Overview of Algorithmic Dynamic Modal Decomposition of Arbitrary Functions

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1 Dynamic Modal Decomposition library

The ModalDecomposition python library makes use of the *modred* python library[1]. ModalDecomposition requires *modred*, *Numpy 1.5+*, *Python 2.7.+*. Documentation is available at `http://www.lepp.cornell.edu/~snh54`

1.1 Primary functionality

1. *decomposeFunction(f, start, stop, delta)*
   Finds the sum of exponentially decreasing modes which add up to create a function f as it appears between *start* and *stop*. The *delta* argument changes the number of points calculated for the function, (0.1 means 10 data points between two integer values) and thus the precision of the modes.

2. *decompose(data, delta)*
   Finds the sum of modes which add to create a function which best matches the provided data points with a *delta* time difference between them.

1.2 Dynamic Modal Decomposition

The library makes use of a mathematical concept called Dynamic Modal Decomposition, or DMD. This process can decompose a dataset created from a function into a set of component modes. DMD differs from other Modal Decomposition methods in that it focuses on which wherein for a small step in time Δt changes the modes exponentially[2]:

$$f(t + \Delta t) \approx e^{-\lambda \Delta t} f(t) \quad (1)$$

1.3 Calculation

The modal decomposition function for f works by making use of the Dynamic Modal Decomposition functionality of the *modred* library. The operations in *modred* take in a matrix made up of vectors representing steps in time. It then returns ritz/eigenvalues and mode norms for each mode. Each mode

[1] Modred documentation: `http://pythonhosted.org/modred/`
[2] Definition: `http://en.wikipedia.org/wiki/Dynamic_mode_decomposition`

can be interpreted from the ritz-values and mode norms returned from the *modred* library. Dynamic modes are represented by the equation:

$$f(t) = Ae^{-\lambda t} \sin \omega t \tag{2}$$

1.3.1 Frequency

The frequency ω can be calculated from the angle θ from the complex ritz-value over the time step between values:

$$\omega = \frac{\theta}{delta} \tag{3}$$

1.3.2 Damping

The preliminary damping coefficient for the mode can be calculated from the magnitude of the complex ritzvalue returned from the *modred* library:

$$\lambda = |z| \tag{4}$$

Errors in this calculation are addressed in section 1.4.1

1.3.3 Amplitude

The amplitude of the modes can be calculated for large damping simply from the mode norm n given by the *modred* library:

$$A = 2\sqrt{n} \tag{5}$$

The errors on this are quite significant for damping values $|\lambda| < 1.5$, see section 1.4.2

1.4 Error Correction

The calculations above provide a good foundation for estimating the values of the various coefficients of the modes, however in different cases small corrections need to be applied in order to further increase accuracy. I did not include these into the previous section because the values found in the error calculations do not relate to any of the given amplitudes, damping coefficients, or frequencies and are found entirely through optimization of the Root Means Square Error (RMSE). All error values calculated here are based upon a comparison between the *raw* calculated values, and a set of given mode coefficient drawn from definition a function f as the sum of several modes.

$$f(t) = A_o e^{-\lambda_o t} sin\omega t + A_1 e^{-\lambda_1 t} sin\omega_1 t + ... \tag{6}$$

The DMD calculation to this point can differentiate between the modes, but cannot easily calculate accurate values for A_i, λ_i, and ω_i under certain circumstances. As such I performed a series of experiments with many values for each variable to plot the changes to its raw calculated value with respect to other variables. This does not provide a solid provable foundation for this work, however in order to operate properly these corrections are necessary. I will further elaborate on the topic in section 1.5, Limitations.

1.4.1 Damping

While the raw calculated damping coefficients are often close to the supplied values in testing, errors occur for large frequencies. After plotting the difference between the calculated values and the supplied testing values a trend emerged, the calculated damping coefficients were off by a constant value for a given *delta* and ω frequency. After plotting the difference in function of ω the best fitting function appeared to be:

$$\Delta\lambda = A^{B\omega + C} + D \tag{7}$$

With:
A = 595.8
B = 0.2234
C = -1.389
D = -0.03057
This function relates to the actual difference with a RMSE of 0.01

1.4.2 Amplitude

In addition to relying on the normals of each mode from the raw amplitude calculations described in section 1.3.3 the error corrections also require a corrected damping coefficient λ in their calculations. We will define the provided mode amplitude as A_o. The difference between A_o and the calculated amplitude $(2\sqrt{n})$ as ΔA. The relation between the relative difference and λ can best be described by:

$$\frac{\Delta A}{A_o} = A\lambda^B + C^{-\lambda+d} + D \tag{8}$$

With:

A = 0.5852
B = -0.5332
C = 0.9640
D = -4.676
d = -37.86

This function relates to the actual $\frac{\Delta A}{A_o}$ with a RMSE of 0.004
The corrected value of the error is more accurate than the raw calculation for values of $\lambda < 1.5$, however then the raw amplitude more accurately represents the actually mode amplitude A_o. A_o is therefore given by:

$$A_o = \begin{cases} \frac{2\sqrt{n}}{1+A\lambda^B+C^{-\lambda+d}+D} & \text{if } 0 < \lambda < 1.5 \\ 2\sqrt{n} & \text{otherwise.} \end{cases} \tag{9}$$

With the same constants A, B, C, D, d as defined above.

1.5 Limitations

The *modred* operations in this library are limited, the DMD process cannot uncover phase shifts in each mode. For example:

$$f(t) = Ae^{-\lambda t}sin(\omega t) \text{ and } f(t) = Ae^{-\lambda t}sin(\omega t + \delta) \tag{10}$$

are indistinguishable to the DMD operation. Taking δ as $\frac{\pi}{2}$; this also means $sin(\omega t)$ and $cos(\omega t)$ are indistinguishable. All the values calculated are subject to some amount of error. The mode frequency, ω least so, requiring no correction yet obtaining results often correct to a thousandth of a percent. The damping coefficients are subject to error dependent on the frequency, and the error on the amplitudes is subject to all other variables. Furthermore the functions found to best describe the errors' trends are based entirely on optimization programs finding the curves of best fit, and as of yet there is no apparent link between the necessary constants (depicted as A through d) and the given values.