Phase Measurements

Suppose a particle is offset at a point in the ring at (x_0, y_0). This displacement will propagate as:

$$x_{ij} = A_i \cos(j2\pi Q_x + \phi_x^i) \tag{1}$$
$$y_{ij} = B_i \cos(j2\pi Q_y + \phi_y^i) \tag{2}$$

where the scripts ij indicate the i^{th} BPM on the j^{th} turn. We assume a large number of turns (~ 1000 or more) are used.

Define C_x^i, S_x^i to be:

$$C_x^i = \sum_{j=1}^{N} x_{ij} \cos(j2\pi Q_x) \tag{3}$$
$$S_x^i = \sum_{j=1}^{N} x_{ij} \sin(j2\pi Q_x) \tag{4}$$

with similar definitions for C_y^i, S_y^i. Further inspect the expression for C_x^i:

$$C_x^i = \sum_{j=1}^{N} A_i \cos(j2\pi Q_x + \phi_x^i)\cos(j2\pi Q_x)$$
$$= A_i \sum_{j=1}^{N} \left(\cos(j2\pi Q_x)\cos\phi_x^i - \sin(j2\pi Q_x)\sin\phi_x^i\right)\cos(j2\pi Q_x)$$
$$= A_i \sum_{j=1}^{N} \left(\cos^2(j2\pi Q_x)\cos\phi_x^i - \sin(j2\pi Q_x)\cos(j2\pi Q_x)\sin\phi_x^i\right)$$
$$= A_i \sum_{j=1}^{N} \left(\frac{1}{2}\left(1+\cos(j4\pi Q_x)\right)\cos\phi_x^i - \frac{1}{2}\sin(j4\pi Q_x)\sin\phi_x^i\right)$$
$$= \frac{1}{2}A_i \left(N\cos(\phi_x^i) + \sum_{j=1}^{N}\left(\cos(j4\pi Q_x)\cos\phi_x^i - \sin(j4\pi Q_x)\sin\phi_x^i\right)\right)$$
$$= \frac{1}{2}A_i N \left(\cos(\phi_x^i) + \frac{1}{N}\sum_{j=1}^{N}\cos(j4\pi Q_x + \phi_x^i)\right)$$

Cosine is constrained to be $\in (-1, 1)$ and we explicitly choose the tunes Q such that we are not on-resonance, therefore as $N \rightarrow$ very large, the remaining sum's contribution to C_x^i goes to zero. Therefore,

$$C_x^i \approx \frac{1}{2}A_i N \cos(\phi_x^i) \tag{5}$$

We attain similar expressions for $S_x^i, C_y^i,$ and S_y^i, where the S terms are negative.

Finding the phase advance is now relatively trivial:

$$\phi_x^i = \tan^{-1}\left(\frac{S_x^i}{C_x^i}\right) \tag{6}$$

$$\phi_y^i = \tan^{-1}\left(\frac{S_y^i}{C_y^i}\right) \tag{7}$$

Note that the arctan values returned by Fortran are constrained to be $\in (-\pi, \pi)$. In order to correctly propagate the phase, we must add factors of 2π appropriately whenever the phase at a BPM "rolls over" from π to $-\pi$.

This only leaves an ambiguity in a constant phase offset from the design values. This can be resolved in the same way CESRv resolves the problem: find the average difference between the "measurement" and the design, and add this to the measured values. This sets the average difference between measured and design to zero.

Coupling Measurements

Although everything in the above section is technically correct, it only holds for the zero-coupling scenario. In the more general case, we have to also include terms arising from coupling:

$$x_{ij} = A_a^i \cos(j2\pi Q_x + \phi_{ax}^i) + A_b^i \cos(j2\pi Q_y + \phi_{bx}^i) \tag{8}$$

$$y_{ij} = B_b^i \cos(j2\pi Q_y + \phi_{by}^i) + B_a^i \cos(j2\pi Q_x + \phi_{ay}^i) \tag{9}$$

We must now introduce new sums ($C_{ax}^i, S_{ax}^i, C_{ay}^i, S_{ay}^i, C_{bx}^i, S_{bx}^i, C_{by}^i, S_{by}^i$) to fully describe the system. These sums are described in the same fashion as in equations (3,4), and the results for $C_{ax}^i, S_{ax}^i, C_{by}^i, S_{by}^i$ are the same as those we originally found. The new cross-terms are what will allow us to find the coupling measurements.

The $\bar{C}$ elements of interest are defined by:

$$\bar{C}_{12} = \frac{B_a}{A_a}\sin(\phi_{ay} - \phi_{ax}) \tag{10}$$

$$\bar{C}_{22} = \frac{B_a}{A_a}\cos(\phi_{ay} - \phi_{ax}) \tag{11}$$

$$\bar{C}_{11} = \frac{A_b}{B_b}\cos(\phi_{bx} - \phi_{by}) \tag{12}$$

We can similarly use the b-mode measurements to define $\bar{C}_{12}$. To obtain the amplitudes A, B from our measurement, consider the sums C_{ax}, S_{ax}:

$$\sqrt{{C_{ax}}^2 + {S_{ax}}^2} = \frac{N}{2}A_a^i \tag{13}$$

We can repeat this method to find all four amplitudes.