Kinematics in rotating frames

24 Apr.

REVIEW: notation, fictitious forces
Focault pendulum

BODY FRAME: UNPRIMED
SPACE FRAME: PRIMED

Vector: $\vec{e} = v_i \hat{x}_i(t) \xrightarrow{\text{more gen.}} v_i(t)\, \hat{x}_i(t)$

$= \underbrace{v_i'(t)\, \hat{x}_i}_{\text{const vec in BODY FRAME}} \longrightarrow \underbrace{v_i'(t)\, \hat{x}_i}_{\text{GENERAL VECTOR}}$

"abstract"

still no t-dep

[apology for the different notations btwn lecture & section]

ROTATION BETWEEN FRAMES: $e_i' = U_{ij} e_j$

TOTAL TIME RATE OF CHANGE:

$$\dot{e}' = \frac{d}{dt}(Ue) = \dot{U}e + U\dot{e}$$

$$= \underbrace{\dot{U}U^T}_{A'} \underbrace{Ue}_{e'} + U\dot{e}$$

$$+ (\dot{e})'$$

$$\dot{\vec{e}} = \vec{\omega} \times \vec{e} + \mathring{\vec{e}}$$

cf HW 8

OPEN DOT : the 'velocity' without the time evolution of the basis vectors

$$\mathring{\vec{e}} = \dot{e}_x \hat{x} + \dot{e}_y \hat{y} + \dot{e}_z \hat{z}$$

time deriv / velocity that we obs as people stuck on earth's surface

BODY FRAME EXPANSION

↳ but BASIS VECTORS ALSO TIME EVOLVE!

so $\mathring{\vec{e}}$ is not the full time deriv.

cf.

$$\dot{\vec{e}} = \dot{e}'_x \hat{x}' + \cdots \qquad \text{(SPACE FRAME)}$$

these never time evolve

carries full time evolution

Q: CAN YOU WRITE $\mathring{\vec{e}}$ in SPACE FRAME?

in principle yes, but have to disentangle the basis evolution from body vector evolution. (& why would this even be useful??)

Q: CAN YOU WRITE $\dot{\vec{e}}$ in BODY FRAME?

$$\dot{\vec{e}} = \dot{e}_x \hat{x} + e_x \dot{\hat{x}} + \cdots$$

So: $\boxed{\dot{\vec{e}} = \vec{\omega} \times \vec{e} + \mathring{\vec{e}}}$ ← relates non-inertial frame physics to inertial frame laws

eg. $\vec{e} = \hat{x}$ (body frame)

↑ const. body frame components

$\Rightarrow \mathring{\vec{e}} = \mathring{\hat{x}} = 0$

WHAT IS THE FULL TIME DERIVATIVE?

$\dot{\vec{e}} = \dot{\hat{x}} = \vec{\omega} \times \hat{x}$

↳ body axes precess about instantaneous angular velocity $\vec{\omega}$

eg. $\dot{\vec{\omega}} = \vec{\omega} \times \vec{\omega} + \mathring{\vec{\omega}} = \mathring{\vec{\omega}}$

<u>then</u>: let $\vec{e} = \vec{v} = \dot{\vec{r}} = \vec{\omega} \times \vec{r} + \mathring{\vec{r}}$

$$\dot{\vec{e}} = \ddot{\vec{r}} = \vec{\omega} \times \vec{e} + \mathring{\vec{e}}$$
$$= \vec{\omega} \times \dot{\vec{r}} + \mathring{\dot{\vec{r}}}$$
$$= \vec{\omega} \times (\vec{\omega} \times \vec{r} + \mathring{\vec{r}}) + (\vec{\omega} \times \vec{r} + \mathring{\vec{r}})^{\circ}$$

Q: $\mathring{\dot{\vec{r}}} \overset{?}{=} \dot{\mathring{\vec{r}}}$

Q: ~~CHAIN~~ PROD. RULE for $\circ$?

cont'd

$$\dot{\vec{e}} = \ddot{\vec{r}} = \vec{\omega}\times(\vec{\omega}\times\vec{r} + \mathring{\vec{r}}) + \mathring{\vec{\omega}}\times\vec{r} + \vec{\omega}\times\mathring{\vec{r}} + \overset{\circ\circ}{\vec{r}}$$

$\mathring{\vec{\omega}} = \dot{\vec{\omega}}$

$$= \overset{\circ\circ}{\vec{r}} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + 2\vec{\omega}\times\mathring{\vec{r}} + \dot{\vec{\omega}}\times\vec{r}$$

multiply across by mass
[fun: bottle rocket on a rope - what if mass had time dependence?]

$$\vec{F} = \vec{F}_{fict} + \underbrace{m\vec{\omega}\times(\vec{\omega}\times\vec{r}) + 2m\vec{\omega}\times\mathring{\vec{r}} + m\dot{\vec{\omega}}\times\vec{r}}$$

"FICTITIOUS FORCE"

BETTER NAME: NON INERTIAL FORCE

these guys are the "fictitious" forces

$$\vec{F}_{fict} = \vec{F} - \underbrace{m\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\text{CENTRIFUGAL } (O(\omega^2))} - 2m\vec{\omega}\times\mathring{\vec{r}} - m\dot{\vec{\omega}}\times\vec{r}$$

true force (eg from V')

CORIOLIS

"EULER" (USUALLY IGNORE B/C $\dot{\vec{\omega}} = 0$)

HEURISTIC PICTURE (wiki)

Foucault Pendulum : human scale pendulum vs. Earth scale rotating frame

last time:

CENTRIFUGAL vs. CORIOLIS IN THIS SYSTEM

$\sim m\omega^2 r$ $\qquad$ $\sim m\omega\dot{r}$

RELATIVE MAGNITUDES $\sim$ $\omega\Delta r : v$

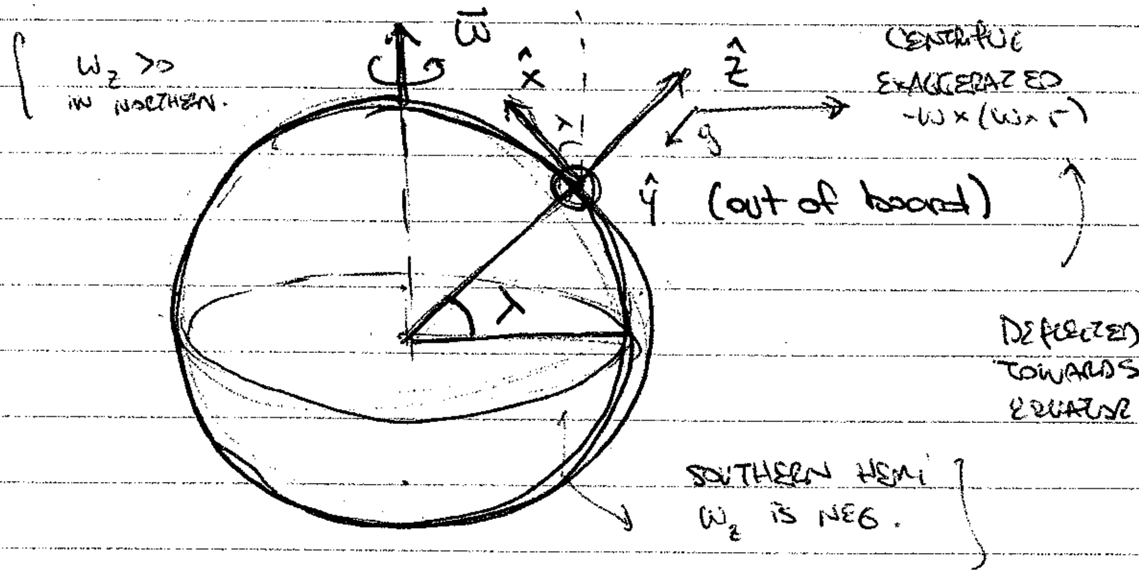
length of string, l $\qquad$ PENDULUM SPEED $\sim\sqrt{gl}$ (DIM ANALYSIS)

$$\omega\Delta r = \omega l = \frac{2\pi}{24\cdot 60^2}\,\frac{1}{s}\cdot 1\,m \approx 10^{-4}\ m/s$$

$$\underbrace{\sqrt{gl}}_{10\ m/s^2} \approx 3\ m/s$$

so Coriolis force is dominant non-inertial effect! (OF COURSE, GRAVITY IS DOMINANT FORCE)

Body axes: (@ latitude given by angle λ)



Q: this is the frame for the Northern Hemisphere ... how do we know that the dir of $\vec{\omega}$ is correct?

$$\vec{\omega} = \Omega \, (\cos\lambda \, \hat{x} + \sin\lambda \, \hat{z})$$

BODY COORDS

$\dfrac{2\pi}{24 \text{ hrs}}$

DON'T REALLY CARE

$\hat{z}$ term is what gives $\hat{x}, \hat{y}$ component of CORIOLIS FORCE $\sim \vec{\omega} \times \vec{r}$

v in non int frame

CORIOLIS 'FORCE' : $-2m\vec{\omega}\times\dot{\vec{r}}$

SMALL ANGLE OSC. ASSUMP: neglect v_z

$$= -2m\Omega\left[\cos\lambda\, v_y\,\hat{z} + \sin\lambda\, v_x\,\hat{y} - \sin\lambda\, v_y\,\hat{x}\right]$$

WE CARE ABOUT $\hat{x}$-$\hat{y}$ PLANE MOTION

<u>Gravitational force:</u>

$$F_x = -m\omega_g^2 x$$
$$F_y = -m\omega_g^2 y$$

$$\omega_g^2 = g/l$$

<u>Gravity + ~~fictitious~~ coriolis 'force'</u>

$$F_{fx} = -m\omega_g^2 x + 2m\Omega\sin\lambda\,\dot{y}$$
$$F_{fy} = -m\omega_g^2 y - \underbrace{2m\Omega\sin\lambda}_{\omega_p}\,\dot{x}$$

then EOM:

$$\begin{aligned} \ddot{x} &= -\omega_g^2 x + 2\omega_p \dot{y} \\ \ddot{y} &= -\omega_g^2 y - 2\omega_p \dot{x} \end{aligned} \Bigg\} (*)$$

in BODY (EARTH) FRAME, THIS IS WHAT WE CARE ABOUT.

Usual 2^{ND} Θ ODE TRICK:

write $z = x + iy$, then $(*) \Rightarrow$

$$\ddot{z} = -\omega_g^2 z - i 2\omega_p \dot{z}$$

Ansatz: $z = Ae^{i\alpha t}$

$$\Rightarrow -\alpha^2 = -\omega_g^2 + 2\omega_p \alpha$$

$\omega_g >> \omega_p$

$$\Rightarrow \alpha = \pm \omega_g + \Theta(\omega_p)$$

refine

$$-\alpha^2 = -\omega_g^2 \pm 2\omega_p \omega_g + \Theta(\omega_p^2)$$

$$\alpha^2 = \omega_g^2 (1 \mp 2\omega_p/\omega_g)$$

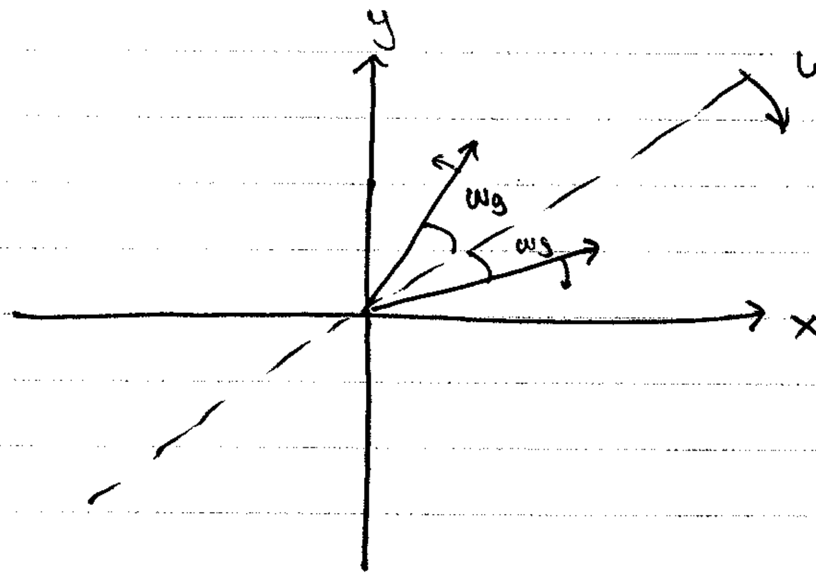
$$\alpha \cong \pm \omega_g (1 \mp \omega_p/\omega_g)$$

Approx sols: $\boxed{\alpha_{\pm} = \pm\omega_g - \omega_p}$

$$z = A\left(e^{i(\omega_g-\omega_p)t} + e^{i(-\omega_g-\omega_p)t}\right)$$

+ from init cond
eg. (A,0,0) @ t=0

two counter rotating phasors w/r to same origin in phase



ω_p slowly clockwise rotating

$\omega_p = \Omega \sin\lambda$

Another way to see (Marion & Thornton)

SUPPOSE $\omega_p = 0$: like a space frame
(no rotation of Earth)

$$\ddot{z}' = -\omega_0^2 \underset{\sim}{z}' \qquad \Big\} \; z_1' : \omega_p = 0$$

$$z = Ae^{i\omega t} + Be^{-i\omega t}$$ ← USUAL MOTION

COMPARE TO Z ABOVE

$$z = e^{-i\omega_p t} z'$$

USUAL MOTION

$$x + iy = (\cos(\omega_p t) - i\sin(\omega_p t))(x' + iy')$$

$$= (x'\cos\omega_p t + y'\sin\omega_p t) \leftarrow x$$
$$+ i(-x'\sin\omega_p t + y'\cos\omega_p t) \leftarrow y$$

or:
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\omega_p t & \sin\omega_p t \\ -\sin\omega_p t & \cos\omega_p t \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

ROTATION OF
PLANE OF PENDULUM