Physics 214 Assignment #3
Due Thursday, September 19 before 5:00 pm

Reading for coming lectures: Young & Freedman, Sections 20.1—20.4; 19.6—19.9; 20.5.

Skills to be mastered

• be able to convert between wavelength ($\lambda$) and wavenumber ($k = 2\pi/\lambda$);

• given the wave speed ($v$) and any one of the quantities ($T$, $f$, $\omega$), be able to find any one of ($\lambda$, $k$), and vice versa;

• be able to distinguish between speed of wave propagation and chunk speed (called particle speed in Y&F);

• understand that the longitudinal component of tension $\tau_x$ is constant under certain conditions, and know what those conditions (assumptions) are;

• be able to get $\tau_y$ knowing $\frac{\partial y}{\partial x}$;

• understand the part of the derivation where the ”chunk” mass is $\mu \Delta x$;

• be able to convert expressions like $\frac{\partial y}{\partial x}(x + \Delta x, t) - \frac{\partial y}{\partial x}(x, t) \Delta x$ to the proper partial derivatives, $\frac{\partial}{\partial x} \left( \frac{\partial y}{\partial x}(x, t) \right)$, in the limit $\Delta x \to 0$;

• be able to confirm a given $y(x, t)$ as a solution to the wave equation;

Problems

1. The motion of a particle is described by $x(t) = \Re[(−5 + 7i)e^{i\omega t}]$ mm where $\omega = 10^5 s^{-1}$.\(^1\)

   (a) What are the position and velocity of the particle at $t = 10^{-5}$ s?

   (b) What is the (real) amplitude of the oscillation?

   (c) Sketch the complex position $\hat{x}(t) = (5 + 7i)e^{i\omega t}$ mm on the complex plane at $t = 0$ and $t = 10^{-5}$ s.

2. In lecture we derived an expression for the complex amplitude of a driven harmonic oscillator: $A = A e^{i\phi_0}$ with is the real amplitude $A$ and the initial phase $\phi_0$ given by

\[
A = \left| A \right| = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\alpha \omega)^2}} \quad \text{and} \quad \phi_0 = \tan^{-1} \left( \frac{\alpha \omega}{\omega^2 - \omega_0^2} \right), \quad (1)
\]

where $f_0 \equiv F_0/m$ is the amplitude of the driving force in units of mass and $\omega_0 \equiv \sqrt{k/m}$ is the natural frequency of the oscillator.

In order to better appreciate the physical significance of these quantities and their frequency dependence, it is useful to plot them versus frequency and to identify certain important parameters of these plots. In this problem you will do this, starting from analyzing $\frac{A}{f_0}(\omega)$ and $\phi_0(\omega)$ in various limits\(^2\):

\(^1\) $\Re[\bar{Q}]$ means the real part of $\bar{Q}$.

\(^2\) The normalized plot $\frac{A}{f_0}(\omega)$ shows how the amplitude of the driven oscillations compares with the amplitude of the driving force.
(a) Evaluate \( \frac{A}{f_0}(\omega = 0) \) and \( \phi_0(\omega = 0) \).

(b) Evaluate \( \lim_{\omega \to \infty} \frac{A}{f_0}(\omega) \) and \( \lim_{\omega \to \infty} \phi_0(\omega) \).

(c) Show that \( \frac{A}{f_0}(\omega) \) has a maximum at \( \omega = \omega_R = \sqrt{\omega_0^2 - \alpha^2/2} \) if \( \alpha^2 < 2\omega_0^2 \). Express the value of \( \frac{A}{f_0} \) at the maximum in terms of \( \omega_0 \) and \( \alpha \).

Hint: Show that the denominator of \( A \) in eq. (1) has a minimum at \( \omega = \omega_R \) by checking that \( \frac{dD(\omega)}{d\omega} \bigg|_{\omega=\omega_R} = 0 \) for the function \( D(\omega) \equiv (\omega_0^2 - \omega^2)^2 + (\alpha \omega)^2 \).

(d) In the limit \( \alpha \ll \omega_0 \), show that \( \omega_R \approx \omega_0 \) and that \( \frac{A}{f_0}(\omega) \) is approximately equal to \( \frac{A}{f_{0\max}}/\sqrt{2} \) at each of the frequencies \( \omega_{1,2} = \omega_0 \pm \alpha/2 \). Compute the frequency band-width \( \Delta \omega \equiv \omega_2 - \omega_1 \).

(e) Sketch the plots of the functions \( (A/f)(\omega) \) and \( \phi_0(\omega) \) in the limit \( \alpha \ll \omega_0 \). Label all important quantities, such as \( \frac{A}{f_{0\max}} \), \( \Delta \omega \), etc., expressed in terms of \( \omega_0 \) and \( \alpha \).

3. Some cordless phones transmit and receive 2.0 GHz microwaves. Find the wavelength \( \lambda \) and the wavenumber \( k \). Microwaves travel through air at speed \( c = 3.0 \times 10^8 \) m/s.

4. The propagation of waves along a stretched DNA molecule is described by the modified wave equation

\[
\frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + a^2 \frac{\partial^4 y}{\partial x^4} = 0
\]

(2)

where \( c \) and \( a \) are constants. \( (c = \sqrt{\tau/\mu}, \) where \( \tau \) is the tension and \( \mu \) the linear mass density as for the string considered in class, and \( a \) is an elastic parameter independent of tension.\)

Show that the standing wave \( y(x, t) = A \cos(kx) \Re[e^{i\omega t + i\phi_0}] \) is a solution to the DNA wave equation (2) and derive the dispersion relation (in other words, find \( \omega \) as a function of \( k \)).

5. A standing wave on a string of mass \( m \) fixed at both ends \( (x = 0 \) and \( x = L) \) is described by

\[
y(x, t) = A \sin(kx) \Re[e^{i\omega t + i\phi_0}] = \sin(kx) \Re[Ae^{i\omega t}] 
\]

(3)

Assume that the amplitude of the wave is small \( (kA \ll 1) \) and make any justifiable approximations based on that assumption. Consider \( m, A, k, \) and \( \omega \) to be known quantities (i.e. you may leave them in your answers).

(a) What is the \( x \)-component of the force due to the string on the fixed point \( x = 0 \)? (Remember, the string is under tension so it pulls on whatever is holding it.)

(b) What is the \( y \)-component of the force due to the string on the fixed point \( x = 0 \)?

(c) Consider a tiny chunk of string of length \( dx \) between \( x \) and \( x + dx \). Find the \( x \)- and \( y \)-components of the force on this chunk due to the rest of the string on the left side only.

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3The frequency band-width is the range of frequencies ‘preferred’ by the driven/damped oscillator. The number \( Q \equiv \frac{1}{\alpha T} \approx \frac{1}{2\pi} \frac{\Delta \omega}{\omega_0} \) is called a ‘quality factor’ or a ‘Q-factor’. It is a measure ‘how sharp’ the resonance is. The value of \( Q \) shows how good a mechanical system is in ‘selecting’ certain frequencies. Large \( Q \)-factors are important for instance in piezo-electric crystals setting the precise timing of quartz watches and computers.
(d) Find the $x$- and $y$-components of the force on this chunk due to the rest of the string on the right side only.

(e) Find the net force on the chunk.

(f) Verify that $\vec{F} = m\vec{a}$ works for the chunk. [Find $a(x, t)$ by differentiating $y(x, t)$.]