Lecture 2

- degrees of freedom
- generalized coordinates ~~and~~ (holonomic constraints)
- generalized velocities
- generalized forces

We say a mechanical system has N degrees of freedom if N (and no less) real-valued variables are required to uniquely specify the state of the system (i.e the positions of all the particles).

Often there are several choices of sets of variables, and the choice we make is usually one of convenience. The standard notation for these variables, in a general discussion, is

$$q_1, q_2, \ldots, q_N.$$

In practice these might be distances, angles, or even the charge on a capacitor plate.

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degree of freedom analysis of the cyclo-heptane molecule:

- point mass in 3D : 3 DoF
- 7 point masses (centers of carbon atoms): $7 \times 3 = 21$ DoF

Each constraint subtracts a degree of freedom:

7 bond length constraints +
7 bond angle constraints =
14 constraints

21 DoF − 14 constraints = 7 DoF

Some degrees of freedom are guaranteed by symmetry:

translation (of molecule): 3 DoF
rotation (of molecule): 3 DoF

So there are 6 DoF just from symmetry, or "rigid-body motion".
But 7-6 = 1, so cyclo-heptane

has 1 non-rigid-motion DoF.

The q's are called "generalized coordinates". By definition, ~~any~~ the position of any particle in our system is uniquely specified by the q's:

$$\vec{r}_i = \vec{r}_i(q_1, \dots, q_N)$$

$$\vec{r}_i = \text{position of } i^{th} \text{ particle}$$

In the case of cyclo-heptane,

$$\vec{r}_1, \dots, \vec{r}_7 = \left(\text{positions of carbon atoms}\right)$$

$$q_1, q_2, q_3 = \left(\text{CM coordinates of molecule}\right)$$

$$q_4, q_5, q_6 = \begin{pmatrix} \text{rotation angles} \\ \text{about three axes} \end{pmatrix}$$

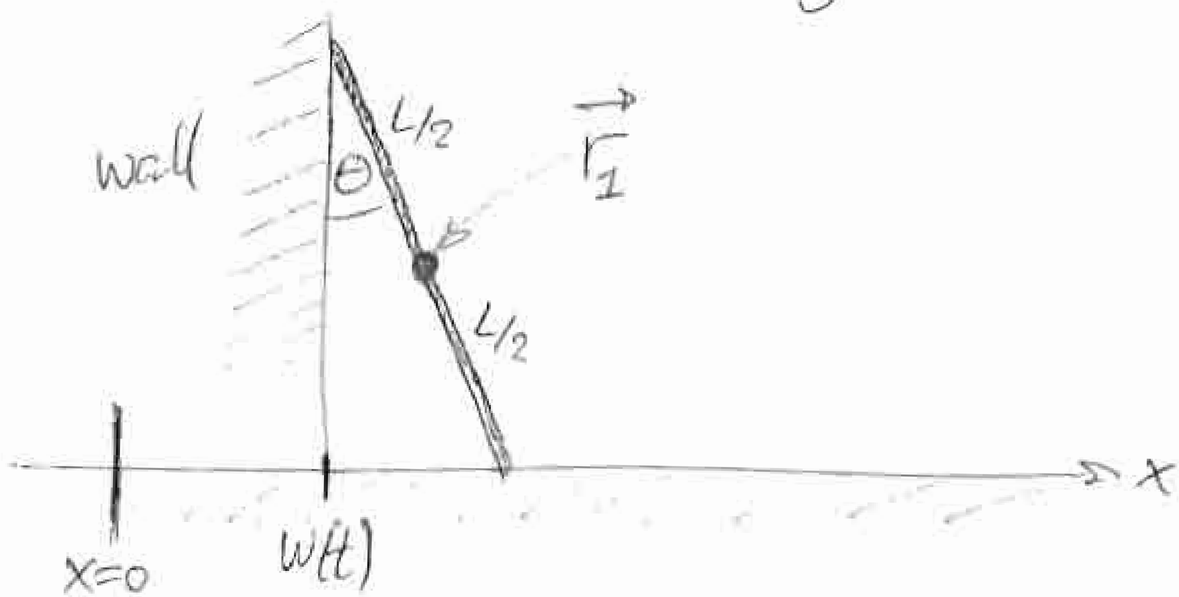
$$q_7 = \begin{pmatrix} \text{coordinate that specifies} \\ \text{the shape of the molecule} \end{pmatrix}$$

Sometimes the "environment" of the system changes with time, in which case we write

$$\vec{r}_i = \vec{r}_i(q_1, \ldots, q_N, t).$$

The environment is not considered part of the system, and hence there are no q's associated with it.

Simple example of a system with 1 DoF : ladder leaning on a frictionless, moving wall



$\vec{r}_1$ = position of ladder CoM

$= x\hat{x} + y\hat{y}$

θ = generalized coordinate

$w(t)$ = "environment"

The position of the wall, $w(t)$, is not "free" (a degree of freedom) but is given to us.

For example:

$$w(t) = v_0 t \quad \text{"moving frame"}$$

$$w(t) = \frac{1}{2} a t^2 \quad \text{"accelerating frame"}$$

$$w(t) = A \cos \omega t \quad \text{"earthquake"}$$

Calculate $\vec{r}_1$:

environment

$$x = \frac{L}{2} \sin\theta + w(t)$$

$$y = \frac{L}{2} \cos\theta$$

q

Time derivatives of the generalized coordinates (q's) are the ~~xx~~ <u>generalized velocities</u>. They are related to the particle velocities by the chain rule:

$$\dot{\vec{r}}_i = \sum_{k=1}^{N} \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}$$

particle velocity

generalized velocity

We should think of the $\dot{q}_k$'s as <u>independent</u> of the q_k's, in the same way that we consider the initial position and velocity of a mechanical system as

* generalized velocity for ladder system :

$$\dot{x} = \frac{L}{2}\cos\theta\,\dot{\theta} + \dot{w}$$

$$\dot{y} = -\frac{L}{2}\sin\theta\,\dot{\theta}$$

check :

$$\frac{\partial \dot{x}}{\partial \dot{\theta}} = \frac{L}{2}\cos\theta = \frac{\partial x}{\partial \theta}$$

$$\frac{\partial \dot{y}}{\partial \dot{\theta}} = -\frac{L}{2}\sin\theta = \frac{\partial y}{\partial \theta}$$

independent. Therefore, since $\vec{r}_i$ is just a function of the q's (not $\dot{q}$'s), $\frac{\partial \vec{r}_i}{\partial q_k}$ and $\frac{\partial \vec{r}_i}{\partial t}$ are also just functions of the q's.

Thus
$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \frac{\partial \vec{r}_i}{\partial q_k}.$$
* ladder example page 8.5

Next consider the forces acting on our system. We will be primarily interested in the <u>work</u> performed by these forces, for all conceivable motions as the generalized coordinates are varied. We can neglect all forces whose role is to impose the constraints on the particles, sometimes called "constraint forces". For example, we do not need to consider the forces that maintain a constant carbon-carbon bond distance, or the normal force that keeps a book from sinking into a table. By definition, these forces perform zero work since there is no displacement in the direction of the force.

Let

$$\delta q_1 , \ldots , \delta q_N$$

be arbitrary, infinitesimal changes in the generalized coordinates. Using the chain rule, the corresponding particle displacements are

$$\delta \vec{r}_i = \sum_{k=1}^{N} \frac{\partial \vec{r}_i}{\partial q_k} \delta q_k$$

Let $\vec{F}_i$ be the (non-constraint) force on particle i; then

$$\text{work} = \delta W = \sum_i \vec{F}_i \cdot \delta \vec{r}_i$$

where the sum is over all the particles.

Substituting the previous expression for $\delta \vec{r}_i$:

$$\delta W = \sum_i \vec{F}_i \cdot \left(\sum_{k=1}^{N} \frac{\partial \vec{r}_i}{\partial q_k} \delta q_k \right)$$

and after some rearranging

$$\delta W = \sum_{k=1}^{N} \underbrace{\left(\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \right)}_{\tilde{F}_k} \delta q_k$$

* → over

$\tilde{F}_k$ = (generalized force associated with gen. coordinate k)

* Emphasize, that in this expression $\vec{F}_i$ could include the constraint forces, but including them has no effect due to the nature of the motion ; the vectors $\frac{\partial \vec{r}_i}{\partial q_k}$ are $\perp$ to all the constraint forces)