ANNOUNCEMENTS

- PRELIM #1 NEXT WK → SUGGESTION: DO PRACTICE PROBLEMS eg. GRIFFITHS EXAMPLES
  - ↳ SHORT HW #5
    MON OH: ASK QUESTIONS ABOUT THE COURSE
- HOMEWORKS | POST STATS.

Review

1. LAPLACE EQ IN $\left\{ \begin{matrix} \text{RECT} \\ \text{SPHR} \\ \text{CYL} \end{matrix} \right\}$ COORDINATES

2. SEPARATION ANSATZ $\underbrace{\Phi(x_1, x_2, x_3)} = X_1(x_1)\, X_2(x_2)\, X_3(x_3)$

   x, y, z
   r, θ, φ
   ρ, φ, z

3. SEPARATE (AS MUCH AS POSSIBLE) LAPLACE EQ., WRITE OUT CONSTANTS (frequencies!)

eg. $\frac{1}{X} X'' + \frac{1}{Y} Y'' + \frac{1}{Z} Z'' = 0$

EACH IS CONSTANT.
IN FACT, $\alpha^2 + \beta^2 + \gamma^2 = 0$.

eg. $\frac{1}{r^2 R} \frac{d}{dr}\left(r^2 R'\right) + \frac{1}{P r^2 \sin\theta} \frac{d}{d\theta}\left(\sin\theta\, P'\right) + \frac{1}{Q r^2 \sin^2\theta} Q'' = 0$

↓ reshuffle

$\left[r \,;\, \theta \right] = \boxed{-\frac{1}{Q} Q''}$

↑ SAME TYPE OF EQ. AS CARTESIAN CASE $\longrightarrow e^{im\varphi}$

SOLUTION: $\left(A_{\ell m} r^{\ell} + B_{\ell m} \frac{1}{r^{\ell+1}}\right) Y_{\ell}^{m}$

↳ $\propto P_\ell(\cos\theta)$ for $m = 0$

4. USE BC TO SOLVE FOR:

① COEFFICIENTS – EASY ONES

↳ eg. NO DIVERGENCE @ $r=0/\infty$
eg. $\Phi=0$ @ SURFACE → KILL cosh or cos term

② FREQUENCIES

↳ eg ONCE ONE BC FIXES rel coef,
PARALLEL BC CAN ONLY FIX FREQ.

or MATCHING TO BNDY w/ FIXED P_ℓ or Y_ℓ^m EXPANSION.

③ COEFFICIENTS – HARD ONES USING FOURIER'S TRICK

5. IF NEEDED, SOLVE FOR ρ USING DISCONTINUITY IN Φ'

CYLINDRICAL COORDS

LAPLACE + SEPARATION:

$$\underbrace{\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)}_{\text{cf. SPHERICAL CASE!}} + \frac{r^2}{Z}\frac{d^2Z}{dz^2} = \underbrace{\frac{-1}{Q}\frac{d^2Q}{d\theta^2}}$$

$Q \sim e^{\pm in\theta}$
[making an assump on sign of n^2]

SOLUTION TO LHS: $Z \sim e^{\pm kz}$ <u>IF $k=0$,</u> then:

$$R_n(r) = \underbrace{A_0}_{\text{const.}} + \underbrace{B_0 \ln r}_{\substack{\text{physical sig?} \\ \infty \text{ line charge}}} + \left(A_n r^n + B_n \underbrace{\frac{1}{r^n}}_{\text{not }(n+1)!}\right)$$

MORE GENERAL SOLUTION IN CYLINDRICAL COORDS

↳ $k \neq 0$, MAKES r EQN MORE DIFFICULT

GEN SOLUTION

$$\sum_{m,n} \left[A_{mn} J_n(k_m r) + B_{mn} N_n(k_m r) \right] e^{\pm i n \theta} \; e^{\pm k_m z}$$

$A \cos(n\theta) + B \sin(n\theta)$

$A \sinh(k_m z) + B \cosh(k_m z)$

KEY PROPERTIES

ORTHOGONALITY FOR FOURIER'S ~~TRICK~~ TRICK:

$$\int_0^a J_n(k_m r)\, J_n(k_{m'} r)\, \underbrace{r\, dr}_{\text{RECALL: CYL. COORD.}} = \underbrace{\frac{a^2}{2} J_{n+1}^2(k_{m'} a)}_{\text{const.}} \delta_{mm'}$$

IN THIS CLASS: MOSTLY FOCUS ON $n=0$

N DIVERGES @ ORIGIN: $B_{mn} = 0$ INSIDE CYLINDERS.

UPSHOT: THERE ARE ONLY A HANDFUL OF SUFFICIENTLY TRACTABLE BESSEL FUNCTION PROBLEMS @ THIS LEVEL.

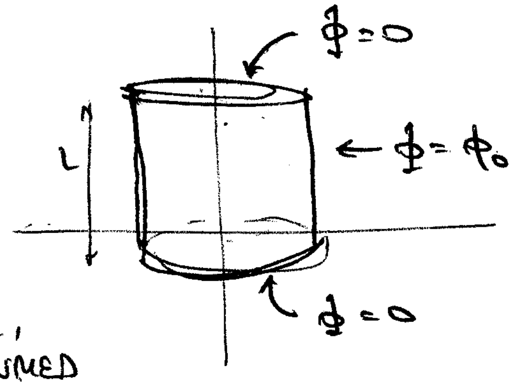
1. EXAMPLE 3.5 IN THE BOOK
2. PROBLEMS 3.36 – 3.38 IN BOOK

1st 2 ON HW

LAST ONE IS PROBABLY A BIT TOO DIFFERENT.

BOOK : 3-38

FIND Φ INSIDE.



DIFFERENT FROM CLASS / ~~eqs~~ 3-5!
↓
CAN YOU SEE WHY?
Z(z) MUST NOW BE PERIODIC, WHEREAS WE PREVIOUSLY ASSUMED EXPONENTIAL DECAY.

↪ SEPARATION CONSTANT IS NEGATIVE → ? MEANS for BESSELS?

BY NOW MAYBE YOU CAN JUST READ off THE Z DEPENDENCE

$$Z(z) = A \sin\left(\frac{m\pi}{L} z\right) \leftarrow m \in \mathbb{Z}_{odd}$$

↑ BC @ L

↑ no cos term by BC @ z = 0

RECALL : IN RECTANGULAR SYSTEM (SAY, 2D), ONE SINUSOID & ONE EXPONENTIAL.

↪ relation btwn separation constants

SO: BEFORE : $Z(z) \sim e^{kz}$

NOW : $Z(z) \sim e^{ikz}$

$k \to ik$

END UP w/ SOMETHING LIKE $J_n(kr) \to J_n(ikr)$

TURNS OUT THERE'S A NAME for THESE:

$$\boxed{I_n(kr) = i^{-n} J_n(ikr)}$$

↑ "Modified Bessel"

~ exponential

(I_n, K_n)

↪ DIVERGE @ ORIGIN!

~ SINUSOID

(J_n, N_n)
↕
Y_n

WHAT ABOUT ANGULAR DEPENDENCE?,

↳ no θ DEPENDENCE $\Leftrightarrow$ $n=0$ in $e^{in\theta}$ term
SAME AS PROJECTING OUT COSINE.

$$\Phi = \sum_{m\ \text{odd}} A_m \sin(k_m z)\, I_0(k_m r)$$

↑ $k_m = \frac{m\pi}{L}$, <u>not</u> zero of BESSEL.

B/C @ $r=a$: $\Phi(r=a,z) = \Phi_0$

$$\hookrightarrow = \sum_{m\ \text{odd}} \underbrace{A_m I_0(k_m a)}_{\equiv B_m} \sin(k_m z)$$

FOURIER'S TRICK:

$$\underbrace{\int_0^L \Phi_0 \sin(k_n z)\, dz}_{\frac{2L}{\pi n}\Phi_0} = \underbrace{\int_0^L \sum_m B_m \sin(k_m z) \sin(k_n z)\, dz}_{B_n \frac{L}{2}}$$

$n \in$ ODD ✓

$$\Rightarrow \boxed{B_m = \frac{4\Phi_0}{\pi m}} \qquad m \in \text{ODD},\ 0 \text{ OTHERWISE}$$

↑ $A_m = \frac{4\Phi_0}{\pi m} \frac{1}{I_0(k_m a)}$

$$\boxed{\Phi = \sum_{m\ \text{odd}} \frac{4\Phi_0}{\pi m} \frac{I_0(k_m r)}{I_0(k_m a)} \sin(k_m z)}$$