P3318 HW8 SOLUTIONS

due: ~~15~~ 22 APRIL

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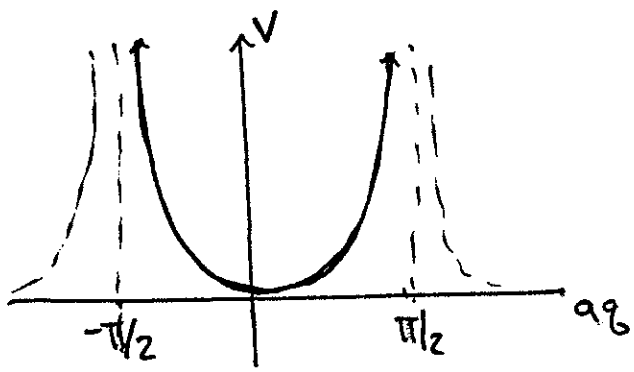
1. (H&F 6.18) tan^2 POTENTIAL

$$V(q) = U \tan^2(aq)$$



a) FIND TURNING POINTS:

SUPPOSE A PARTICLE HAS ENERGY E; the TURNING POINTS q_0 CORRESPOND TO $V(q_0) = E$.

$$E = U \tan^2(a q_0)$$

$$\Rightarrow \tan^2(a q_0) = \frac{E}{U}$$

$$\boxed{q_0 = \frac{1}{a} \tan^{-1}\sqrt{\frac{E}{U}}}$$

b) PROVE: $\frac{1}{\sqrt{2}} a I = \sqrt{E+U} - \sqrt{U}$:

(6.94): $I = \frac{1}{2\pi} \oint p \, dq$

p GIVEN BY ENERGY CONDITION

$$E = \frac{p^2}{2m} + U \tan^2(aq)$$

$$p = \sqrt{2m} \sqrt{E - U \tan^2(aq)}$$

further, can take $\oint \longrightarrow 2\int_{-q_0}^{q_0}$

$$I = \frac{\sqrt{2m}}{\pi a} \int_{-aq_0}^{aq_0} \sqrt{E - U \tan^2(aq)} \, d(aq)$$

now change variables:

$$x = \tan(aq) \qquad dx = \frac{d(aq)}{\cos^2(aq)}$$

note: $\tan^2(aq) + 1 = \frac{1}{\cos^2(aq)} \Rightarrow dx = (x^2+1)\, d(aq)$

LIMITS of INTEGRATION: $\pm aq_0 \longrightarrow \tan(\pm aq_0) = \pm\sqrt{\frac{E}{U}}$

from previous page

$$I = \frac{\sqrt{2m}}{\pi a} \int_{-\sqrt{E/U}}^{\sqrt{E/U}} \frac{\sqrt{E - Ux^2}}{1 + x^2}\, dx$$

Many students had difficulty integrating this in Mathematica. As a general rule, you want to feed Mathematica the most simplified integral you can muster.

IN THIS INTEGRAL, Mathematica GETS WORRIED WHEN THE ARGUMENT OF THE SQUARE ROOT IS NEGATIVE. ONE WAY TO MAKE IT CLEAR WHAT'S GOING ON IS TO WRITE EVERYTHING IN TERMS OF $E/U \equiv A$.

$$I = \frac{\sqrt{2mU}}{\pi a} \int_{-\sqrt{A}}^{\sqrt{A}} \frac{\sqrt{A - x^2}}{1 + x^2}\, dx$$

IN THIS FORM IT'S CLEAR THAT YOU NEVER INTEGRATE IN THE REGION WHERE THE INTEGRAND IS IMAGINARY. HERE'S THE Mathematica CODE:

In[344]:= Assuming$\left[\{A \in \text{Reals}, A > 0\}, \text{Integrate}\left[\frac{\sqrt{A - x^2}}{1 + x^2}, \{x, -\sqrt{A}, \sqrt{A}\}\right]\right]$

Out[344]= $\left(-1 + \sqrt{1 + A}\right) \pi$

$$I = \frac{\sqrt{2mU}}{\pi a} \cdot \pi \left(\sqrt{1 + \frac{E}{U}} - 1 \right)$$

$$= \frac{\sqrt{2m}}{a} \left(\sqrt{U+E} - \sqrt{U} \right)$$

as noted in problem $m = 1$.

$$\Rightarrow \boxed{\frac{aI}{\sqrt{2}} = \sqrt{E+U} - \sqrt{U}} \qquad (*)$$

c) SHOW THAT $\omega(E) = \frac{\omega}{a\sqrt{2}} = \sqrt{E+U}$

$$(*) \Rightarrow E + U = \left(\frac{aI}{\sqrt{2}} + \sqrt{U} \right)^2$$

$$E = \frac{a^2}{2} I^2 + a\sqrt{2} I \sqrt{U} \qquad (\because)$$

$E = H$ for this system

IN THE ACTION-ANGLE REPRESENTATION, $H = \omega I$

$$\omega = \frac{\partial H}{\partial I} = a^2 I + a\sqrt{2U}$$

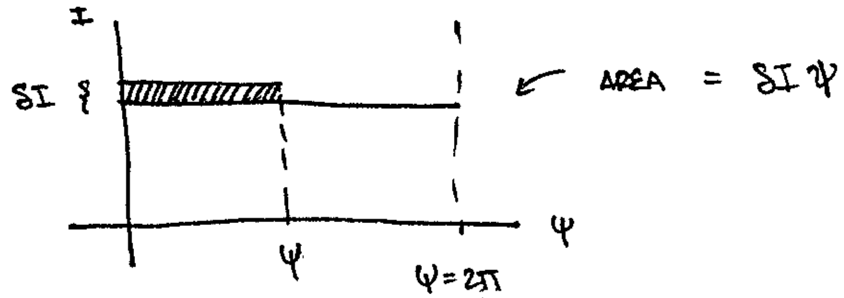
$$= a^2 \left(\frac{\sqrt{2}}{a} \sqrt{E+U} - \frac{\sqrt{2}}{a} \sqrt{U} \right) - a\sqrt{2U}$$

$$= a\sqrt{2}\sqrt{E+U}$$

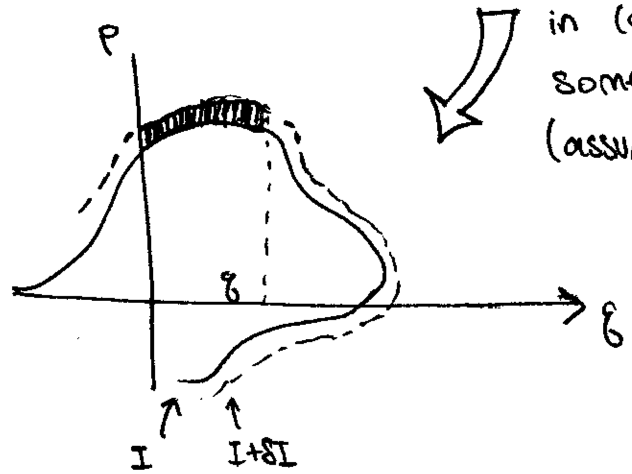
$$\Rightarrow \boxed{\frac{\omega}{a\sqrt{2}} = \sqrt{E+U}}$$

2 (H&F 6.19) <u>Area in PHASE SPACE & CAN. TRANSF</u>

a) The infinitesmal area described by the problem is:



in (q,p) space this is something like this. (assuming $\psi = 0 \leftrightarrow q = 0,\ p > 0$)



WRITE $p(q, I) = p(q, E(I))$

$$\text{AREA in } (q,p)\text{-SPACE} = \int_0^q \left[p(q', I+\delta I) - p(q', I) \right] dq'$$

$$= \boxed{\delta I \int_0^q \frac{\partial p(q', I)}{\partial I}\, dq'}$$

$$= \delta I\, \psi$$

THIS GIVES A CLOSED FORM EXPRESSION FOR THE ANGLE VARIABLES

REMARK: the problem notes that this is precisely what we have for an F_2-type generating function. EXPLICITLY,

$$F_2(I,q) = \int_0^q p(q',I)\,dq'$$

$$\Rightarrow \quad p = \frac{\partial F_2}{\partial q} \quad \checkmark$$

$$\Rightarrow \quad \psi = \frac{\partial F_2}{\partial I} = \int_0^q \frac{\partial p}{\partial I}(q',I)\,dq' \quad \checkmark$$

b) NOW WE APPLY THIS RESULT TO THE $\tan^2$ POTENTIAL of PROBLEM 1.

RESULT OF (a): $\psi = \int_0^q \frac{\partial}{\partial I} p(q',I)\,dq'$

$$p = \sqrt{2m}\sqrt{E(I) - V(q')}$$

FROM PROBLEM 1c), eqn (ii)

$$E = \frac{a^2}{2}I^2 + a\sqrt{2U}\,I$$

SO WE JUST HAVE TO PLUG & CHUG ...

$$\psi = \sqrt{2m} \int_0^q \frac{\partial}{\partial I} \sqrt{E(I) - V(q')} \, dq'$$

$$= \sqrt{2m} \int_0^q \frac{\partial/\partial I \, E(I)}{\sqrt{E(I) - V(q')}} \cdot \frac{1}{2} \, dq'$$

$$\frac{\partial}{\partial I} E(I) = a^2 I + a\sqrt{2U}$$

PROB 1 (*): $I = \frac{\sqrt{2}}{a}\left(\sqrt{E+U} - \sqrt{U}\right)$

$$\Rightarrow \frac{\partial}{\partial I} E(I) = a\sqrt{2}\sqrt{E+U}$$

$$= a\sqrt{m} \int_0^q \sqrt{\frac{E+U}{E - U\tan^2 aq'}} \, dq'$$

$$= \sqrt{m(A+1)} \int_0^{aq} \frac{d(aq')}{\sqrt{A - \tan^2(aq')}}$$

$A = E/U$

set $m=1$

ACCORDING TO PERCIVAL & RICHARDS <u>INTRO TO DYNAMICS</u> eq (A.29)

$$= \sqrt{A+1} \times \frac{1}{\sqrt{A+1}} \sin^{-1}\left[\sqrt{\frac{A+1}{A}} \sin aq\right]$$

$$= \boxed{\sin^{-1}\left[\sqrt{\frac{A+1}{A}} \sin aq\right]}$$

NOTE: SLIGHTLY DIFFERENT FROM THE BOOK!

REMARK: THIS INTEGRAL IS ALSO NONTRIVIAL TO DO IN MATHEMATICA. THE 'OUT OF THE BOX' SOLUTION FROM MATHEMATICA DOESN'T AGREE WITH THE $\sin^{-1}$ SOLUTION OVER THE ENTIRE DOMAIN:

In[56]:= **Assuming**$\left[\{A \in \text{Reals}, A > 0\}, \text{Integrate}\left[\frac{\sqrt{A+1}}{\sqrt{A - \text{Tan}[k]^2}}, k\right]\right]$

Out[56]=
$$\frac{\text{ArcTan}\left[\frac{\sqrt{2}\,\sqrt{1+A}\,\text{Sin}[k]}{\sqrt{-1+A+(1+A)\,\text{Cos}[2\,k]}}\right]\sqrt{-1+A+\text{Cos}[2\,k]+A\,\text{Cos}[2\,k]}\;\text{Sec}[k]}{\sqrt{2}\,\sqrt{1+A}\,\sqrt{\frac{A-\text{Tan}[k]^2}{1+A}}}$$

In[59]:= $f[k\_, A\_] := \frac{\text{ArcTan}\left[\frac{\sqrt{2}\,\sqrt{1+A}\,\text{Sin}[k]}{\sqrt{-1+A+(1+A)\,\text{Cos}[2\,k]}}\right]\sqrt{-1+A+\text{Cos}[2\,k]+A\,\text{Cos}[2\,k]}\;\text{Sec}[k]}{\sqrt{2}\,\sqrt{1+A}\,\sqrt{\frac{A-\text{Tan}[k]^2}{1+A}}}$

In[61]:= $g[k\_, A\_] := \text{ArcSin}\left[\sqrt{\frac{A+1}{A}}\;\text{Sin}[k]\right]$

In[67]:=
```
Plot[{f[k, 4], g[k, 4]}, {k, -2 π, 2 π},
 Frame → True,
 PlotStyle → {{Thick, Red}, {Thick, Blue, Dashed}}
]
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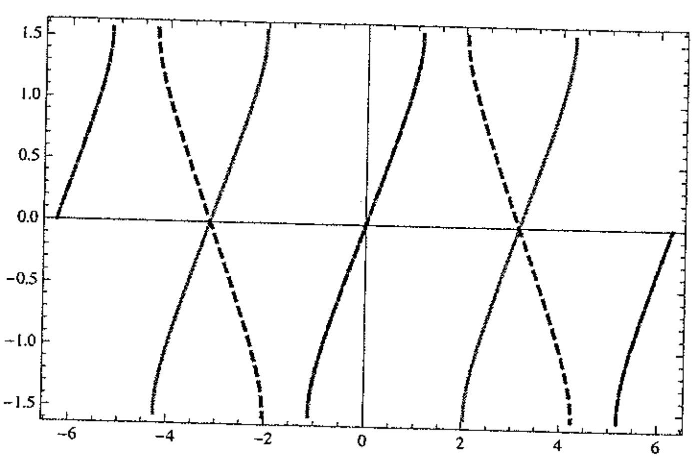
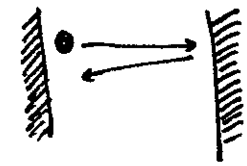
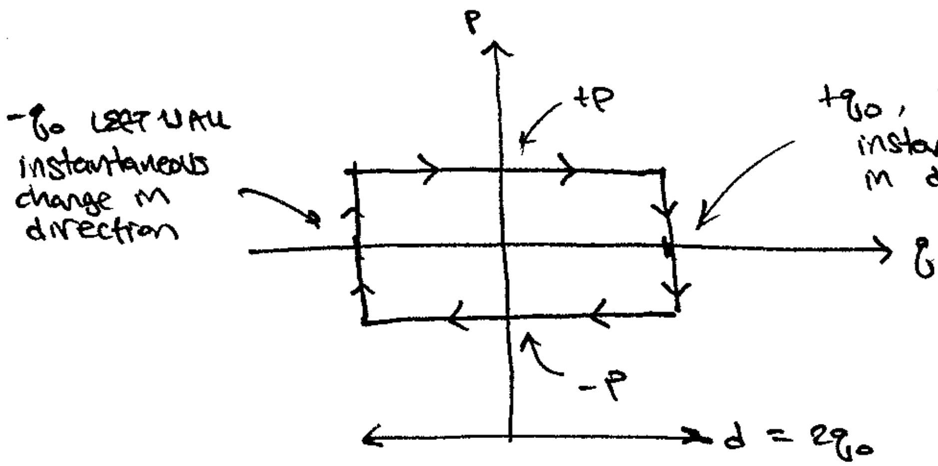
Out[67]=


3. (H&F 6.20) Bouncing ball btwn two walls

config space:  (assuming $m = 1$)

BALL TRAVELS WITH CONSTANT MOMENTUM EXCEPT @ THE WALL WHERE IT CHANGES DIRECTION INSTANTANEOUSLY.

a) PHASE SPACE TRAJECTORY



ACTION VARIABLE: $I = \frac{1}{2\pi} d \times 2p$

$$= \boxed{\frac{d}{\pi}\sqrt{2ME}}$$ ($m = 1$)

$H = f(I) \rightarrow \omega = \partial H / \partial I$

$H = \frac{1}{2m}\left(\frac{\pi}{d} I\right)^2$

$\rightarrow \omega = \frac{1}{m}\left(\frac{\pi}{d}\right)^2 I = \boxed{\frac{\pi}{d}\sqrt{\frac{2E}{m}}}$

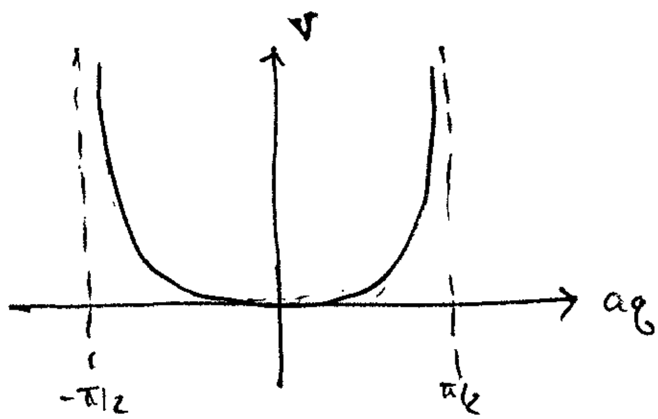
COMPARE TO $\omega = \frac{2\pi}{T} = \frac{\pi}{d}\sqrt{\frac{2E}{m}}$ $\longrightarrow$ matches!

$T = 2d/v, \quad v = \sqrt{2E/m}$

AS NOTED BY THE QUESTION, ONE CANNOT WRITE A HAMILTONIAN FOR THIS MODEL SINCE THE FORCE IS DISCONTINUOUS.

WE SHOWED IN THE PREVIOUS PROBLEM THAT THE ANGLE VARIABLE CAN BE EXPRESSED AS AN INTEGRAL OF $\partial p(q,I)/\partial I$ & THIS IS WELL DEFINED.

b) RECALL: $V(q) = U \tan^2(aq)$



INCREASING U: ROUNDER @ BOTTOM:

INCREASING a: tightens domain: (in q space)

WANT LIMIT: $\boxed{U \to 0}$ (flat bottom)

a s.t. DISTANCE is $2q_0 = d$

$\Rightarrow \boxed{a = \pi/d}$

THEN: (6.134) $\rightarrow$ $\frac{1}{\sqrt{2}}\frac{\pi}{d} I = \sqrt{E}$

$$\boxed{I = \frac{d}{\pi}\sqrt{2E}}$$

matches part (a).

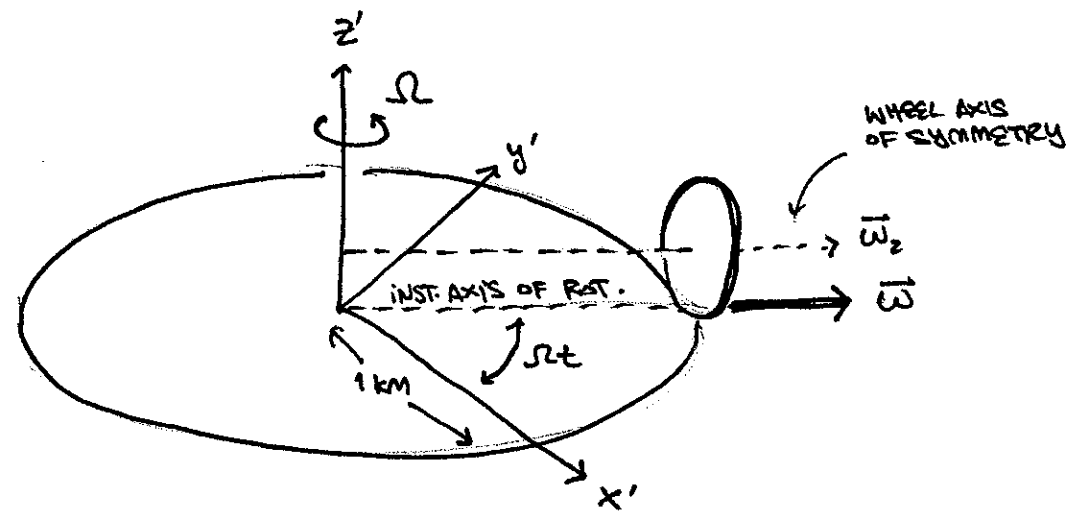
(6.135) $\rightarrow$ $\frac{d}{\pi}\frac{\omega}{\sqrt{2}} = \sqrt{E}$

$$\boxed{\omega = \frac{\pi}{d}\sqrt{2E}}$$

~~9 (H&F 7.1) LOCOMOTIVE~~

~~2a~~

4 (H&F 7.1) <u>LOCOMOTIVE</u>



(COMPARE TO H&F FIG 7.4)

$$\Omega = \left(80\ \frac{km}{hr}\right)\cdot\frac{1\ \text{rotation}}{2\pi\ km}\cdot\frac{1\ hr}{60\ min} = \frac{2}{3\pi}\ rpm$$

a) (H&F eq 7.11)

$$\vec{\omega} = -\left|1000\ rpm\right|\left[(\cos\Omega t)\,\hat{x}' + (\sin\Omega t)\,\hat{y}'\right]$$

$$\boxed{\vec{\omega} = \begin{pmatrix} -1000\cos\frac{2}{3\pi}t \\ -1000\sin\frac{2}{3\pi}t \\ 0 \end{pmatrix}}$$

$\Big\}$ t in minutes, $\vec{\omega}$ in RPM

b) <u>INTUITIVE ANSWER</u>: @ $t=0$ the body frame axes align w/ the space frame. In this frame

$$\boxed{\vec{\omega} = (-1000\text{ rpm}, 0, 0)}$$

(H&F 7.29) $A = U^T A' U = U^T \dot{U} \Rightarrow A' = \dot{U} U^T$

where $\vec{r}\,' = \underline{U \vec{r}}$

RELEVANT 'SPACE FRAME'
↳ TRAIN FRAME
(in this problem)

BODY FRAME (no rotation)
↳ ie WHEEL FRAME

THE ROTATION FROM WHEEL → TRAIN FRAME IS:

$$U_{W \to T} = \begin{pmatrix} 1 & & \\ & \cos \omega t & -\sin \omega t \\ & \sin \omega t & \cos \omega t \end{pmatrix}$$

then (Mathematica):

$$A' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{pmatrix} \rightarrow \boxed{\vec{\omega} = (\omega, 0, 0)} \checkmark$$

b) continued: OUR ANSWER MATCHES OUR INTUITION. NOTE THAT ~~WE~~ CONTRADICTIONS CAN POP UP IF WE USE THE WRONG ROTATION – eg A $\hat{z}$ COMPONENT FROM INCLUDING THE TRAIN TRAJECTORY

TRAIN → SPACE

WHEEL → TRAIN

c) $U = U_2 U_1$

$$A' = \dot{U} U^T$$

$$\dot{U} = \dot{U}_2 U_1 + U_2 \dot{U}_1$$

$$U^T = U_1^T U_2^T$$

$$= (\dot{U}_2 U_1 + U_2 \dot{U}_1) U_1^T U_2^T$$

$$= \underbrace{\dot{U}_2 U_2^T}_{A_2'} + U_2 \underbrace{\dot{U}_1 U_1^T}_{A_1'} U_2^T$$

TRAIN ANG. VELOCITY IN SPACE FRAME } A_2'

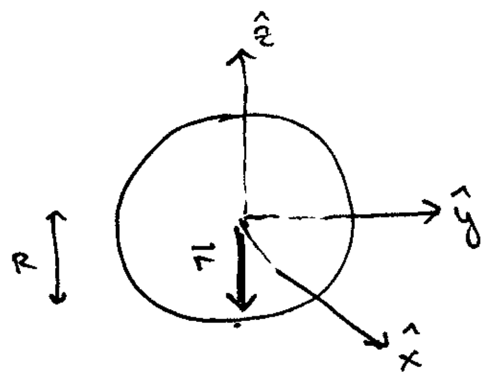
A_1' { WHEEL ANGULAR VELOCITY IN TRAIN FRAME

$$= A_2' + \underbrace{U_2 (A_1') U_2^T}_{\text{TRANSFORMS } A_1' \text{ TO SPACE FRAME}}$$

$$= A_2' + A_1''$$

$$\leftrightarrow (\vec{\Omega} + \vec{\omega})$$

5 (H+F 7.4) <u>Rolling sphere</u>



LET $\vec r = -R\hat z$ BE THE POINT OF CONTACT WITH THE SURFACE.

THE ROTATION IN THE $\hat z$ DIRECTION (ω_z) IS NOT RELATED TO THE CM MOTION. ON THE OTHER HAND,

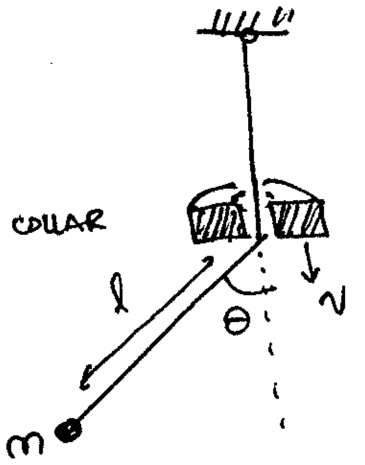
$$\omega_x R = -(V_{cm})_y$$
$$\omega_y R = (V_{cm})_x$$

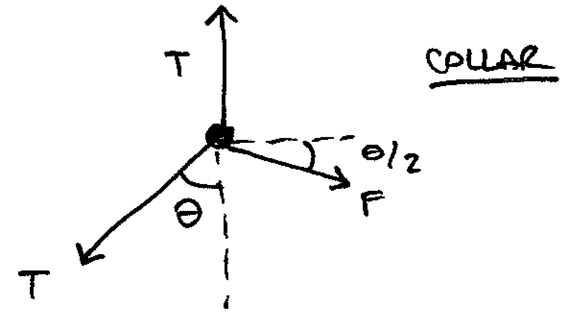
WE CAN WRITE THIS AS $\boxed{\vec\omega \times \vec r = -\vec V_{cm}}$

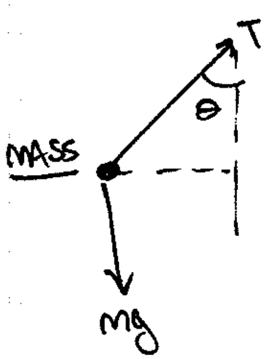
BUT RECALL THAT SINCE $\vec r \sim \hat z$ AND $(\vec V_{cm})_z = 0$, THIS ONLY CONTAINS 2 SCALAR EQUATIONS.

$$\text{DoF} = \underset{\vec V_{cm}}{3} + \underset{\vec\omega}{3} - \underset{\vec\omega\times\vec r = -\vec V_{cm}}{2} - \underset{(V_{cm})_z = 0}{1} = \boxed{3}$$

6. SEE LECTURE NOTES FOR FREE BODY DIAGRAMS







from collar diagram: $T\sin\theta = F\cos\frac{\theta}{2}$

from mass diagram: $T\cos\theta = mg$

a) $P = \vec{F}\cdot\vec{v} = F\sin\frac{\theta}{2}\, v$

SMALL ANGLE:
$$F = T\theta$$
$$\sin\frac{\theta}{2} = \frac{\theta}{2}$$
$$T = mg$$

$$\rightarrow \boxed{P = \frac{1}{2} mg\theta^2 v}$$

b) $\theta(t) = \theta_0(\ell) \cos(\omega(\ell)\, t)$

$$\langle P(\ell) \rangle = \frac{1}{2} mgv \langle \theta(t)^2 \rangle$$

$$= \frac{1}{2} mgv\, \theta_0(\ell)^2 \underbrace{\langle \cos^2(\omega(\ell)\, t) \rangle}_{1/2}$$

$$\boxed{\langle P(\ell) \rangle = \frac{1}{4} mgv\, \theta_0(\ell)^2}$$

c) $E(\ell) = \frac{1}{2} mg\ell \left(\theta_0(\ell)\right)^2$

$$\dot{E} = \frac{1}{2} mg \left[\dot{\ell}\, \theta_0^2 + 2\ell\, \theta_0\, \theta_0'\, \dot{\ell} \right]$$

$$= \frac{1}{2} mg(-v)\, \theta_0 \left(\theta_0 + 2\ell\, \theta_0' \right)$$

$$= \frac{1}{4} mgv\, \theta_0^2$$

$$\Rightarrow \frac{1}{2} \theta_0 = -\theta_0 - 2\ell\, \theta_0'$$

$$\Rightarrow \boxed{\frac{d\theta_0}{d\ell} = -\frac{3}{4} \frac{\theta_0}{\ell}}$$

d)

$$\frac{d\theta_0}{\theta} = -\frac{3}{4}\,\frac{d\ell}{\ell}$$

$$\ln\theta_0 = -\frac{3}{4}\ln\ell + \tilde{C}$$

$$\theta_0 = \tilde{C} e^{-\frac{3}{4}\ln\ell}$$

$$= \boxed{\tilde{C}\,\ell^{-3/4}}$$