{\partial(q,p)} \cdot \underbrace{\frac{\partial(q,p)}{\partial(Q,P)}}_{\text{Jacobian}} \right| = \left| \frac{\partial(F,G)}{\partial(q,p)} \right| = \{F,G\}_{q,p}$$

BUT RECALL THAT CANONICAL
TRANSFORMATIONS PRESERVE
AREA $\Rightarrow$ THIS FACTOR IS 1.

4 (H&F 6.10) <u>finding generating functions</u>

$$Q = \log \frac{\sin p}{q} \qquad P = q \cot p$$

a) $\partial Q/\partial q = (q/\sin p) \cdot \sin p \, (-1/q^2) = -1/q$

$\partial Q/\partial p = (q/\sin p) \cdot \frac{1}{q} \cos p = \cot p$

$\partial P/\partial q = \qquad = \cot p$

$\partial P/\partial p = -q \csc^2 p = -q/\sin^2 p$

$$\{Q, P\}_{q,p} = \frac{1}{\sin^2 p} - \frac{\cos^2 p}{\sin^2 p} = \boxed{1} \Rightarrow \boxed{\text{canonical}}$$

b) $dQ = \left(\frac{-1}{q}\right) dq + (\cot p)\, dp$

$-P\,dQ = \cot p \, dq - q \cot^2 p \, dp$

$d(q \cot p) = \cot p \, dq - q \cot^2 p \, dp$

<u>then</u>: LHS $= p\,dq - P\,dQ = p\,dq + \cot p \, dq - q \cot^2 p \, dp$

RHS $= d(pq + q\cot p) = p\,dq + q\,dp + d(q \cot p)$

$+ \cot p \, dq - q(\cot^2 p - 1) dp$

$= p\,dq + \cot p \, dq - q \cot^2 p \, dp$

$\Rightarrow \boxed{\text{LHS} = \text{RHS}}$ ✓

c) Now construct $F_1(q, Q)$ using the explicit formulae for $Q(q,p)$ & $P(q,p)$ &

$$p = \frac{\partial F_1}{\partial q} \qquad P = \frac{-\partial F}{\partial Q}$$

$$Q = \log \frac{\sin p}{q} \Rightarrow p = \sin^{-1}(q e^{Q}) = \frac{\partial F_1}{\partial q} \qquad (**)$$

then integrate using the hint:

$$(*) \quad F_1(q,Q) = \int \frac{\partial F_1}{\partial q}\, dq = e^{-Q}\sqrt{1-q^2 e^{2Q}} + q \sin^{-1}(q e^{Q}) + C(Q)$$

NOW fix $C(Q)$ BY USING $P = q \cot p = -\partial F_1/\partial Q$

PLUG IN EXPRESSION for p; use $\cot(\sin^{-1} x) = \sqrt{1-x^2}/x$

$$q \cot\left[\sin^{-1}(q e^{Q})\right] = + e^{-Q}\sqrt{1 - e^{2Q} q^2} + C'(Q)$$

$$q\, \frac{\sqrt{1-q^2 e^{2Q}}}{q e^{Q}} \qquad\qquad -\frac{\partial}{\partial Q}(*)$$

$$\cancelto{0}{e^{-Q}\sqrt{1-q^2 e^{2Q}}} = \cancelto{0}{e^{-Q}\sqrt{1-q^2 e^{2Q}}} + C'(Q)$$

$\Rightarrow C'(Q) = \text{CONST}$, can pick to be zero.
(we only use derivatives of generating functions, so we're free to shift by a constant)

So:

$$\boxed{F_1(q,Q) = e^{-Q}\sqrt{1-q^2e^{2Q}} + q\sin^{-1}(qe^{Q})},$$

REMARK: YOU COULD HAVE ALSO SKIPPED THE INTEGRAL AND JUST PLUGGED THE RESULT (**) INTO THE RESULT OF PART (b).

+ constant.

5 (H & F 6.14) Poisson Brackets & the Bianchi identity

$$F(x,p,t) = x - \frac{1}{m}pt$$

$$H(p,q,t) = \frac{1}{2m}p^2$$

a) $\partial F/\partial t = -\frac{1}{m}p$

$$\{H,F\} = \cancel{\frac{\partial H}{\partial q}\frac{\partial F}{\partial p}}^{\,0} - \frac{\partial H}{\partial p}\frac{\partial F}{\partial q} = -\frac{p}{m}$$

$$\boxed{\frac{\partial F}{\partial t} = \{H,F\}}$$

$$\frac{d}{dt}F = \frac{\partial F}{\partial t} + \{F,H\} = \frac{\partial F}{\partial t} - \{H,F\} = \boxed{0}$$

b) [the Bianchi Identity]

we want to show that

$$\{\{F,G\},H\} = -\frac{\partial}{\partial t}\{F,G\}$$

$$= -\{\partial F/\partial t, G\} - \{F, \partial G/\partial t\}$$

$$= \{\{F,H\},G\} + \{F,\{G,H\}\}$$

using $\dot{F} = \dot{G} = 0$

A nice way to write this (WHAT WE WANT TO SHOW) is:

$$\{\{F,G\},H\}+\{\{H,F\},G\}+\{\{G,H\},F\}=0$$

this is nice because it is simply:

$$\{\{F,G\},H\} + (\text{cyclic permutations of } [F,G,H]).$$

<u>SO</u>: a nice strategy is to just brute force calculate <u>one</u> term & obtain all the others from cyclic permutations.

As further shorthand:

$\left.\begin{array}{l}\text{WRITE A DOT } \bullet \text{ FOR } \partial/\partial q \\ \text{WRITE A TICK } ' \text{ FOR } \partial/\partial p\end{array}\right\}$ these commute

$$\{\{F,G\},H\} = \{\dot F G' - F'\dot G, H\}$$

$$= (\dot F G' - F'\dot G)^{\bullet} H' - (\dot F G' - F'\dot G)' \dot H$$

$$= (\ddot F G' + \dot F \dot G' - \dot F' \dot G - F'\ddot G) H' - (\dot F' G' + \dot F G'' - F''\dot G - F'\dot G')\dot H$$

$$= \ddot F G'H' + \dot F\dot G' H' - \dot F'\dot G H' - F'\ddot G H' - \dot F' G'\dot H - \dot F G''\dot H + F''\dot G\dot H + F'\dot G'\dot H$$

SO WE HAVE ((THAT MESS)) + ((CYCLIC PERMUTATIONS)) ;
WE WANT TO SHOW THAT IT VANISHES!
OBSERVE THAT EACH TERM IS OF THE FORM FGH
WITH 2 •'s AND 2 ′'s : EACH F, G, H GETS ONE
DERIVATIVE, ONE GETS TWO. LET US MAKE A
CHART OF EACH TERM, KEEPING TRACK OF THE
DERIVATIVES & SIGN:

	F	G	H	SIGN	CANCELLED BY cyclic permutation of
1	••	′	′	+	#4
2	•	•′	′	+	#5
3	•′	•	′	−	#8
4	′	••	′	−	#1
5	•′	′	•	−	#2
6	•	′′	•	−	#7
7	′′	•	•	+	#6
8	′	•′	•	+	#3

the sum of these terms does not yet cancel.
However, when we include cyclic permutations,
each term has a counterpart that cancels it.

FOR EXAMPLE: TERM 1 REPRESENTS 3 TERMS UPON
CYCLIC PERMUTATIONS:

$$\ddot{F}G'H' + F'\ddot{G}H' + F'G'\ddot{H}$$

THESE ARE CANCELLED BY THE CYCLIC PERMUTATIONS of TERM 4 : (note the relative sign)

$$-F'\ddot{G}H' - F'G'\ddot{H} - \ddot{F}G'H'$$

& SIMILARLY WITH THE OTHER TERMS LISTED IN THE TABLE ABOVE.

Alternately, you could try to brute force it all the way. :)

(c) suppose $\dot{H} = \dot{F} = 0$

$$\dot{F} = 0 = \frac{\partial F}{\partial t} + \{F, H\}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial t}\right) = -\{\dot{F}, H\} - \{F, \dot{H}\} = \boxed{0}$$ ✓

6. Point transformation

Given $Q = Q(q,t)$; $L'(Q,\dot{Q},t) = L(q,\dot{q},t)$

& $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} = \dfrac{\partial L}{\partial q}$ $\qquad \dot{Q} = \dfrac{\partial Q}{\partial q}\dot{q} + \dfrac{\partial Q}{\partial t}$

RHS: $$\frac{\partial L}{\partial q} = \frac{\partial L'}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial L'}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial q} \qquad = \frac{d}{dt}\frac{\partial Q}{\partial q}$$

LHS: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}\left(\frac{\partial L'}{\partial Q}\cancelto{0}{\frac{\partial Q}{\partial \dot{q}}} + \frac{\partial L'}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial \dot{q}}\right) \qquad = \frac{\partial Q}{\partial q}$$

{ By DOT CANCELLATION RULE, cf H&F p. 14.

$$= \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{Q}}\frac{\partial Q}{\partial q}\right)$$

$$= \left(\frac{d}{dt}\frac{\partial L'}{\partial \dot{Q}}\right)\frac{\partial Q}{\partial q} + \frac{\partial L'}{\partial \dot{Q}}\frac{d}{dt}\frac{\partial Q}{\partial q}$$

LHS = RHS $\Rightarrow$ $$\frac{\partial L'}{\partial Q}\cancel{\frac{\partial Q}{\partial q}} + \cancel{\frac{\partial L'}{\partial \dot{Q}}\frac{d}{dt}\frac{\partial Q}{\partial q}} = \left(\frac{d}{dt}\frac{\partial L'}{\partial \dot{Q}}\right)\cancel{\frac{\partial Q}{\partial q}} + \cancel{\frac{\partial L'}{\partial \dot{Q}}\frac{d}{dt}\frac{\partial Q}{\partial q}}$$

$$\Rightarrow \boxed{\frac{d}{dt}\frac{\partial L'}{\partial \dot{Q}} = \frac{\partial L'}{\partial Q}}$$