Free-precession of a rigid body

- equations of motion in the body frame
- $I_1 = I_2$: the "symmetric top"
- precession of the symmetric top in body & space frames

The relationship between angular velocity and angular momentum is simplest when expressed in the body frame and is especially simple when the body frame axes are aligned with principal axes of the inertia tensor:

$L_1 = I_1 \omega_1$, $L_2 = I_2 \omega_2$, $L_3 = I_3 \omega_3$.

Except for the special case $I_1 = I_2 = I_3$, or special initial conditions ($\omega_1 = \omega_2 = 0$, $\omega_1 = \omega_3 = 0$, $\omega_2 = \omega_3 = 0$), we will find that $\vec{\omega}$ is not constant in time. The most direct way of obtaining the equations of motion for $\vec{\omega}$ is to examine the equation $0 = \frac{d\vec{L}}{dt}$ in the <u>body frame</u>:

$$0 = \dot{\vec{L}} = \overset{\circ}{\vec{L}} + \vec{\omega} \times \vec{L}$$

$$\vec{L} = I_1 \omega_1 \hat{1} + I_2 \omega_2 \hat{2} + I_3 \omega_3 \hat{3}$$

$$\overset{\circ}{\vec{L}} = I_1 \dot{\omega}_1 \hat{1} + I_2 \dot{\omega}_2 \hat{2} + I_3 \dot{\omega}_3 \hat{3}$$

(Recall that $\overset{\circ}{\vec{L}}$ ignores the fact that $\hat{1}, \hat{2}, \hat{3}$ are time-dependent basis vectors.)

$$\vec{\omega} \times \vec{L} = \big(\omega_2 (I_3 \omega_3) - \omega_3 (I_2 \omega_2)\big)\hat{1}$$
$$+ \big(\omega_3 (I_1 \omega_1) - \omega_1 (I_3 \omega_3)\big)\hat{2}$$
$$+ \big(\omega_1 (I_2 \omega_2) - \omega_2 (I_1 \omega_1)\big)\hat{3}$$

$$0 = \vec{\tau} \Rightarrow \begin{cases} I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3) \\ I_2 \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1) \\ I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2) \end{cases}$$

These equations determine $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$ given initial values $\omega_1(0)$, $\omega_2(0)$, $\omega_3(0)$.

As claimed, when the principal moments are equal we have

$\mathring{\omega}_1 = \mathring{\omega}_2 = \mathring{\omega}_3 = 0 \implies \dot{\vec{\omega}} = \mathring{\vec{\omega}} = 0$

and the body rotates about a fixed axis.

Let's turn to the general case, where all three principal moments are different, say $I_1 < I_2 < I_3$. From the equations of motion we see that

$$\omega_1 I_1 \mathring{\omega}_1 + \omega_2 I_2 \mathring{\omega}_2 + \omega_3 I_3 \mathring{\omega}_3 = 0$$

which should not be a surprise since this is just

$$\frac{d}{dt} T_{rot} = \frac{d}{dt}\left(\tfrac{1}{2} I_1 \omega_1^2 + \tfrac{1}{2} I_2 \omega_2^2 + \tfrac{1}{2} I_3 \omega_3^2\right)$$

i.e. the time-derivative of the conserved energy.

The trajectories $\omega_1(t), \omega_2(t), \omega_3(t)$ therefore lie on an ellipsoid defined by $T_{rot} = E$.

Notice that whenever two ω's are initially zero, say $\omega_1(0) = \omega_2(0) = 0$, then $\dot{\omega}_1(0) = \dot{\omega}_2(0) = \dot{\omega}_3(0) = 0$ and so the angular velocity is constant with the body rotating uniformly about the third body axis, in this case $\hat{3}$. Let's go a step further and examine the case where ω_1 and ω_2 are both very small. Then the equation for $\dot{\omega}_3$ tells us the change in ω_3

is smaller still (of $O(\omega_1\omega_2)$) and so we approximate ω_3 as a constant. The equations for $\dot{\omega}_1$ and $\dot{\omega}_2$ are linear in this approximation and we find

$$I_1 \ddot{\omega}_1 \cong \omega_3 (I_2 - I_3) \dot{\omega}_2$$

$$= \omega_3 (I_2 - I_3) \omega_3 \left(\frac{I_3 - I_1}{I_2} \right) \omega_1$$

$$\Longrightarrow \quad I_1 I_2 \ddot{\omega}_1 = - \omega_3^2 (I_2 - I_3)(I_1 - I_3) \omega_1$$

Since $(I_2 - I_3)(I_1 - I_3)$ is <u>positive</u>, we see that ω_1 (and also ω_2) will oscillate and remain small. The special solution $\omega_1(t) = \omega_2(t) = 0$ is therefore <u>stable</u> to perturbations.

We can do the same analysis for the case $\omega_2 \cong \omega_3 \cong 0$, $\omega_1 \cong$ constant by cyclically shifting indices:

$$I_2 I_3 \ddot{\omega}_2 = -\omega_1^2 (I_3 - I_1)(I_2 - I_1)\omega_2$$

This oscillates, because $(I_3 - I_1)(I_2 - I_1)$ is positive, so the solution $\omega_2(t) = \omega_3(t) = 0$ is also stable.

Finally, consider the third case:

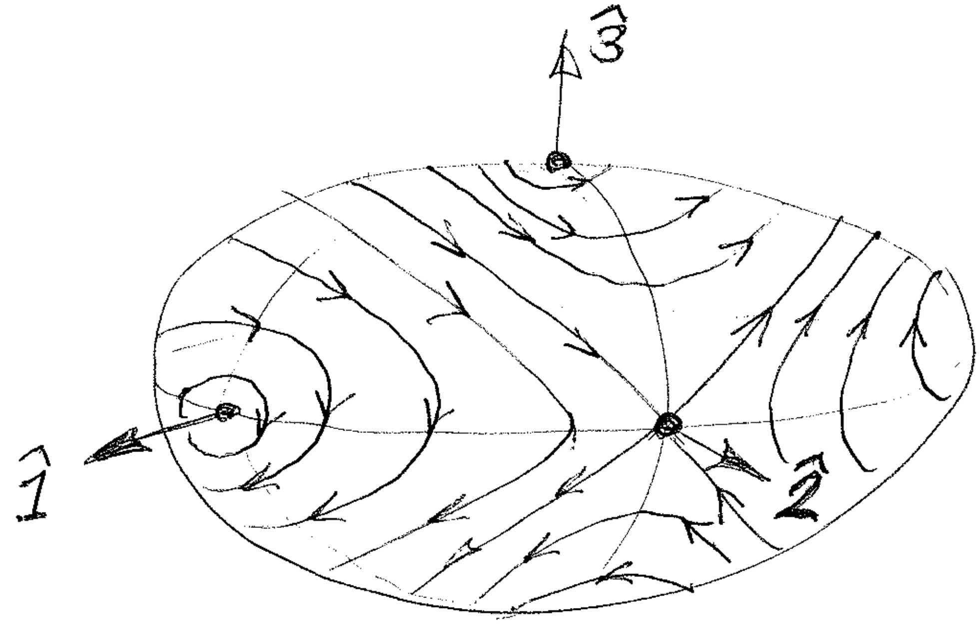
$$I_3 I_1 \ddot{\omega}_3 = -\omega_2^2 (I_1 - I_2)(I_3 - I_2)\omega_3$$

However, now ~~the~~ because $(I_1 - I_2)(I_3 - I_2)$ is negative, ω_3 does <u>not</u> oscillate the solution $\omega_1(t) = \omega_3(t) = 0$ is <u>not</u> <u>stable</u>.

These observations are summarized by the <u>anti</u>-<u>Goldilocks</u> <u>theorem</u>:

Fixed-axis rotation of a rigid body is stable only about the principal axes of <u>smallest</u> and <u>largest</u> moments of inertia.

The stability of motion about the principal axes is also reflected in the trajectory plot over the whole energy-ellipsoid:



There are many instances of rigid bodies where two of the principal moments are equal and different the third. Such bodies are called "symmetric tops". Here "symmetry" applies to the equality of two principal moments, $I_1 = I_2$ (without loss of generality), and does <u>not</u> require rotational symmetry of the body about the third axis ($\hat{3}$). For example, four equal masses forming a rigid square is a "symmetric top" (with $\hat{3}$ perpendicular to the plane of the square). The rotational motion of a symmetric top is simpler than the general case, and will be the last system we study in detail.

Return to the general equations and set $I_1 = I_2 = I$. First observe that $\dot{\omega}_3 = 0$, so ω_3 is constant. Here are the two remaining equations:

$$\dot{\omega}_1 = \omega_3\left(1 - \frac{I_3}{I}\right)\omega_2$$

$$\dot{\omega}_2 = -\omega_3\left(1 - \frac{I_3}{I}\right)\omega_1$$

Define the (constant) angular frequency

$$\Omega = \omega_3\left(\frac{I_3}{I_1} - 1\right)$$

so the coupled linear equations become

$$\dot{\omega}_1 = -\Omega\,\omega_2$$
$$\dot{\omega}_2 = +\Omega\,\omega_1$$
$$\dot{\omega}_3 = 0$$

All three equations are written compactly as

$$\dot{\vec{\omega}} = \overset{\circ}{\vec{\omega}} = (\Omega\,\hat{3}) \times \vec{\omega} .$$

As viewed in the body frame the angular velocity vector precesses about the body $\hat{3}$ axis with precession frequency Ω. We also see there are two types of symmetric top, with opposite signs of Ω:

"prolate"	"oblate"
$I_3 < I$	$I_3 > I$
$\Omega < 0$	$\Omega > 0$
football	frisbee

The earth is, to a rough approximation, an oblate rigid body with

$$\frac{I_3}{I} - 1 \cong \frac{1}{300}$$

In the absence of torques (from the moon), damping (from fluid motions) and effects due to the core, the angular velocity vector of the earth would maintain a fixed angle (currently about 10^{-6} radians) as it precesses about the symmetry ($\hat{3}$) axis

The period of the precession would be about

$$\frac{2\pi}{\Omega} \simeq \frac{2\pi}{\omega_3} \times 300$$

$$= (24\ \text{hr}) \times 300$$

or roughly one year. Astronomical observations give the direction of $\vec{\omega}$ in terms of the point in the sky about which the stars appear to be rotating. Seth Carlo Chandler (1891) first reported the precessional motion of this point and determined the period was somewhat longer, about 430 days, than predicted by the perfectly-rigid-earth model.

Body-frame analysis of free precession makes sense in the case of body-bound observations, such as the "Chandler wobble". However, to describe the motion of a football we prefer to know ~~what~~ how the axis ($\hat{3}$) is moving relative to the inertial (space) frame. We will do this for the case of the symmetric top.

For a symmetric top with $I_1 = I_2 = I$, we can rewrite

$$\vec{L} = I\omega_1 \hat{1} + I\omega_2 \hat{2} + I_3 \omega_3 \hat{3}$$

as $$\vec{L} = I\vec{\omega} + (I_3 - I)\omega_3 \hat{3}$$

Using our previous definition

$$\Omega = \left(\frac{I_3}{I} - 1\right)\omega_3$$

we get

$$\vec{L} = I\vec{\omega} + I\Omega\,\hat{3}.$$

Since $L = |\vec{L}|$ is constant, we can define another constant:

$$\omega_p = \frac{L}{I}$$

This gives us a vector relationship among three angular velocities:

$$\omega_p \hat{L} = \vec{\omega} + \Omega\,\hat{3}$$

Of these, only $\omega_p \hat{L}$ is constant as a vector. Simple algebra shows the other two vectors are precessing about $\hat{L}$:

$$0 = \frac{d}{dt}(\omega_p \hat{L}) = \mathring{\vec{\omega}} + \Omega\, \vec{\omega} \times \hat{3}$$

$$\Rightarrow \quad \mathring{\vec{\omega}} = \Omega\, \hat{3} \times \vec{\omega}$$

(re-derivation of body-frame precession)

$$\Rightarrow \quad \mathring{\vec{\omega}} = (\omega_p \hat{L} - \vec{\omega}) \times \vec{\omega}$$

$$\Rightarrow \quad \mathring{\vec{\omega}} = \omega_p \hat{L} \times \vec{\omega}$$

So $\vec{\omega}$ precesses about $\hat{L}$ (a fixed direction of space) with frequency ω_p.

We also know that

$$\dot{\hat{3}} = \vec{\omega} \times \hat{3}$$

$$= (\omega_p \hat{L} - \Omega \hat{3}) \times \hat{3}$$

$$\Longrightarrow \dot{\hat{3}} = \omega_p \hat{L} \times \hat{3}$$

So $\hat{3}$ also precesses about $\hat{L}$, and with the same frequency, ω_p. Summarizing: the vectors $\hat{L}$, $\vec{\omega}$, and $\hat{3}$ lie in one plane and this plane rotates with angular velocity ω_p about $\hat{L}$.

Let's apply this analysis to a football thrown so that it wobbles (a Tebow throw).

Let θ be the angle between the fixed precession axis $\hat{L}$ and the symmetry axis of the football, $\hat{3}$. Then

$$\cos\theta = \frac{\vec{L}}{L} \cdot \hat{3}$$

$$= \frac{I_3 \omega_3}{L} = \frac{I_3 \omega_3}{I \omega_p} .$$

Measurements of actual footballs have determined

$$I_3 / I \cong 0.6 .$$

Unlike the earth, a football is <u>prolate</u>, making ω_3 and Ω have opposite signs:

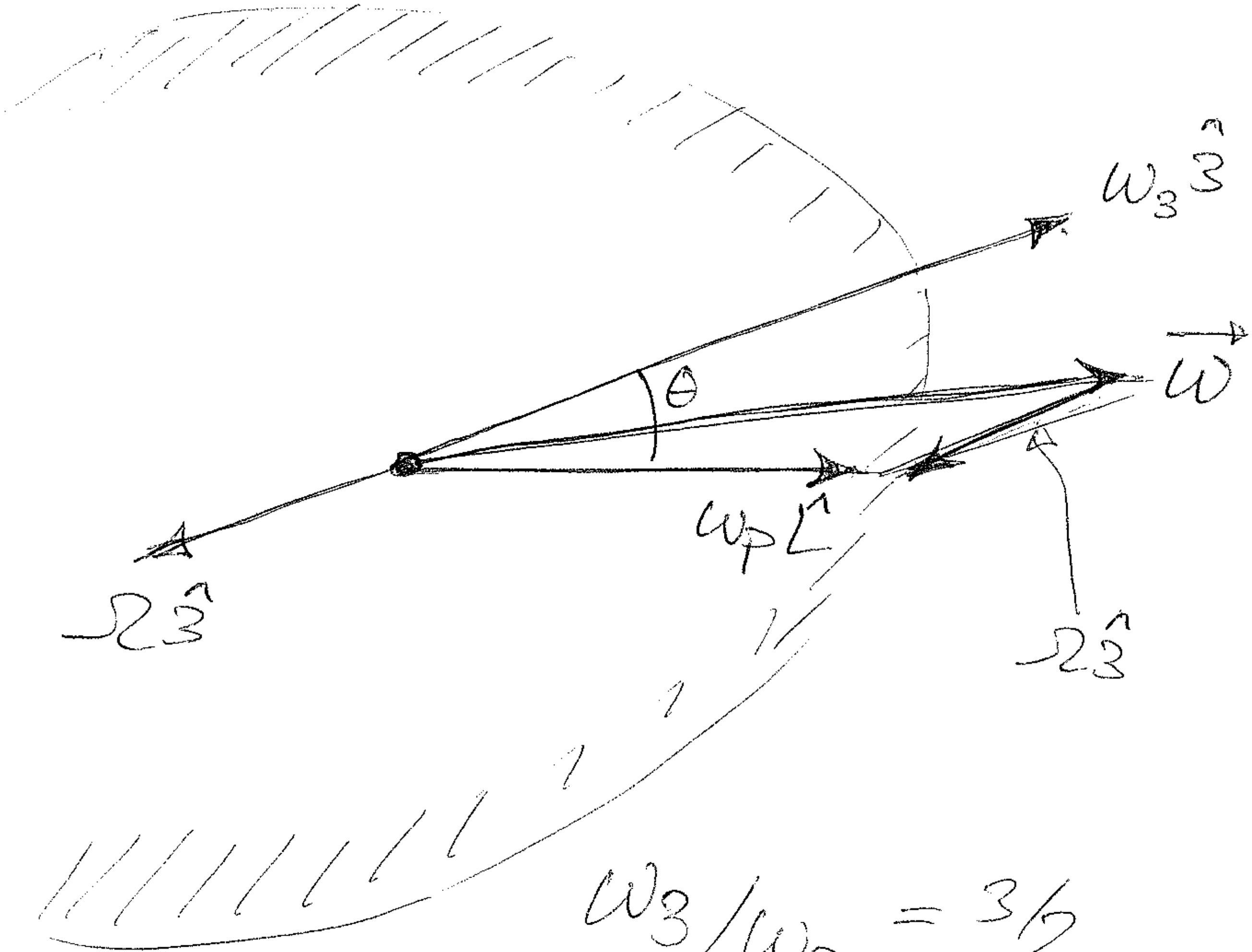
$$\Omega = \left(\frac{I_3}{I} - 1\right)\omega_3 \cong -0.4\,\omega_3$$

Suppose the football axis is tipped by 25° from the precession axis, so $\cos\theta \cong .9$. Then

$$\frac{\omega_3}{\omega_p} = \cos\theta \cdot \frac{I}{I_3}$$

$$\cong (.9)/(.6) = 3/2$$

The axis precesses twice during the time the football spins three times about its axis. The vector relationship among the angular velocities is shown on the next page:



$\omega_3 / \omega_p = 3/2$

$\cos\theta = 0.9$