P318 : SECTION 4   F FEB 15

RETURN HW #2

PLAN : DO CH.4 READING §4.3-4.7

REMARKS: what we're skipping in ch.3

↳ why GREEN'S FUNCTIONS MATTER

FUN : CONTINUUM LAGRANGIAN MECHANICS } PUT THESE TOGETHER & GET QFT

NEXT WEEK M: QM

W: CENTRAL FORCES

remark

IN LEC: HOLONOMIC

$\rightarrow \vec{U} = \vec{\nabla} g$

$\leftrightarrow$ integrable (CONSERVATIVE)

THE SHO   simple harmonic oscillator

↑ the most important system in physics

WHY?

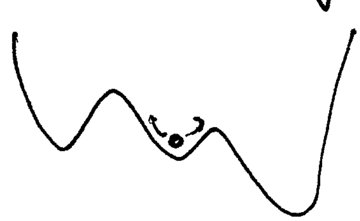
↳ we USUALLY PERTURB AROUND THE MINIMA of OUR SYSTEMS

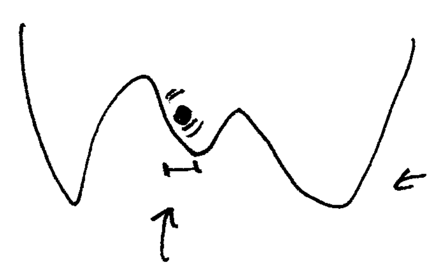
⇕

Taylor expansion

] SO MUCH OF PHYSICS REDUCES TO THIS!

V





← not nec true min!

take some system in equilibrium

DISPLACE SOMETHING A LITTLE & SEE WHAT HAPPENS

$$V(q_0+\Delta q) = V(q_0) + \cancel{q V'(q_0)}^{\,0} + \tfrac{1}{2} q^2 V''(q_0)$$

YOU'VE SEEN SHO LAGRANGIANS IN YOUR HW

... YOU'VE STARED AT THE NASTY ELLIPTIC INTEGRALS

for many things, the small angle approx is ~~good~~
→ even things which are not SHO!

WHAT ABOUT THE REST OF $V(q)$? PERTURBATION THY.
many fancy ways... all basically Taylor expansion

H7F p.88 What's so great about SHO? ✓ 2nd ⊙ stops here

① EoM is LINEAR : only 1st power of $q, \dot{q}, \ddot{q}, \dots$

② EoM is HOMOGENEOUS : no "+ constant"

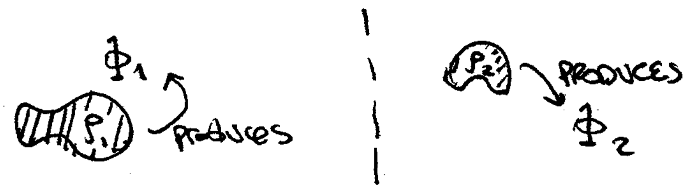
↪ if $q(t)$ a solution, so is $\alpha q(t)$

$\boxed{\text{SUPERPOSITION}}$ if q_1 & q_2 solutions, so is $\alpha_1 q_1 + \alpha_2 q_2$

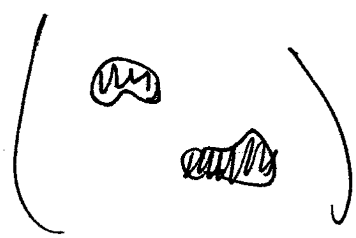
this is 'physically obvious' for things like waves on a string... BUT PERHAPS LESS OBVIOUS IN OTHER SITUATIONS eg. LAPLACE EQ: $\nabla^2 \Phi = 0$ of HW

3327 students know this story (& it's E&M generalizations well)

in fact: $\nabla^2 \Phi = \rho$ ← charge density (static)



QUESTION: ? Φ for

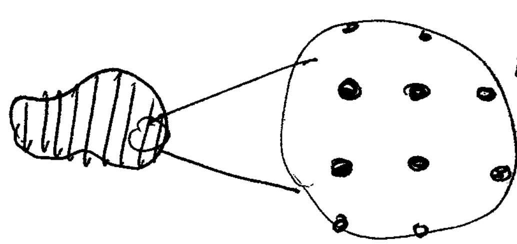


?

$$\boxed{\Phi_{1+2} = \Phi_1 + \Phi_2}$$ SUPERPOSITION.

→ lead to all sorts of cute tricks.

BUT WHAT is Φ_1 TO BEGIN WITH?



(static) POINT CHARGES! ← equal charge

[you can take this literally or figuratively]

$$\Phi(\text{POINT CHARGE @ } \vec{s}) \sim \frac{q_i}{|\vec{r} - \vec{s}|}$$

$$\rho = \sum_{\vec{s}_i} \underset{\Delta q_i}{\boxed{}} \, \delta_{\vec{s}_i \vec{r}}$$ ← 'coarse grained'

then: $\Phi = \sum_{\vec{s}_i} \frac{\Delta q_i}{|\vec{r} - \vec{s}_i|}$ ← $\Delta q_i = \underbrace{d^3\vec{s}}_{\text{VOL}}\ \underbrace{\rho(\vec{s}_i)}_{\text{CH-DENSITY}}$

↓ pass to continuum

$$= \int d^3\vec{s}\, \frac{\rho(\vec{s})}{|\vec{r} - \vec{s}|}$$

↳ P3327 : LEGENDRE POLYNOMIALS...

BUT: WHAT IS $\nabla^2 \frac{1}{|\vec{r} - \vec{s}|}$? (or $\nabla^2 \frac{1}{r}$)

↳ recall: $= -\frac{1}{|\vec{r} - \vec{s}|^2} * \boxed{\delta(\vec{r} - \vec{s})}$ ← δ function SOURCE ↓ "POINT PARTICLE"

makes sense, $\nabla^2 \Phi = \rho(\vec{r})$ ✓

SO $|\vec{r} - \vec{s}|^{-1}$ IS <u>SPECIAL</u> w/rt ∇^2

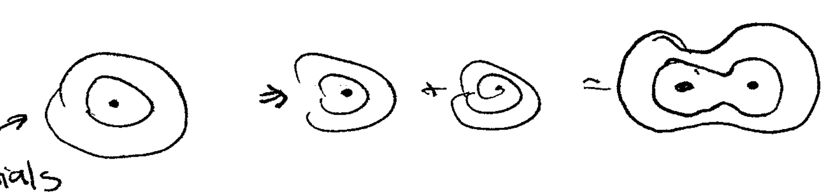
IT IS THE "ATOM" OF POTENTIAL THAT YOU CAN USE TO BUILD UP COMPLICATED POTENTIALS, EXACTLY AS ONE WOULD USE POINT CHARGES AS "ATOMS" TO BUILD UP SOURCES.

↳ "<u>Green's function</u>"

the book applies this to the <u>forced</u> HO.

[GREEN'S FUNCTIONS POP UP ALL OVER]

eg POINT POTENTIAL



Green's function : WAY TO SOLVE INHOMOGENEOUS DIFF EQ. BY BREAKING DOWN INHOMOGENEITY INTO "ATOMS" WHICH EACH CONTRIBUTE A GREEN'S FUNCTION TO THE SUPERPOSITION WHICH GIVES THE SOLUTION.

inhomogeneous harmonic oscillator :

$$\ddot{q} + q = g(t)$$

what is this?

comes from:

$$L = \tfrac{1}{2} m \dot{q}^2 - \tfrac{1}{2} m q^2 + m q g$$

FROM EOM : $g(t)$ is a driving force

IN LAGRANGIAN : ALMOST A CONSTRAINT FORCE

(mg) WOULD BE A LAGRANGE MULTIPLIER (λ) FOR THE CONSTRAINT $q = 0$ (or $q = \text{const}$) IF WE TREATED IT AS AN AUXILIARY DOF - (we don't)

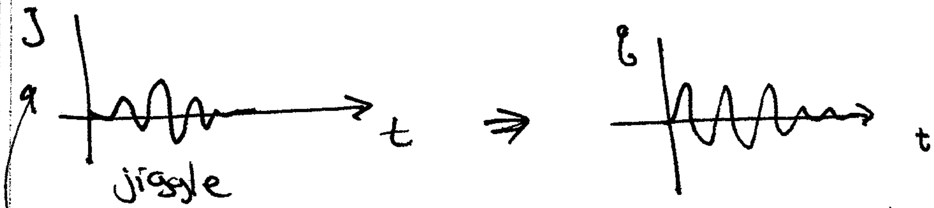
↓

but it is clear that this is an additional force

So GREEN'S FUNCTION; $G(t)$ SATISFIES $\ddot{G} + G = \delta(t-t')$

} can build up the solution for any impulse

<u>Remark</u> terms in $V(q)$ like $J(t)\,q$ are often called <u>sources</u>. WIGGLING THE SOURCE, eg



causes wiggles in q

DRIVING FORCE for $q(t)$

IF THE SYSTEM HAS MANY DOF, CAN IMAGINE A TERM:

$$V(q_1, q_2, \dots) \supset q_1(t)\, q_2(t)$$

q_1 ACTS AS A <u>SOURCE</u>/<u>DRIVING FORCE</u> FOR q_2! (& VICE VERSA)

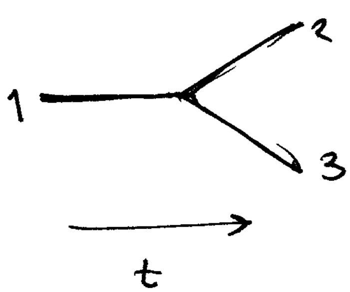
WIGGLES IN $q_1 \longleftrightarrow$ WIGGLES IN q_2

WHAT ABOUT: $q_1 q_2 q_3$?

WIGGLES IN (q_1 AND q_2) $\to$ WIGGLES IN q_3

or WIGGLES IN q_1 $\longrightarrow$ WIGGLES IN q_2 AND q_3

LINE = WIGGLENESS

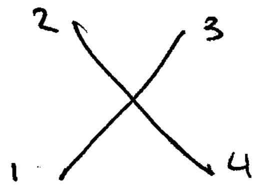


JIM'S SUMMER STUDENTS WILL RECOGNIZE THIS AS A PROTO FEYNMAN DIAGRAM

this is a <u>PERTURBATION</u> TO THE SHO L

by the way:

ok. CLEAR HOW TO GENERALIZE TO, eg. $q_1 q_2 q_3 q_4$



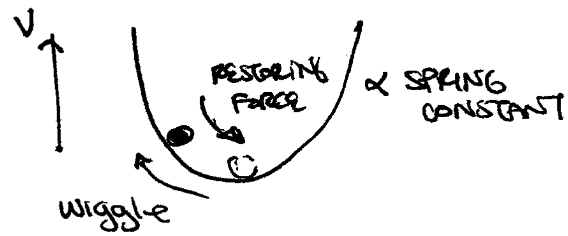
"WIGGLES CAUSE OTHER WIGGLES"

WHAT ABOUT TERMS THAT WE ALREADY KNOW & LOVE?

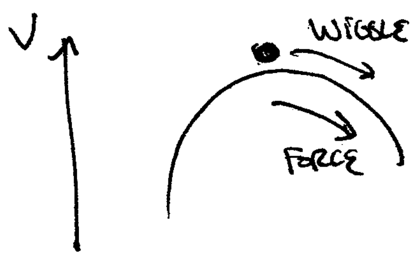
$$V \supset -\underbrace{\tfrac{1}{2} m \omega^2}_{\text{const.}} q^2$$

wiggles in q cause "opposite" wiggles in itself

makes sense, right?



What if $+\frac{1}{2} m \omega^2 q^2$?

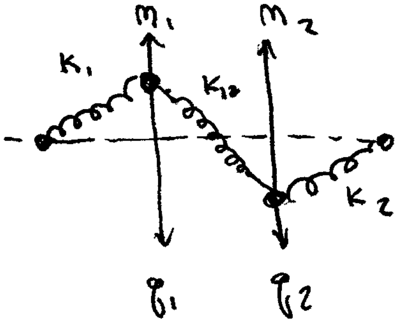


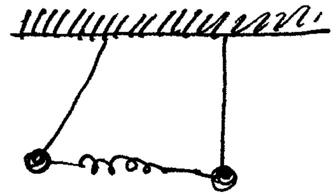
<u>not</u> a minimum!

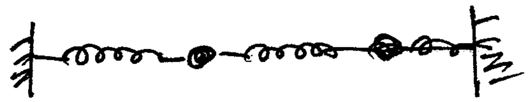
<u>Remark</u>: $-\frac{1}{2} m \omega^2 q_1 q_2$ 1 —— 2

↪ probably want to diagonalize...

EXAMPLE: COUPLED H.O.



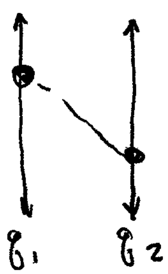




FOR SIMPLICITY: $k_1 = k_2 = 0$ $\quad k_{12} \mapsto k$

$m_1 = m_2$

[compare this to polymer model, ch 4 central force prob]



$V = V(q_1 - q_2)$

$$L = \frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2 - \frac{1}{2} \underbrace{m \omega_0^2}_{k} (q_2 - q_1)^2$$

$\leftarrow$ "coupling"

or: $\ddot{q}_1 + \omega^2 (q_1 - q_2) = 0$

EOM: $\ddot{q}_1 - \omega_0^2 (q_2 - q_1) = 0$

$\ddot{q}_2 - \omega_0^2 (q_1 - q_2) = 0$

ansatz: $q_i(t) = A_i e^{i\omega t}$ $\leftarrow$ $\omega_1 = \omega_2 = \omega$

$$\rightarrow -\omega^2 A_1 - \omega_0^2 (A_2 - A_1) = 0$$

$$-\omega^2 A_2 - \omega_0^2 (A_1 - A_2) = 0$$

$$\Rightarrow \quad (-\omega^2+\omega_0^2)A_1 + \omega_0^2 A_2 = 0$$
$$(-\omega^2+\omega_0^2)A_2 + \omega_0^2 A_1 = 0$$

$$M\begin{pmatrix}A_1\\A_2\end{pmatrix} = 0$$

$$\Rightarrow \det M = 0 \qquad \text{(don't care about } \vec{A}=0 \text{ sol.)}$$

$$= \begin{vmatrix} -(\omega^2-\omega_0^2) & \omega_0^2 \\ \omega_0^2 & -(\omega^2-\omega_0^2) \end{vmatrix} = (\omega^2-\omega_0^2)^2 - (\omega_0^2)^2$$

$$\cancel{= \omega^4 + 2\omega^2\omega_0^2}$$

signs!

$$\omega^2 - \omega_0^2 = \pm\omega_0^2$$

$$\boxed{\omega^2 = \omega_0^2 \pm \omega_0^2} \quad 0,\ 2\omega_0^2$$

<u>interpretation</u>

2 <u>EIGENMODES</u> (<u>normal modes</u>)

ch 4 INTUITION

$$q_{swing} = \tfrac{1}{2}(q_1+q_2) \leftarrow \text{CENTER of MASS}$$
$$q_{STRETCH} = \tfrac{1}{2}(q_1-q_2) \leftarrow \text{RELATIVE MOTION}$$

could have seen this in the L

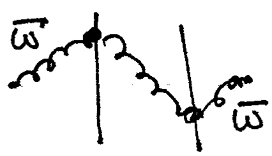
$$L = \tfrac{1}{2}(2m)(\dot{q}_{swing}^2 + \dot{q}_{STRETCH}^2) - \tfrac{1}{2}m\omega_0^2 q_{STRETCH}^2$$

total mass

not a func of q_{SWING} (FREE MOTION)

Q: what if we tied it down?

remark: more generally



$$\Rightarrow \quad \omega^2 = \bar{\omega}^2 + \omega_0^2 \pm \omega_0^2$$

same EIGENMODES.

WHAT ABOUT 3 COUPLED OSCILLATORS?

$$L = \sum_i^3 \frac{1}{2} m \dot{q}_i^2 - \frac{m}{2}\omega_{12}^2 (q_1 - q_2)^2 - \frac{m}{2}\omega_{23}^2 (q_2 - q_3)^2$$

for now: no $\frac{m}{2}\omega_{13}$ term
$\longrightarrow$ what would it mean?

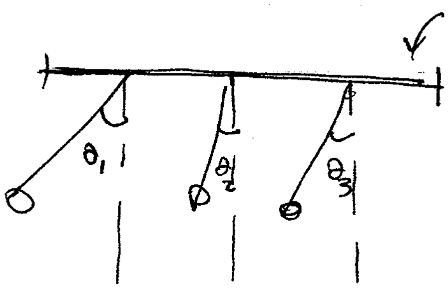
$$\frac{2}{m} V(\vec{q}) = +\omega_{12}^2 \left(q_1^2 - 2q_1 q_2 + q_2^2\right) + \omega_{23}^2 \left(q_2^2 - 2q_1 q_2 + q_3^2\right)$$

then: let $q_i = A_i e^{i\omega t}$; write: $\omega_{12}^2 = \omega_{23}^2 = \varepsilon$

$$\frac{2}{m} L = \vec{q}^{\,T} \begin{pmatrix} -\omega^2 & & \\ & -\omega^2 & \\ & & -\omega^2 \end{pmatrix} \vec{q} - \varepsilon \vec{q}^{\,T} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \vec{q}$$

You can also add "individual" potentials,
eg. $\Delta V = \Sigma \frac{1}{2}\omega_0^2 q_i^2$

eg.



WOBBLY ROD $\rightarrow$ ε COUPLING

BUT ANYWAY, YOU CAN SOLVE THIS.

YOU CAN SEE WHERE THIS IS GOING

$$L = \sum_i \left\{ \frac{1}{2} m \dot{q}_i^2 - \frac{1}{2} m \omega_0^2 (q_i - q_{i+1})^2 \right\}$$

EOM: q_j^{th}: $\ddot{q}_j - \omega_0^2 (q_j - q_{j+1}) + \omega^2 (q_{j-1} - q_j) = 0$

note signs!

$$\ddot{q}_j - \omega_0^2 (q_{j-1} - 2q_j + q_{j+1}) = 0$$

~~this gives a big matrix~~

ANSATZ: $q_j = A_j e^{i\omega t}$

EOM: $(-\omega^2 + 2\omega_0^2) q_j - \omega_0^2 q_{j-1} - \omega_0^2 q_{j+1}$

$$\Rightarrow \vec{q}^{\,T} \underline{M} \vec{q} = 0$$

solve: $\det \underline{M} = 0$ for ω

↳ TRICK: ANSATZ: $A_j = A e^{i(j\gamma + \delta)}$

$$(2\omega_0^2 - \omega^2) A e^{i(j\gamma+\delta)} - \omega_0^2 A e^{i((j-1)\gamma+\delta)} - \omega_0^2 A e^{i((j+1)\gamma+\delta)}$$

$$= (2\omega_0^2 - \omega^2) - \omega_0^2 e^{-i\gamma} - \omega_0^2 e^{+i\gamma} = 0$$

⇒

Section 4, cont'd.

$$\omega^2 = 2\omega_0^2 - \omega_0^2 \underbrace{(e^{i\gamma} + e^{-i\gamma})}_{2\cos\gamma}$$

$$= 2\omega_0^2 (1 - \cos\gamma)$$
$$= 4\omega_0^2 \sin^2\gamma$$

EXPECT n SOLUTIONS TO $\det M = 0$, expect n VALUES OF γ

NOW ASSUME ENDPOINTS OF STRING ARE FIXED:

$$A_0 = A_{n+1} = 0$$

$e^{i((n+1)\gamma - \delta)}$, but only Re PART MATTERS

$\Rightarrow \cos[(n+1)\gamma - \delta] \rightarrow$ need $\delta = \pi/2$ ($\times$ odd $\mathbb{Z}$)

$= \sin(\frac{s}{?}\gamma)$

$(n+1)$ $(n+1)\gamma = s\pi$ $s \in 1, 2, \ldots$

$$\Rightarrow A_j = A_{(s)} \sin\left(j \frac{s}{n+1}\right)$$

s LABELS $(n+1)$ SOLUTIONS for EIGENVALUES.

So what? we have a wave in the "j" direction

↳ "EMERGENT SPATIAL DIRECTION"

(cf "deconstruction"

WHAT'S GOING ON? LARGE n LIMIT

$$L = \sum \tfrac{1}{2} m \underline{\underline{\dot{q}(t,x)}} - \tfrac{1}{2} m \omega_0^2 \underbrace{\left(q(t,x) - q(t,x+\Delta x)\right)^2}$$

$$- \tfrac{1}{2} m \omega_0^2 \left(\frac{\Delta q}{\Delta x}\right)^2$$

$$\rightarrow \mathcal{L} = \tfrac{1}{2} m \dot{q}^2 - \tfrac{1}{2} m \omega_0^2 \left(\frac{dq}{dx}\right)^2$$

$$\boxed{S = \int dt\, dx\, \mathcal{L}}$$

we'll get back to EOM LATER
BUT YOU ALREADY KNOW THEM!
cf E&M PROBLEM IN HW #2
→ 2 DEP. VARS.

relevant aside

what is the L ← one particle
for a relativistic particle (free)

think: should be LORENTZ INVARIANT
what lorentz invariants are there?
τ "kinetic"

$\Rightarrow$ PROPER TIME $\quad \tau = \frac{1}{c^2} x^\mu x_\mu$

PROPOSE $\quad S \propto \int d\tau$

$$\propto \sqrt{\Delta t^2 - \frac{\Delta x^2}{c^2}}$$

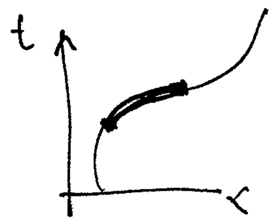
$$\propto \Delta t \sqrt{1 - \frac{v^2}{c^2}}$$ ← SMALL

$$\to (\text{const}) \left(1 - \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{4}\frac{v^4}{c^3} + \cdots\right)$$

CONST. IGNORE

if const $= -mc^2$

SO FOR 1 (RELATIVISTIC) DEPENDENT PARAMETER, 1 PARTICLE
the ACTION IS THE PROPER TIME.



VOLUME of WORLD LINE

for free field? ~ world sheet

EOM for A FIELD (4D MINKOWSKI)

$$S = \int d^4x \, \mathcal{L}$$

for FIELD, INCLUDE "SELF-INTERACTIONS"

$$\mathcal{L} \sim \left(\frac{dq}{dt}\right)^2 - \left(\frac{dq}{dx}\right)^2 \sim (\partial q)^\mu (\partial q)_\mu$$

$$\frac{\partial}{\partial x^\mu} \frac{\partial L}{\partial(\partial_\mu q)} - \frac{\partial L}{\partial q} = 0$$

interactions

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial(\partial_t q)} - \frac{\partial}{\partial x} \frac{\partial L}{\partial(\partial_x q)} - \cdots$$

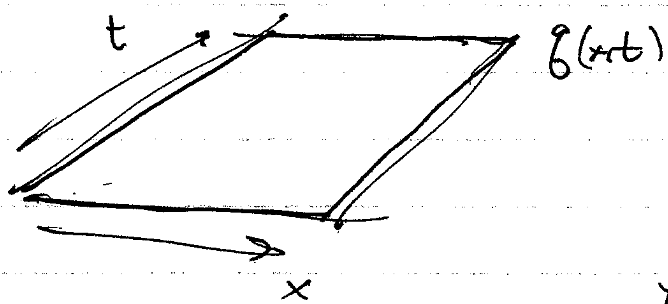
$$\partial_t \partial_t q - \partial_x \partial_x q \cdots$$

note that q POSITION IS NOT RELATED TO SPACE, BUT FIELD DISPLACEMENT

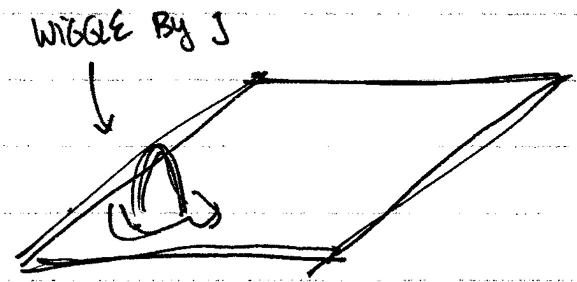
cf SCALAR POT.

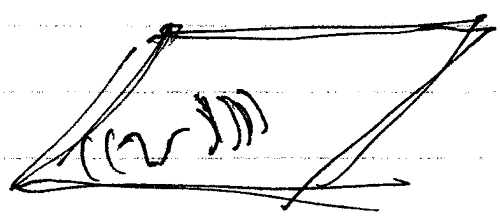
$\Rightarrow \Box q = 0$ for "free" field $\Rightarrow$ WAVE IN SPACETIME

REMARK: SOURCES: $J(t,\vec{x})$



$\mathcal{L} \supset J(t,\vec{x})\, q(x,t)$





WIGGLES PROPAGATE
THROUGH THE FIELD
IN SPACE (& AND IN TIME)

EXAMPLES IN NATURE?

cf Dylan's Q in 3327: L for em?

YOU SAW PART OF THIS IN HW: PARTICLE COUPLED TO $\vec{A}$

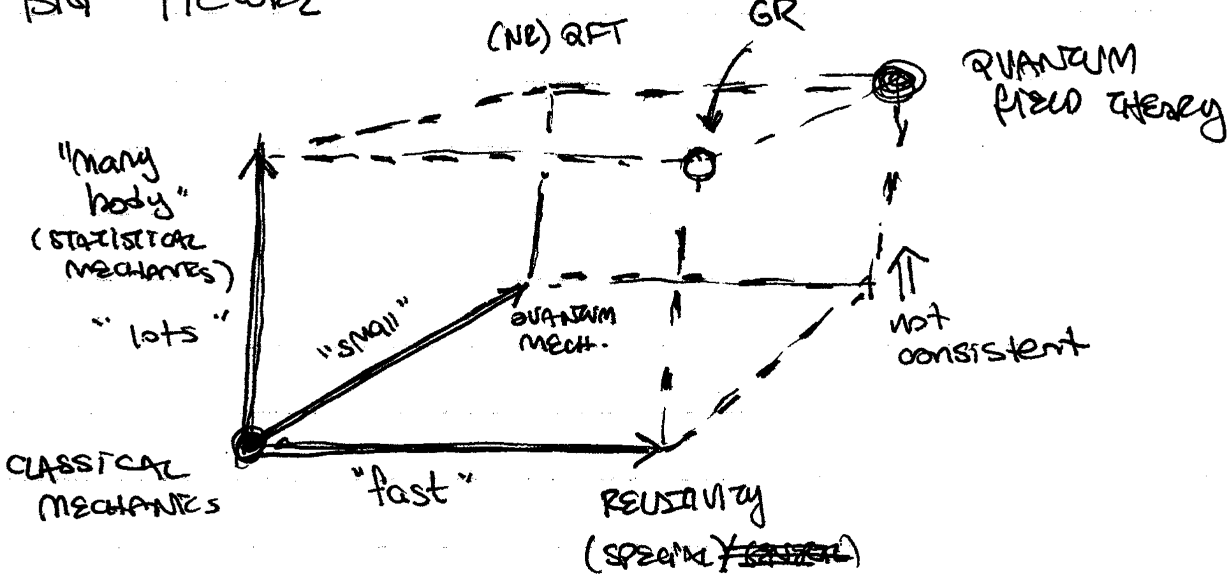
BUT WHAT ABOUT DYNAMICS of FIELDS?

eg. SCALAR POTENTIAL.

Does $(\partial\Phi)^2 = \mathcal{L}$ work?

TURNS OUT NO! $\Phi \in A_\mu$.

BIG PICTURE


(NR) QFT
GR
QUANTUM
field theory
"many
body"
(STATISTICAL
MECHANICS)
" lots "
"small"
QUANTUM
MECH.
not
consistent
CLASSICAL
MECHANICS
"fast"
RELATIVITY
(SPECIAL ~~GENERAL~~)

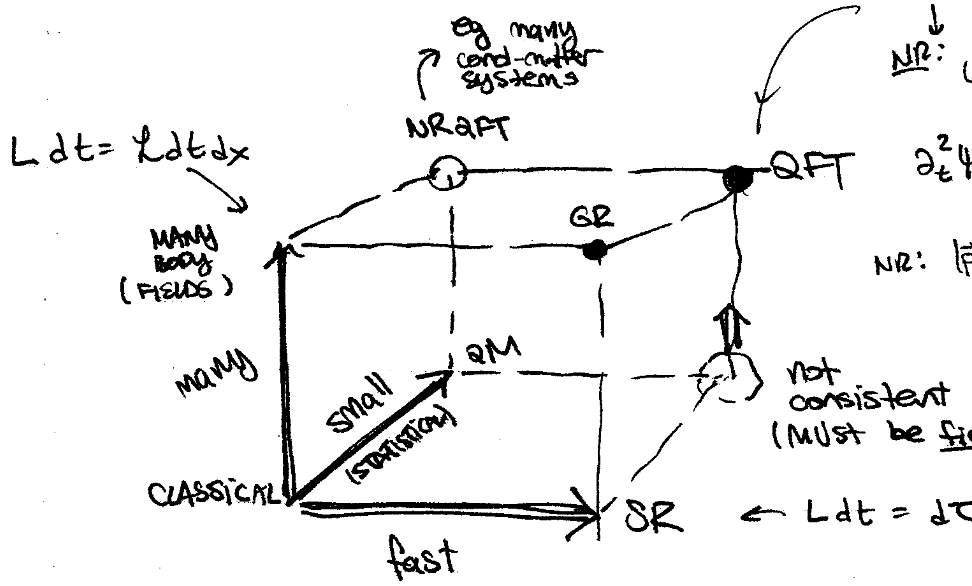
Recap

$$\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m\phi^2 + V(\phi)$$

<u>NR</u>: $\psi = e^{-imt}\tilde{\psi}$

$$\partial_t^2\psi - \nabla^2\psi + m^2\psi = e^{-imt}\left(\ddot{\tilde{\psi}} - 2im\dot{\tilde{\psi}} - \nabla^2\tilde{\psi}\right)$$

NR: $|\vec{p}| \ll m \rightarrow |\ddot{\tilde{\psi}}| \ll m|\dot{\tilde{\psi}}|$



1. EVERYTHING IS (S)HO
2. "DRIVING FORCES" → "SOURCES", represent couplings of diff HO. & most important: it's SOLVABLE EXACTLY (most quadratic things are)
   → APPEAR AS INHOMOGENEOUS TERMS
3. CLASSICAL → ~~SPECIAL~~ MANY : "EMERGENT DIMENSION"

   eg many COUPLED OSCILLATORS

$$L = \sum \tfrac{1}{2}m\dot{q}_i^2 - \underbrace{\tfrac{1}{2}m\omega_0^2(q_i - q_{i+1})^2}$$

begins to look like $\left(\frac{\Delta q}{\Delta x}\right)^2$
if $q_i \to q(t,x)$

END UP W/ $S = \int dt\, dx\, \mathcal{L}$

4. RELATIVITY (CLASSICAL) : L should be LORENTZ INVARIANT.

remark: statistical uncertainty $\leftrightarrow$ quantum uncertainty

Lecture addendum

Feynman diagrams

- SHO (quadratic) solvable
- Higher order terms are expansions in P.T.

↳ act like sources (mathematically & intuitively)

eg $\frac{1}{2}\left((\partial\phi)^2 - m\phi^2\right) + \mathcal{O}(\phi^3)$ ← assume small

$$\frac{\partial}{\partial x^\mu}\frac{\partial L}{\partial(\partial_\mu\phi)} - \frac{\partial L}{\partial\phi} = \underbrace{\partial^2\phi + m\phi^2}_{\text{wave eq.}}$$

(Klein Gordon eq.)

ϕ ——— ϕ

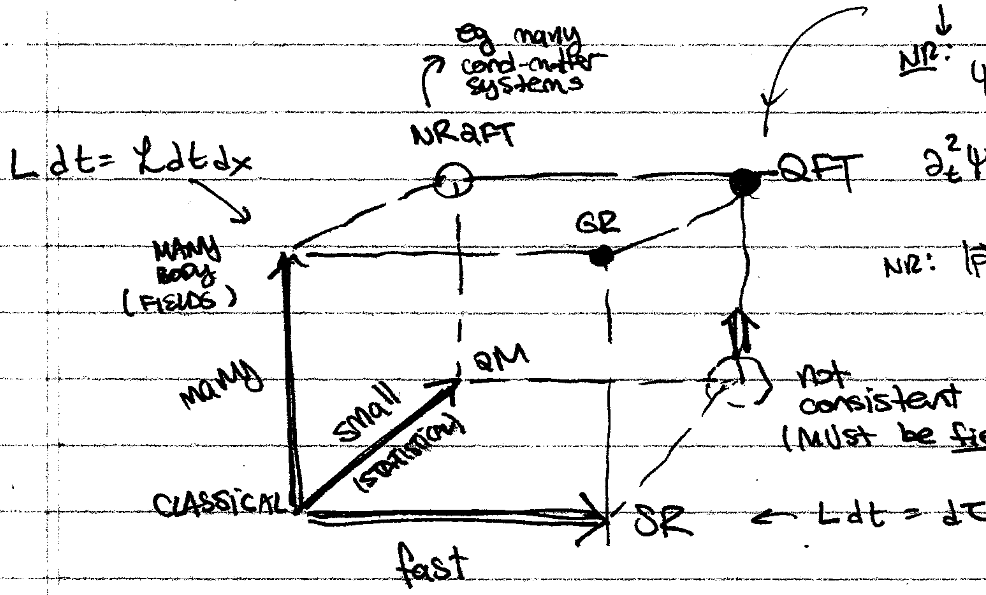
Recap

$$\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m\phi^2 + V(\phi)$$

NR: $\psi = e^{-imt}\tilde{\psi}$

$$\partial_t^2\psi - \nabla^2\psi + m^2\psi = e^{-imt}\left(\ddot{\tilde{\psi}} - 2im\dot{\tilde{\psi}} - \nabla^2\tilde{\psi}\right)$$

NR: $|\vec{p}| \ll M \rightarrow |\ddot{\tilde{\psi}}| \ll M|\dot{\tilde{\psi}}|$



& most important: it's SOLVABLE EXACTLY (most quadratic things are)

1. EVERYTHING IS (S)HO
2. "DRIVING FORCES" → "SOURCES", represent couplings of DIFF HO.
   → APPEAR AS INHOMOGENEOUS TERMS
3. CLASSICAL → ~~SPECTRAL~~ MANY : "EMERGENT DIMENSION"

   eg many coupled oscillators

$$L = \sum \tfrac{1}{2}m\dot{q}_i^2 - \underbrace{\tfrac{1}{2}m\omega_0^2(q_i - q_{i+1})^2}$$

begins to look like $\left(\frac{\Delta q}{\Delta x}\right)^2$
if $q_i \rightarrow q(t,x)$

END UP W/ $S = \int dt\, dx\, \mathcal{L}$

4. CLASSICAL RELATIVITY : L should be LORENTZ INVARIANT.

remark: statistical uncertainty $\leftrightarrow$ quantum uncertainty