Dynamics in phase space, cont.

The only way the time-dependent transformation defined by the generating function

$$F = F(q, Q, t)$$

can fail is if the equations

$$p = \frac{\partial F}{\partial q}, \quad P = -\frac{\partial F}{\partial Q}$$

do not define Q and P uniquely in terms of q and p (the trans. is singular). This happens, for example, when $F = f(q) + g(Q)$ since then

$$p = f'(q), \quad P = -g'(Q)$$

and Q, P are independent of

q, P. To get a general condition on F, we note

$$1 = \frac{\partial p}{\partial p} = \frac{\partial}{\partial p}\left(\frac{\partial F}{\partial q}\right) = \left(\frac{\partial^2 F}{\partial Q \partial q}\right)\left(\frac{\partial Q}{\partial p}\right)$$

so the mixed partial of F must be non-zero (which was not the case in the previous example).

Time was just a fixed parameter in the canonical transformation generated by F, and the fact that time varies in a dynamical process played no role. This changes as soon as we consider the dynamics in phase space first in terms of q, p

and then in terms of Q, P.
That the dynamics satisfies Hamilton's equations is equivalent to the requirement that action in phase-space is extremal for arbitrary variations δq and, independently, δp:

$$S = \int_{t_1}^{t_2} (p\dot{q} - H)\,dt$$

We now express this in terms of the transformed variables Q and P to see if these variables have the same form of action principle and therefore also satisfy ~~the~~ Hamilton's equations. But unlike our earlier work with the generating function, now we must take into consideration the <u>changes</u> <u>in</u> <u>time</u>:

$$S = \int_{t_1}^{t_2} (pdq - Hdt)$$

$$= \int_{t_1}^{t_2} \left(PdQ\big|_t + dF\big|_t - Hdt \right)$$

We would like to be able to replace $dQ|_t$ by dQ, so $PdQ = P\dot{Q}dt$ has the correct form for the action principle. Similarly, we would like to replace

$dF|_t$ by dF, so its integral just gives irrelevant endpoint terms. Let's start with $dQ|_t$:

$$dQ = \dot{Q}\,dt = \left(\frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} + \frac{\partial Q}{\partial t}\right)dt$$

$$= dQ|_t + \frac{\partial Q}{\partial t}dt$$

$$\Rightarrow dQ|_t = dQ - \frac{\partial Q}{\partial t}dt$$

Now work on $dF|_t$:

$$dF = \dot{F}\,dt = \ldots$$

In this calculation we recall that F is defined as a function of three variables like this:

$$F = F(q, Q(q,P,t), t)$$

$$dF = \left(\frac{\partial F}{\partial q} \dot{q} + \frac{\partial F}{\partial Q} \left(\frac{\partial Q}{\partial q} \dot{q} + \frac{\partial Q}{\partial p} \dot{p} + \frac{\partial Q}{\partial t} \right) + \frac{\partial F}{\partial t} \right) dt$$

All the terms with $t = \text{const.}$ are what we called $dF|_t$:

$$dF = \overbrace{\frac{\partial F}{\partial q} dq + \frac{\partial F}{\partial Q} \left(\frac{\partial Q}{\partial q} dq + \frac{\partial Q}{\partial p} dp \right)}^{dF|_t} + \left(\underbrace{\frac{\partial F}{\partial Q}}_{-P} \frac{\partial Q}{\partial t} + \frac{\partial F}{\partial t} \right) dt$$

$$\Rightarrow dF|_t = dF + P \frac{\partial Q}{\partial t} dt - \frac{\partial F}{\partial t} dt$$

Let's substitute $dQ|_t$ and $dF|_t$ into the action expression:

$$S = \int_{t_1}^{t_2} \left(PdQ - \cancel{P\frac{\partial Q}{\partial t}dt} + dF + \cancel{P\frac{\partial Q}{\partial t}dt} - \frac{\partial F}{\partial t}dt - Hdt \right)$$

$$= \int_{t_1}^{t_2} \left(PdQ - \left(\frac{\partial F}{\partial t} + H \right) dt \right) + F(q, Q, t) \Big|_{t_1}^{t_2}$$

As in the past, the variations of the coordinates δq and δQ

are constrained to vanish at the endpoints, so the term

$$F(q, Q, t)\Big|_{t_1}^{t_2}$$

always has zero variation and may be ignored. We therefore see that the form of the action principle (and hence Hamilton's equations) are preserved provided we transform the Hamiltonian as:

$$H'(Q, P) = \frac{\partial}{\partial t} F(q, Q, t) + H(q, p)$$

where $q = q(Q, P, t)$, $p = p(Q, P, t)$.

Return to previous example:

$$F = m\frac{qQ}{t}$$

$$\Rightarrow Q = \frac{p}{m}t\ ,\ P = -m\frac{q}{t}$$

Suppose the Hamiltonian for the original variables was that of a free particle:

$$H = \frac{p^2}{2m}$$

(The mass is the same as the "m" in our generating function, but these could have been different.)

The transformed Hamiltonian is then

$$H' = -\frac{mqQ}{t^2} + \frac{p^2}{2m}$$

$$= + \frac{PQ}{t} + \frac{1}{2} m \left(Q/t\right)^2$$

Hamilton's equations for the new variables:

$$\dot{Q} = \frac{\partial H'}{\partial P} = \frac{Q}{t}$$

$$\dot{P} = -\frac{\partial H'}{\partial Q} = -\frac{P}{t} - m\frac{Q}{t^2}$$

Solutions: $Q(t) = v_0 t$ (v_0 = const.)

$P(t) = -mv_0 - \frac{mq_0}{t}$

(q_0 = const.)

These are just transformations of the free particle solution:

$p = \text{const.} = mv_0$, $q = v_0 t + q_0$.