Physics 214 Assignment #3
Due Thursday, September 19 before 5:00 pm

Reading for coming lectures: *Young & Freedman*, Sections 20.1—20-4; 19.6—19.9; 20.5.

Skills to be mastered

- be able to convert between wavelength (λ) and wavenumber ($k = 2\pi/\lambda$);
- given the wave speed (v) and any one of the quantities (T, f, ω), be able to find any one of (λ, k), and vice versa;
- be able to distinguish between speed of wave propagation and chunk speed (called particle speed in Y&F);
- understand that the longitudinal component of tension τ_x is constant under certain conditions, and know what those conditions (assumptions) are;
- be able to get τ_y knowing $\frac{\partial y}{\partial x}$;
- understand the part of the derivation where the "chunk" mass is $\mu \Delta x$;
- be able to convert expressions like $\frac{\frac{\partial y}{\partial x}(x+\Delta x,t) - \frac{\partial y}{\partial x}(x,t)}{\Delta x}$ to the proper partial derivatives, $\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}(x,t)\right)$, in the limit $\Delta x \to 0$;
- be able to confirm a given $y(x,t)$ as a solution to the wave equation;

Problems

1. The motion of a particle is described by $x(t) = \Re[(-5+7i)e^{i\omega t}]$ mm where $\omega = 10^5 s^{-1}$.[1]

   (a) What are the position and velocity of the particle at $t = 10^{-5}$ s?

   (b) What is the (real) amplitude of the oscillation?

   (c) Sketch the complex position $\breve{x}(t) = (5+7i)e^{i\omega t}$ mm on the complex plane at $t = 0$ and $t = 10^{-5}$ s.

2. In lecture we derived an expression for the complex amplitude of a driven harmonic oscillator: $\breve{A} = Ae^{i\phi_0}$ with is the *real amplitude* A and the *initial phase* ϕ_0 given by

$$A = \left|\breve{A}\right| = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\alpha\omega)^2}} \qquad \text{and} \qquad \phi_0 = \tan^{-1}\left(\frac{\alpha\omega}{\omega^2 - \omega_0^2}\right) \quad , \tag{1}$$

where $f_0 \equiv F_0/m$ is the amplitude of the driving force in units of mass and $\omega_0 \equiv \sqrt{k/m}$ is the natural frequency of the oscillator.

In order to better appreciate the physical significance of these quantities and their frequency dependence, it is useful to *plot* them versus frequency and to identify certain important parameters of these plots. In this problem you will do this, starting from analyzing $\frac{A}{f_0}(\omega)$ and $\phi_0(\omega)$ in various limits[2]:

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[1] $\Re[\breve{Q}]$ means *the real part of* $\breve{Q}$.

[2] The *normalized* plot $\frac{A}{f_0}(\omega)$ shows how the amplitude of the driven oscillations compares with the amplitude of the driving force.

(a) Evaluate $\frac{A}{f_0}(\omega = 0)$ and $\phi_0(\omega = 0)$.

(b) Evaluate $\lim\limits_{\omega\to\infty} \dfrac{A}{f_0}(\omega)$ and $\lim\limits_{\omega\to\infty} \phi_0(\omega)$.

(c) Show that $\frac{A}{f_0}(\omega)$ has a *maximum* at $\omega = \omega_R = \sqrt{\omega_0^2 - \alpha^2/2}$ if $\alpha^2 < 2\omega_0^2$. Express the value of $\frac{A}{f_0}$ at the maximum in terms of ω_0 and α.

Hint: Show that the denominator of A in eq. (1) has a minimum at $\omega = \omega_R$ by checking that $\left.\dfrac{dD(\omega)}{d\omega}\right|_{\omega=\omega_R} = 0$ for the function $D(\omega) \equiv (\omega_0^2 - \omega^2)^2 + (\alpha\omega)^2$.

(d) In the limit $\alpha \ll \omega_0$, show that $\omega_R \approx \omega_0$ and that $\frac{A}{f_0}(\omega)$ is approximately equal to $\frac{A}{f_0}_{\max}/\sqrt{2}$ at each of the frequencies $\omega_{1,2} = \omega_0 \pm \alpha/2$. Compute the 'frequency band-width'[3] $\Delta\omega \equiv \omega_2 - \omega_1$.

(e) Sketch the plots of the functions $(A/f)(\omega)$ and $\phi_0(\omega)$ in the limit $\alpha \ll \omega_0$. Label all important quantities, such as $\frac{A}{f_0}_{\max}$, $\Delta\omega$, etc., expressed in terms of ω_0 and α.

3. Some cordless phones transmit and receive 2.0 GHz microwaves. Find the wavelength λ and the wavenumber k. Microwaves travel through air at speed $c = 3.0 \times 10^8$ m/s.

4. The propagation of waves along a stretched DNA molecule is described by the modified wave equation

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + a^2\frac{\partial^4 y}{\partial x^4} = 0 \tag{2}$$

where c and a are constants. ($c = \sqrt{\tau/\mu}$, where τ is the tension and μ the linear mass density as for the string considered in class, and a is an elastic parameter independent of tension.)

Show that the standing wave $y(x,t) = A\cos(kx)\Re[e^{i\omega t + i\phi_0}]$ is a solution to the DNA wave equation (2) and derive the dispersion relation (in other words, find ω as a function of k).

5. A standing wave on a string of mass m fixed at both ends ($x = 0$ and $x = L$) is described by

$$y(x,t) = A\sin(kx)\Re[e^{i\omega t + i\phi_0}] = \sin(kx)\Re[\breve{A}e^{i\omega t}] \tag{3}$$

Assume that the amplitude of the wave is small ($kA \ll 1$) and make any justifiable approximations based on that assumption. Consider m, A, k, and ω to be known quantities (i.e. you may leave them in your answers).

(a) What is the x-component of the force due to the string on the fixed point $x = 0$? (Remember, the string is under tension so it pulls on whatever is holding it.)

(b) What is the y-component of the force due to the string on the fixed point $x = 0$?

(c) Consider a tiny *chunk* of string of length dx between x and $x + dx$. Find the x- and y-components of the force on this chunk due to the rest of the string on the left side only.

[3] The frequency band-width is the range of frequencies 'preferred' by the driven/damped oscillator. The number $Q \equiv \frac{1}{\alpha T} \approx \frac{1}{2\pi}\frac{\omega_0}{\Delta\omega}$ is called a 'quality factor' or a 'Q-factor'. It is a measure 'how sharp' the resonance is. The value of Q shows how good a mechanical system is in 'selecting' certain frequencies. Large Q-factors are important for instance in piezo-electric crystals setting the precise timing of quartz watches and computers.

(d) Find the x- and y-components of the force on this chunk due to the rest of the string on the right side only.

(e) Find the net force on the chunk.

(f) Verify that $\vec{F} = m\vec{a}$ works for the chunk. [Find $a(x,t)$ by differentiating $y(x,t)$.]