ANNOUNCEMENTS

- PRELUDE #1 NEXT WK
  - SHORT PROJ #5
  - MON OH: ASK QUESTIONS ABOUT THE COURSE
- HOMEWORKS

REVIEW

1. LAPLACE EQ IN \{RECT \setminus ORIG\} COORDINATES:

2. SEPARATION ANSatz \( \Phi(x_1,x_2,x_3) = \prod_{i=1}^3 x_i(x_i) \)

3. SEPARATE (AS MUCH AS POSSIBLE) LAPLACE EQ., WRITE OUT CONSTANTS (Frequencies!)

\[ \frac{1}{x} \frac{d}{dx} x \frac{d}{dx} + \frac{1}{y} \frac{d}{dy} y \frac{d}{dy} + \frac{1}{z} \frac{d}{dz} z \frac{d}{dz} = 0 \]

Sqrt is CONSTANT.
IN FACT, \( x^2 + y^2 + z^2 = 0 \).

\[ \frac{1}{r^2 R} \frac{d}{dr} (r^2 R') + P \frac{1}{r^2 \sin \theta} \frac{d}{d \theta} (\sin \theta P') + \frac{1}{r^2 \sin^2 \theta} Q'' = 0 \]

RESOLVE \( Q \)

\[ \begin{bmatrix} r & \theta \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q'' \end{bmatrix} \]

SAME TYPE OF EQ. AS ELLIPTIC PROFILE CASE \( \rightarrow \) ELLiptic

SOLUTION: \( \left( A m \Gamma^l + B m \Gamma^h \right) \Gamma^M \)

\( \rightarrow \psi P_e(\psi) \) FOR \( M = \infty \)
4. Use BC to solve for:

- **Coefficients - easy ones**
  
  \[ \text{eq: } \text{no divergence } \phi \mid r = 0 \text{ or } \infty \]
  
  \[ \text{eg: } \phi = 0 \text{ at surface } \rightarrow \text{full } \cos \theta \text{ or cos term} \]

- **Frequencies**
  
  \[ \text{eq once one BC fixes the other, parallel BC can only fix } \theta \text{ fixed.} \]
  
  or matching to body w/ fixed \( \theta = \psi \text{ expansion}. \]

- **Coefficients - hard ones using Fourier's trick**

5. If needed, solve for \( f \) using discontinuity in \( \phi \)

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**Cylindrical Coords**


Laplace + Separation:

\[
\frac{\gamma}{r} \frac{\partial}{\partial \theta} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{\gamma} \frac{\partial^2 \phi}{\partial z^2} = \frac{-1}{\delta} \frac{\partial^2 \phi}{\partial \gamma^2}
\]

\[ \uparrow \]

cf. spherical case!

\[ \phi = e^{\pm \gamma \theta} \]

\[ \text{Making an assumption on sign of } \gamma \text{ for solution to HS: } Z = e^{\pm k \gamma} \text{ if } \gamma = 0 \text{, then:} \]

\[ R_n(r) = A_n + B_n \ln r + \left( A_n r^n + B_n \frac{1}{r^n} \right) \]

\[ \text{const.} \]

\[ \text{not } (n \gamma) \text{!} \]

\[ \text{Physical sig? } \]

\[ \text{or line charge?} \]
More general solution in cylindrical coors

\( k \neq 0 \) makes \( \imath \) even more difficult.

**Gen Solution**

\[ \sum_{m,n} \left( A_{mn} J_n(kr) + B_{mn} N_n(kr) \right) e^{\pm in\theta} e^{\pm kmz} \]

\[ A(\cos n\theta) + B\sin(n\theta) \]

\[ A_{m,n} \sinh(kmz) + B_{m,n} \cosh(kmz) \]

**Key Properties.**

**Orthogonality** for Fourier's trick:

\[ \int_{0}^{a} J_{n}(k_{m}r) J_{n}(k_{m}r) \, r \, dr = \frac{a^{2}}{2} J_{n+1}(k_{m}a) \delta_{m,n} \]

Recall cylindrical.

**In this class:** Mostly focus on \( n=0 \)

**In** Dirichlet @ origin: \( B_{m,n} = 0 \) inside cylinders.

**Upshot:** There are only a handful of sufficiently tractable
Kessel function problems @ this level.

1. Example 3.5 in the book
2. Problems 3.36 - 3.38 in book

1st 2 on HW. Last one is probably a bit too different.
Book: 3-38

Find \( \Phi \) inside.

different from class 3-35!

\( \Phi \) must now be periodic, whereas we previously assumed exponential decay.

\( \Rightarrow \) separation constant is negative \( \Rightarrow \) ? means Bessel?

So now maybe you can just read off the \( z \) dependence

\[ Z(z) = A \sin \left( \frac{\pi m}{L} z \right) \quad m \in \mathbb{Z} \]

BC @ \( L \)

no Bessel form

by BC @ \( z = 0 \)

recall: in rectangular system (x,y,z), one Sine&

1 exponential.

(2) relation btw. separation constants

so: before: \( Z(z) \sim e^{kz} \to ik \)

now: \( Z(z) \sim e^{ikz} \)

end up w/ something like \( J_n(kr) \to J_n(ikr) \)

Turns out there's a name for these:

\[ I_n(ikr) = i^{-n} J_n(ikr) \]

\( \sim \text{modified Bessel} \)

\( \sim \text{exponential} \)

\( (I_n, k_n) \)

\( \sim \text{sinusoid} \)

\( (J_n, N_n) \)

\( \Leftrightarrow \text{physical at origin!} \)
WHAT ABOUT ANGULAR DEPENDENCE?

\[ \phi = \sum_{m=0}^{\infty} A_m \sin(k_m z) I_0(k_m r) \]

\[ c_k = \frac{m \pi}{L}, \text{ not zero for } B_m = 0 \]

B/c @ r = a: \[ \phi(r=a, z) = \phi_0 \]

\[ L = \sum_{m=0}^{\infty} A_m I_0(\frac{k_m a}{2}) \sin(k_m z) \]

\[ \Rightarrow B_m \frac{L}{2} \]

**FOURIER'S THEOREM:**

\[ \int_0^L \phi_0 \sin(k_m z) \, dz = \int_0^L \frac{B_m}{m} \sin(k_m z) \, dz \]

\[ \Rightarrow B_m = \frac{4\phi_0}{\pi m} \]

\[ \text{if } m \text{ is odd, otherwise} \]

\[ A_m = \frac{4\phi_0}{\pi m I_0(k_m a)} \]

\[ \phi = \sum_{m=0}^{\infty} \frac{4\phi_0}{\pi m I_0(k_m a)} \sin(k_m z) \]