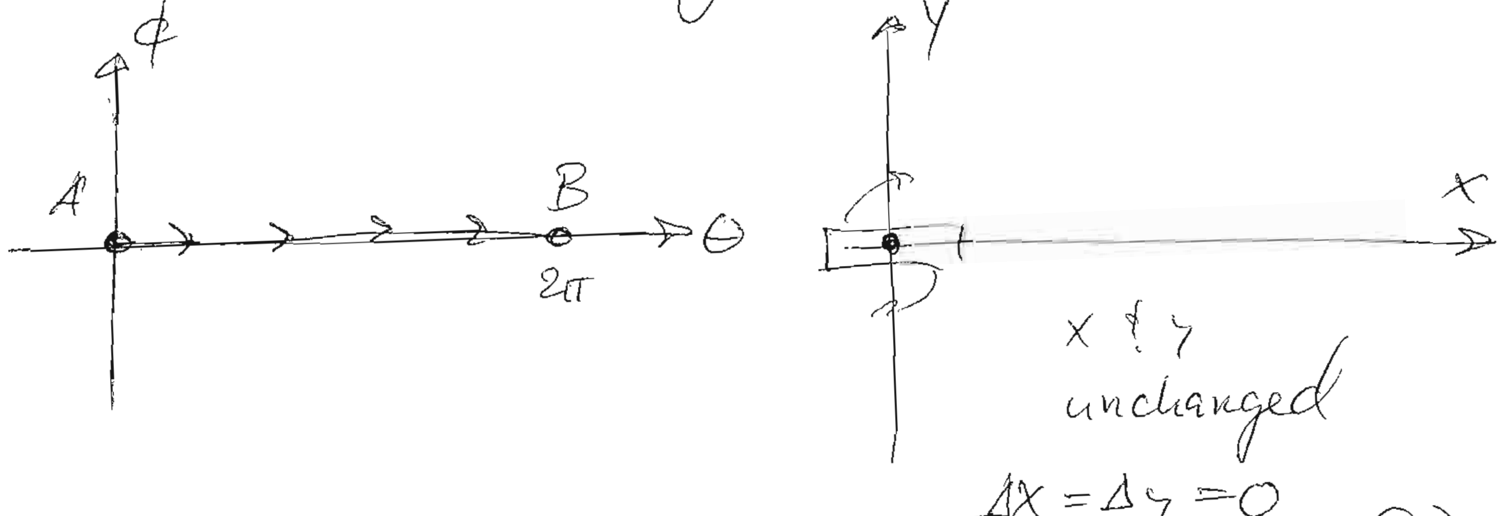
Lecture 6

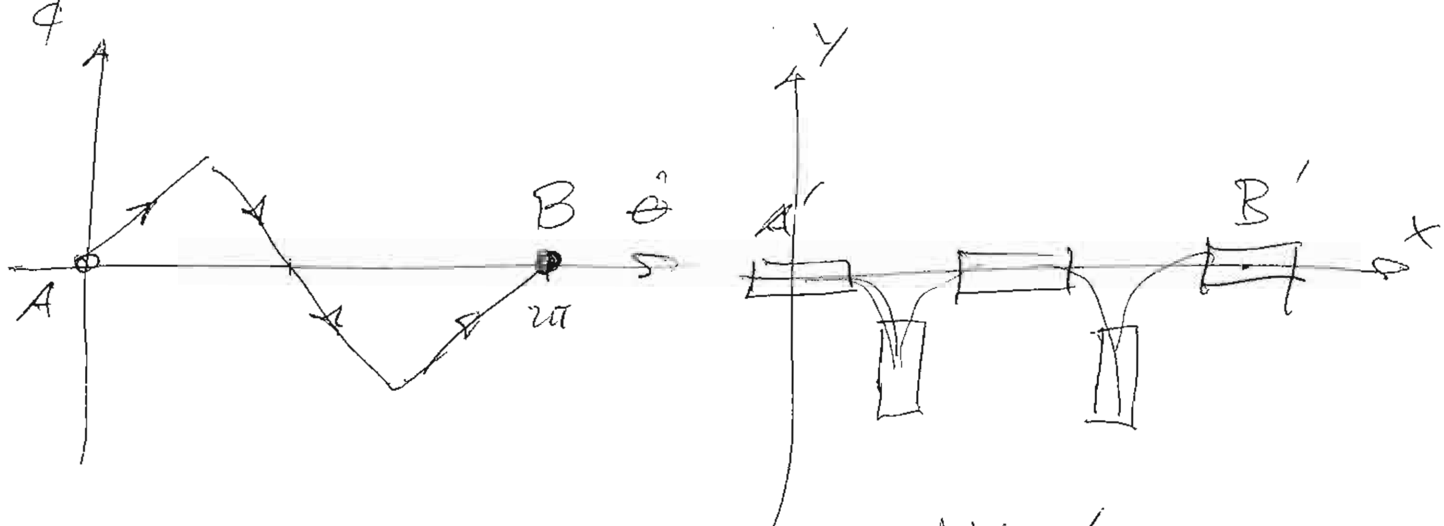
- non-holonomic constraints
- Brachistochrone problem
- calculus of variations

There are sequences of motion that demonstrate it is impossible for formulas

$$x = x(\theta, \phi)$$
$$y = y(\theta, \phi)$$

to exist (rolling wheel system).





$\Delta X \neq 0$

This shows the x, y position depends not just on θ, ϕ but on the <u>history</u> of the motion.

Here's an argument based on calculus. Recall constraint equation (for just x):

$$\dot{x} \cancel{\frac{dx}{dt}} - r\sin\theta \cancel{\frac{d\phi}{dt}} \dot{\phi} = 0$$

If there were a formula

$$x = x(\theta, \phi)$$

we could construct a function

$$f(x,\theta,\phi) = x - x(\theta,\phi)$$

of three variables that satifies the constraint

$$f(x,\theta,\phi) = 0$$

Take the time derivative:

$$0 = \frac{df}{dt} = \underbrace{\frac{\partial f}{\partial x}}_{=1} \dot{x} + \frac{\partial f}{\partial \theta}\dot{\theta} + \frac{\partial f}{\partial \phi}\dot{\phi}$$

We recognize this as our constraint equation for the velocities, so

$$\frac{\partial f}{\partial \theta} = 0, \quad \frac{\partial f}{\partial \phi} = -r\sin\theta$$

But now we have a problem, since

$$\frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial \theta}\right) = 0, \quad \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \phi}\right) = -r\cos\theta$$

which cannot happen for well-behaved functions f, so a (well-behaved) function $X(\theta, \phi)$ does not exist.

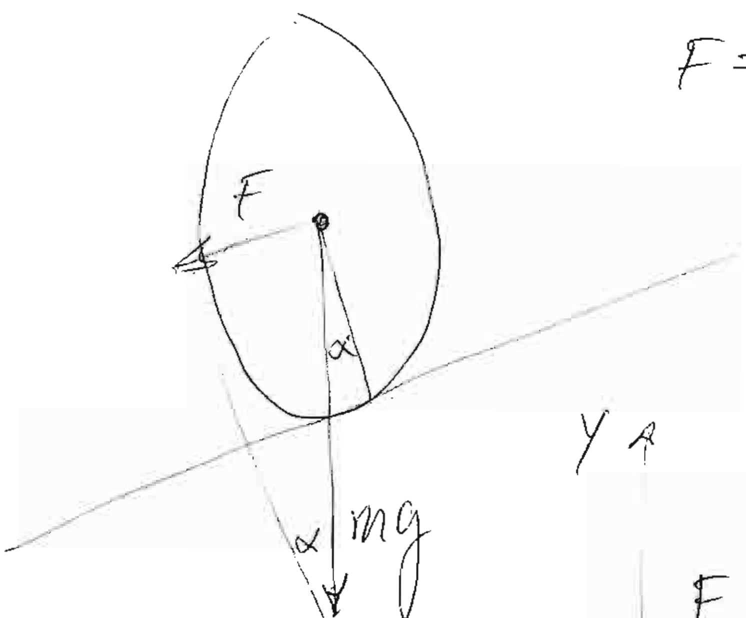
More generally, a system of N degrees of freedom whose variables cannot be expressed uniquely in terms of N generalized coordinates are said to have <u>non-holonomic constraints</u>. We will see how to work with non-holonomic constraints when we formulate mechanics in terms of a <u>variational principle</u>.

In this example ~~xxx~~ it is actually quite simple to solve for the motion directly in terms of Newton's 2nd law for forces and torques:

- No torques about axis normal to table $\Longrightarrow \dot{\theta} = \text{const.} = \omega$

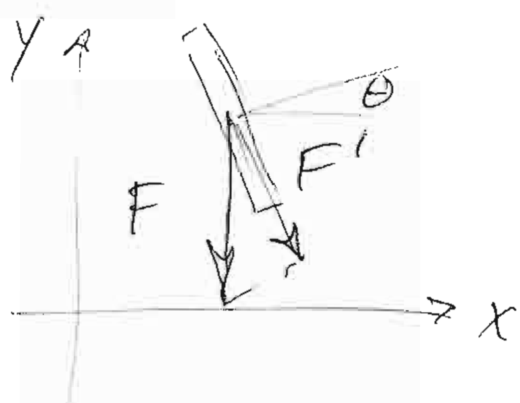
$$\theta(t) = \omega t + \theta_0$$

- Can have torque about axis of wheel. We consider a table tilted by angle α in y-direction:



$F =$ grav. + normal force

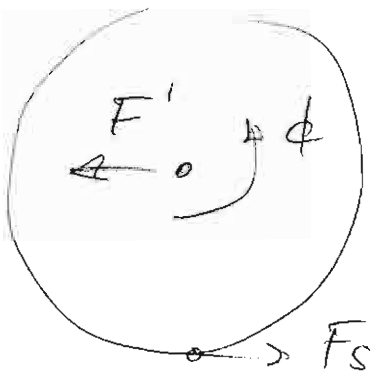
$= mg \sin\alpha$



F' = component of F in plane of wheel

$$= F\cos\theta = mg\sin\alpha\cos\theta$$

$$r\ddot{\phi} = -\ddot{y}$$



$\rightarrow y'$ (plane of wheel)

$$m\ddot{y}' = F_s - F' = -mr\ddot{\phi}$$

$$I\ddot{\phi} = rF_s$$

eliminate F_s : $I\ddot{\phi} = rF' - mr^2\ddot{\phi}$

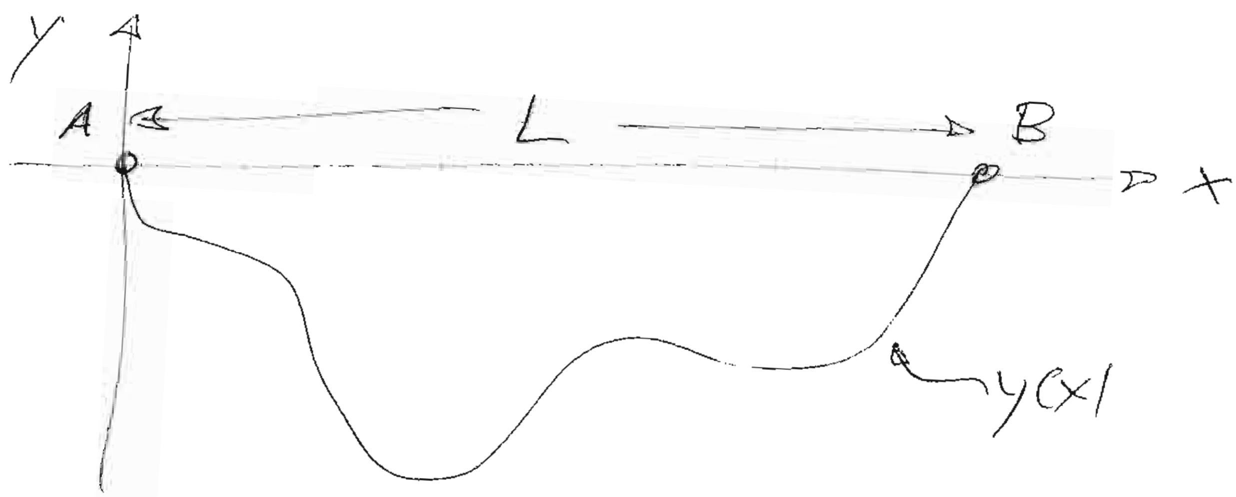
$$(I+mr^2)\ddot{\phi} = rF' = rmg\sin\alpha\cos\theta$$

$$(I+mr^2)\ddot{\phi} = rmg\sin\alpha\cos(\omega t+\theta_0)$$

Integrate twice to get $\phi(t)$, then get $x(t), y(t)$ from $\dot{x} = r\sin\theta\,\dot{\phi}$
$\dot{y} = -r\cos\theta\,\dot{\phi}$

Brachistochrone Problem

Find the curve that a particle acted upon by gravity and starting from rest should take to get from A to B in the least time.



Derive formula for the time for a general path $y(x)$:

$$\text{speed} = v(y)$$

$$\frac{1}{2}mv^2 + mgy = 0$$

$$v(y) = \sqrt{-2gy}$$

$$v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v}$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$T = \int_{t_0}^{t_f} dt = \int_0^L \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-2gy}}\, dx$$

$$= \int_0^L F\left(y, \frac{dy}{dx}, x\right) dx$$

In this case F does not depend explicitly on x, but we will consider that in our general discussion since in mechanics we work with

$$L(q, \dot{q}, t).$$

Brachistochrone problem:

Minimize the "functional"

$$T[y(x)] = \int_0^L F\left(y, \frac{dy}{dx}, x\right) dx$$

with respect to all functions $y(x)$ on the interval $[0, L]$ which have end-point values $y(0) = y(L) = 0$, and

$$F = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-gy}}$$

We will follow a general approach, for which the F above is just a special case.

Suppose $\tilde{y}(x)$ is the function for which $T[\tilde{y}(x)]$ is minimized.

If that is the case, then it should <u>not</u> be possible to slightly perturb $\tilde{y}$ and thereby decrease T.

Let's calculate the time T when $\tilde{y}$ is slightly perturbed:

$$T[\tilde{y}+\delta y] = \int_0^L \Bigg[\underbrace{F(\tilde{y}, \tfrac{d\tilde{y}}{dx}, x)}_{F|_{\tilde{y}}} + \underbrace{\left(\frac{\partial F}{\partial y}(\tilde{y}, \tfrac{d\tilde{y}}{dx}, x)\right)}_{\frac{\partial F}{\partial y}\big|_{\tilde{y}}} \delta y$$

$$+ \underbrace{\frac{\partial F}{\partial\left(\frac{dy}{dx}\right)}\Bigg|_{\tilde{y}} \left(\frac{d\,\delta y}{dx}\right)} + \left(\begin{array}{c}\text{2nd + higher order} \\ \text{terms in } \delta y\end{array}\right) \Bigg] dx$$

integrate by parts

$$\int_0^L \frac{\partial F}{\partial\left(\frac{dy}{dx}\right)}\Bigg|_{\tilde{y}} \frac{d}{dx}(\delta y)\, dx = \underbrace{\frac{\partial F}{\partial\left(\frac{dy}{dx}\right)}\Bigg|_{\tilde{y}} \delta y \,\Bigg|_{x=0}^{x=L}}_{=\,0} \quad \ldots$$

$$-\int_0^L \frac{d}{dx}\left(\left.\frac{\partial F}{\partial \frac{dy}{dx}}\right|_{\tilde{y}}\right)\delta y \, dx$$

Subtract $T[\tilde{y}]$ and combine terms:

$$T[\tilde{y}+\delta y]-T[\tilde{y}] = \delta T$$

$$= \int_0^L \underbrace{\left[\left.\frac{\partial F}{\partial y}\right|_{\tilde{y}} - \frac{d}{dx}\left(\left.\frac{\partial F}{\partial \frac{dy}{dx}}\right|_{\tilde{y}}\right)\right]}_{v(x)\ =\ \text{a function of } x} \delta y(x) \, dx$$

If $v(x)$ is non-zero in some small interval of x, then by making $\delta y(x)$ non-zero and of the correct sign we can decrease T. But this should be impossible, hence $v(x)=0$.

There is a better name for $v(x)$:

$$\frac{\delta T}{\delta y(x)} = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \frac{dy}{dx}}\right)$$

= variational derivative of the <u>functional</u> ~~F~~ T with respect to $y(x)$

Informally: $\frac{\delta T}{\delta y(x)}$ is the rate-of-change (slope) of T when we vary ~~just~~ y ~~a~~ at the point x.

The special function $\tilde{y}$ has the property that all ~~its~~ the variational derivatives of T are zero at this $\tilde{y}$

$$\tilde{y} \text{ minimizes } T \iff \left.\frac{\delta T}{\partial y(x)}\right|_{\tilde{y}} = 0 \quad \underline{\text{all } x}$$

This condition is like saying all the partial derivatives of a function of several variables are zero, only here there is an ~~variable~~ independent variable — called $y(x)$ — at each x!