REMARKS ON SECTION 2 => DISCUSSION

Main goal is for us to talk to each other about material.

No kind of small contribution to your course grade.

Read material ahead of time, bring your questions.

Rule: It's Fri afternoon, you all have busy lives.

I won't be personally offended if you fall asleep.

**But**: I reserve the right to take a picture of you if I post it on my Facebook wall.

A few remarks about vectors / tensors.

In Lec 1 I made a big deal about indices.

Indices are kind of a crutch that physicists use.

That mathematicians kind of look down upon.

The actual things we care about are usually scalars.

Eq: Stress tensor: How much \( P_2 \) emitted in \( \hat{e}_x \) dir?

\[
T_{2x} = \hat{e}_2 \cdot T \cdot \hat{e}_x
\]

So more generally: How much momentum in dir \( \hat{u}_1 \) is emitted in the \( \hat{u}_2 \) dir?

\[
= \hat{u}_1 \cdot T \cdot \hat{u}_2
\]

Now let's be more grown up w/ indices.

Convention choice: Column vector has **upper** index:

\( i \)

\( \hat{u}_i \)

Row vector has **lower** index:

\( \hat{v}^i \)

Dot products:

\[
(\hat{u}^T \cdot \hat{v}) = \sum \hat{u}_i \hat{v}_i
\]

Allowed contractions of indices:

Only go from top to bot.

So: \( \hat{u}_i \hat{v}_i \) is not allowed as a scalar!

(btw: \( \hat{u}_i \hat{v}_j \) is a valid tensor)
**Rotations**

\[ v^i \rightarrow v'^i = R^i_{\ j} v^j \]
\[ w^i \rightarrow w'^i = w^i R^j_{\ i} = R^j_{\ i} w^j \]

⇒ Upper & Lower Indices Transform Differently!

⇒ This is precisely what’s required to keep inner product invariant

\[ \text{dual: } R^i_{\ j} R^j_{\ k} \ldots \text{why?} \]

**Dot/Inner Products**

⇒ "The Metric"

Start w/ Vectors: \( w^i, v^j \). How do we form a scalar?

Need to lower the index somehow

⇒ go from vector → dual vector

BM: \( \text{bra} \rightarrow \text{ket} \)
GR: \( \text{vector} \rightarrow \text{one-form} \)

This is done w/ the metric tensor: \( g^{ij} \)

\[ v \cdot w = (v^T w) = g^{ij} v^j w^i \]
\[ v_i \]

**Note:** Metric does not transform (despite having indices)
(by definition)

\[ \rightarrow R^k_{\ i} R^j_{\ k} g^{kl} = g^{ij} , \text{ also: } g^{ij} = g^{ji} \]

\[ v \cdot w = g^{ij} v^j w^i \rightarrow g^{ij} (R^i_{\ k} v^k) (R^j_{\ l} w^l) = v^k R^i_{\ k} g^{ij} R^j_{\ l} w^l \]
\[ = v^k R^i_{\ k} g^{ij} R^j_{\ l} w^l \]
\[ = R^k_{\ i} v^k R^i_{\ l} w^l \]
\[ = R^k_{\ i} R^i_{\ l} w^l \]

⇒ Also inverse metric: \( g^{ij} = g^{ji} \) s.t. \( g^{ij} g_{jk} = g^{ik} \)
In 3D Euclidean space w/ Cartesian coordinates: $g_{ij} = δ_{ij}$

Why is it called a metric? It measures distance. An alternate way of writing it is:

$$dl^2 = dl_i dl_i = g_{ij} dx^i dx^j = dx^2 + dq^2 + dz^2$$

But we know this is different for different coordinates!

$$dl^2 = dr^2 + r^2 dθ^2 + r^2 \sin^2 θ dφ^2$$

$$= dp^2 + p^2 dp^2 + dz^2$$

So, e.g., in spherical coordinates:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 θ \end{pmatrix}$$

$

\text{depends on position!}$
GRADIENT: WHAT KIND OF INDEX?

\[ \nabla \sim \frac{\partial}{\partial x_i} \Rightarrow \nabla_i, \quad \text{this kind of heuristic logic drives mathematicians crazy} \]
\[ \nabla \times \text{not just x} \]

BUT USUALLY WHEN WE TAKE GRADIENT, WE WANT A COLUMN VECTOR

so:
\[ (\nabla f)_i = g_{ij} \nabla_j f \]

This is where all the weird coefficients come from!

\[ \nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \phi} \frac{\partial f}{\partial \phi} \hat{\phi} \]

THE OTHER DERIVATIVES ARE MORE COMPLICATED BECAUSE THEY ARE RELATED TO DIFFERENTIAL FORMS.

the full expression for div:

\[ \nabla \cdot \mathbf{V} = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left( \sqrt{\det g} \mathbf{V}^i \right) \]

WHY THESE DETERMINANTS? VOLUME FORM.

THESE COME FROM JACOBIANS OF THE VOLUME ELEMENT.

YOU DON'T NEED TO UNDERSTAND THE DERIVATION OF THIS

BUT DO APPRECIATE THE GEOMETRIC FOUNDATION OF EVERYTHING WE'RE DOING!

by the way: \[ df = j \quad d\omega F = 0 \]
Ok. Now we move on: Multiples.

Recall Electric Dipole (Griffiths 8.4)

![Diagram of electric dipole](image)

**Intuition:** Far away, looks like no net charge but there's a charge "moment" (cf. "moment" of inertia, etc...)

- Point at \( \Phi_{\text{point}} \approx \frac{1}{r} \)
- Dipole \( \Phi_{\text{dipole}} \approx \cos \theta / r^2 \)

\[
\frac{1}{r} \quad \text{because } \quad \frac{1}{r} \rightarrow \text{net source} \\
\text{(b/c } \frac{1}{r^2} \approx \epsilon) \\
\text{so must be weaker than this}
\]

Angular distribution:

- \( \Phi @ \theta = \frac{1}{2} \) still must see
- \( \Phi \) because \( \theta \approx \frac{1}{2} \) must cancel.
- Still \( E = \frac{\vec{d}}{r^2} \), of course!

**Physicist:** We make the relevant Taylor expansions!

I think the dipole is usually taught w/ eg, law of cosines & then Taylor expansion later

**What is expandable?** (Small): \( \frac{d}{r} \)

So we're thinking of \( r \ll \) limit.
I'm not going to do the thing → see textbook

\[ \Phi = \frac{2e ( \frac{1}{2} ) \cos \theta}{r^2} \equiv \frac{\Phi \cdot \Phi_r}{r^2} \]

\( \Phi \) dipole moment

But more generally (see Heald + Marion 2.3)

\[ \Phi_a = \Phi^{(0)} (\text{assuming \ a \ e \ origin}) \uparrow \text{ corrections} \]

\[ = \Phi^{(0)} \]

contrib from all charge in the config

\[ R = \begin{vmatrix} \mathbf{r} \end{vmatrix} - \begin{vmatrix} \mathbf{r}_a' \end{vmatrix} \]

\( R \) is the distance for the \( \Phi \sim \frac{1}{r} \) law, but \( a \) is not at the origin \( \Rightarrow \ R \neq r \).

\[ \text{expansion point} \]

\[ \Phi_a(\mathbf{r}) = \Phi^{(0)}(\mathbf{r}) + \begin{vmatrix} \mathbf{r} \end{vmatrix} \cdot \nabla' \Phi^{(0)} + \frac{1}{2} \begin{vmatrix} \mathbf{r} \end{vmatrix} (\nabla' \Phi^{(0)})^2 \nabla_i \nabla_j \Phi^{(0)} + \ldots \]

\[ \text{switch to index notation} \]

\[ \text{otherwise hard to write} \]

\[ \text{check: this is precisely} \]

\[ \text{"half" the dipole term in the simple system above!} \]

\[ \Phi = \sum a \Phi_a \]

\[ \Rightarrow \text{becomes integral for continuous dist.} \]
We'll talk about this more next week, after you've had more lectures (I have more questions) & a new HW. ;)

Note that we're taking derivatives of what is basically

$$\dot{\chi} = \frac{1}{1 - \chi^2}$$

There is a very useful expansion of this function in terms of Legendre polynomials:

$$\frac{1}{\sqrt{1 - \chi^2}} = \frac{1}{\pi \chi^0} \left( \frac{\pi}{\chi} \right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(\pi \chi)^n}{n!} P_n(\cos \theta)$$

Each term is a "nice" function rather than $1/\sqrt{\chi}$.

As you know from quantum, it is related to angular momentum stuff.

$$P_0 = 1$$
$$P_1 = \cos \theta$$
$$P_2 = \frac{1}{2} (3 \cos^2 \theta - 1) = \frac{1}{2} \left( 3 \left( \vec{\hat{r}} \cdot \vec{\hat{r}}' \right)^2 - 1 \right)$$

So often it's nice to write higher pole terms with Legendre poly.

$$= \frac{1}{2} \left( 3 \vec{r}_i \cdot \vec{r}_j - 8 \delta_{ij} \right) \vec{r}_i \cdot \vec{r}_j$$

Traceless matrix ("irreducible")

What are these $P$s? They are just a basis for the angular parts of SO functions.

Well, more appropriately, the $\gamma$'s are these.

This is a vector space (as you learned in quantum?) in its own metric, sense or dual vectors, etc.!