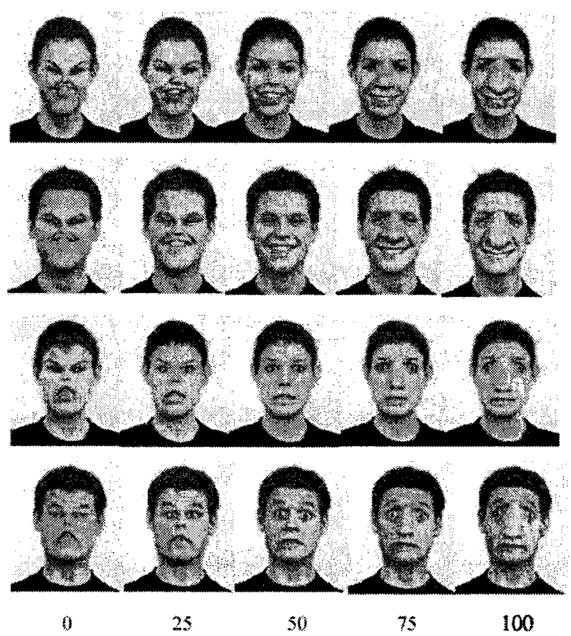
Begin the exam when instructed to do so. You have 50 minutes to complete four problems. The maximum point credit is shown at the beginning of each problem. All work must be on the exam pages.

Rules: No sources (notes, texts, homework solutions, etc.) or calculators are allowed. A few formulas are given at the end of the exam.

The Cornell Code of Academic Integrity is in effect, as always.

SOLUTIONS

Face Space



1. (30 points) As an alternative to the usual circular-orbit Bohr model of the hydrogen atom, suppose the electron moves on a *single axis* that passes through the proton. The Hamiltonian for this system is

$$H(x,p) = \frac{p^2}{2m} - \frac{A}{|x|},$$

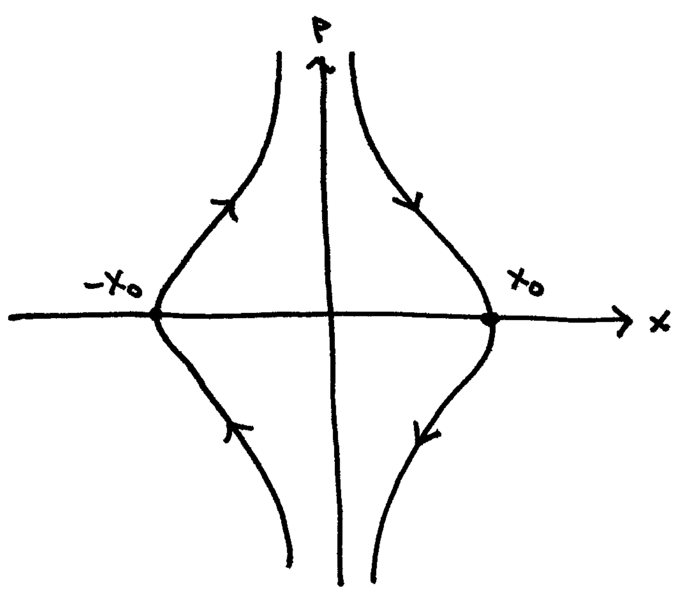
where x is the electron position along the axis, p is the conjugate momentum, m is the electron mass, and $A = e^2/(4\pi\epsilon_0)$ in MKS.

(a) Write down Hamilton's equations of motion for $H(x,p)$.

$$\boxed{\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} &= \frac{-\partial H}{\partial x} = -\operatorname{sgn}(x)\,\frac{A}{x^2} \end{aligned}}$$

Consider a periodic orbit where, at $t = 0$, the electron is at rest at position $x = x_0 > 0$. Its energy on this orbit will be $E = -A/x_0$. After one-quarter of the orbital period, $t = T/4$, the electron will have just passed through the proton at $x = 0$. Later, after reaching $x = -x_0$ at $t = T/2$, the electron reverses its direction and retraces its motion back to $x = x_0$ to complete one period.

(b) Sketch one complete orbit in phase space:



(c) For the orbit with energy $H = E = -A/x_0$, calculate the action (phase-space area) enclosed by the orbit, $S = \oint p\,dx$. Helpful fact:

$$\int_0^{x_0} \sqrt{(1/x) - (1/x_0)}\,dx = \pi\sqrt{x_0}/2.$$

$\frac{1}{4}$ (PERIOD)

$0 < x < x_0$: $p^2 = 2mA\left(\frac{1}{x} - \frac{1}{x_0}\right)$

$$\frac{S}{4} = \sqrt{2mA}\int_0^{x_0}\sqrt{\frac{1}{x} - \frac{1}{x_0}}\,dx = \frac{\pi}{2}\sqrt{2mAx_0}$$

$$\Rightarrow \boxed{S = 2\pi\sqrt{2mAx_0}}$$

(d) Using your answer to (c) you should be able to express the energy $E = -A/x_0$ of the orbit only in terms of S (and constants) as $E = -B/S^2$; find the value of the constant B.

$$x_0 = \frac{S^2}{(2\pi)^2\,2mA}$$

$$E = \frac{-A}{x_0} = \frac{-8\pi^2 mA^2}{S^2}$$

$$\boxed{B = 8\pi^2 mA^2}$$

Notice that this energy expression is consistent in form with the Rydberg series if S is quantized in integer multiples of Planck's constant.

2. (30 points) The Hamiltonian below describes a free particle with a position-dependent mass ($m \propto 1/q$):

$$H(q,p) = Cqp^2.$$

In this problem you will simplify this Hamiltonian by applying the canonical transformation generated by $F(q,Q) = q/Q$.

(a) First find the canonical transformation,

$$\boxed{\begin{aligned} Q(q,p) &= \frac{1}{p} \\ P(q,p) &= qp^2 \end{aligned}} \quad \begin{aligned} &\leftarrow p = \frac{\partial F}{\partial q} = \frac{1}{Q} \\ &\leftarrow P = \frac{-\partial F}{\partial Q} = q/Q^2 \end{aligned}$$

F indep. of t (explicitly)

$H'(Q,P) = H(q(Q,P), p(Q,P))$

(b) Now write down the transformed Hamiltonian:

$$\boxed{H'(Q,P) = CP}$$

(c) Write down Hamilton's equations for Q and P and their most general solution.

$$\dot{Q} = \frac{\partial H}{\partial P} = C \longrightarrow Q(t) = Ct + Q_0$$

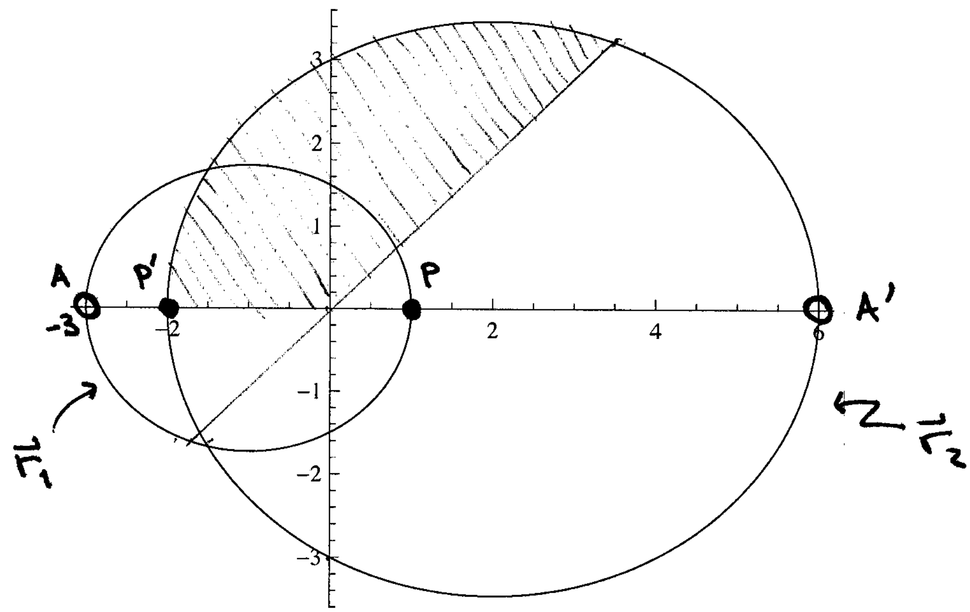
$$\dot{P} = \frac{-\partial H}{\partial Q} = 0 \longrightarrow P(t) = P_0$$

(d) Using your general solution in (c), write down the most general solution for the original position variable, $q(t)$:

$$\boxed{q = PQ^2 = P_0(Ct + Q_0)^2}$$

$$P = \frac{-\partial F}{\partial Q} = q/Q^2$$

3. (20 points) The figure below is a time-lapse image of a binary star system, where the view direction happens to be normal to the plane of the motion. A coordinate system has been overlaid to aid you in answering the questions below. The origin of the coordinate system is at a focus of each ellipse.



(a) Obtain the mass ratio of the two stars. On which ellipse, the small one or the large one, does the more massive star move?

$$\vec{r}_{1,2} = \frac{\pm m_{2,1}}{m_1+m_2} \Rightarrow \frac{r_1}{r_2} = \frac{m_2}{m_1} = \frac{3}{6} \Rightarrow \boxed{\frac{m_2}{m_1} = \frac{1}{2}}$$

massive star on small ellipse

or $\frac{m_1}{m_2} = 2$

(b) Determine the eccentricity ϵ of the larger ellipse.

$r(\theta) = \frac{r_0}{1+\varepsilon\cos\theta} \rightarrow r_2(\theta=0) = \frac{r_0}{1+\varepsilon}$, $r_2(\pi/2) = r_0$

$\theta = 0$

$\Rightarrow 1+\varepsilon = \frac{r(\pi/2)}{r(0)} = \frac{3}{2} \Rightarrow \boxed{\varepsilon = 1/2}$

OR: $\frac{r_{max}}{r_{min}} = \frac{1+\varepsilon}{1-\varepsilon} = \frac{6}{2}$

$\rightarrow 3 - 3\varepsilon = 1+\varepsilon$

$\varepsilon = 1/2$

(c) Mark with P and P' the positions on the orbits at the instant when the two stars are nearest, or "periapsis". Mark with A and A' the positions on the orbits at the instant when the two stars are furthest, or "apoapsis". If it takes 10 days for the stars to move from periapsis to apoapsis, mark with Q and Q' their approximate positions 5 days after periapsis and **briefly explain your reasoning**.

area of shaded region = $\frac{1}{4}$ ellipse
(as per kepler's area law)

4. (20 points) Consider a Lagrangian with no explicit time dependence:

$$L = L(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N).$$

Let L' be the Lagrangian where the following continuous transformation, parameterized by s, is applied to the coordinates:

$$q_i(t) \to Q_i(t, s) = q_i(t + s) \qquad i = 1, \ldots, N. \tag{1}$$

(a) Show that

$$\left.\frac{dL'}{ds}\right|_{s=0} = \frac{dF}{dt} \tag{2}$$

for some function F; find F.

$$L'(Q, \cdots) = L(q(t+s), \cdots)$$

$$\left.\frac{dL'}{ds}\right|_{s=0} = \frac{dL}{dt} \qquad \text{choose } \boxed{F = L}$$

up to overall constant

(b) By the generalized Noether's theorem, Lagrangians with property (2) automatically lead to the conserved quantity

$$I = \sum_{i=1}^{N} p_i \left.\frac{dQ_i}{ds}\right|_{s=0} - F.$$

For the transformation (1) and the corresponding F you found in (a), the quantity I has another name: what is it?

$$\left.\frac{dQ_i}{ds}\right|_{s=0} = \dot{q}_i$$

$$I = \sum_{i=1}^{N} p_i \dot{q}_i - L = H, \quad \boxed{\text{HAMILTONIAN}}$$

Formulas

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$H = \sum_i p_i \dot{q}_i - L$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

$$S = \int_{t_1}^{t_2} L\, dt$$

$$\mathbf{r}_1 = \left(\frac{m_2}{m_1 + m_2}\right)\mathbf{r} \qquad \mathbf{r}_2 = -\left(\frac{m_1}{m_1 + m_2}\right)\mathbf{r}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$A = G m_1 m_2$$

$$L_z = \mu r^2 \dot{\theta}$$

$$\mu \ddot{r} = -\frac{dU}{dr} \qquad U(r) = \frac{L_z^2}{2\mu r^2} - \frac{A}{r}$$

$$r(\theta) = \frac{r_0}{1 + \epsilon \cos\theta}$$

$$r_0 = \frac{L_z^2}{\mu A} \qquad E = \frac{A}{2r_0}(\epsilon^2 - 1)$$

$$b = r_0/\sqrt{\epsilon^2 - 1} \quad (\epsilon > 1)$$

$$a = r_0/(1 - \epsilon^2) \qquad b = \sqrt{1 - \epsilon^2}\, a$$

$$\dot{\mathcal{A}} = \frac{L_z}{2\mu}$$

$$a^3 = \frac{G(m_1 + m_2)}{4\pi^2} T^2$$

$$I = \sum_{i=1}^{N} p_i \left. \frac{dQ_i}{ds} \right|_{s=0} - F$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$S = \int_{t_1}^{t_2} \left(\sum_{i=1}^{N} p_i \dot{q}_i - H \right) dt$$

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}$$

$$\{Q, P\} = 1$$

$$\dot{A} = \{A, H\} + \frac{\partial A}{\partial t}$$

$$F = F(q, Q, t) \qquad p = \frac{\partial F}{\partial q} \qquad P = -\frac{\partial F}{\partial Q}$$

$$H'(Q, P, t) = H(q(Q, P, t), p(Q, P, t), t) + \frac{\partial F}{\partial t}$$