Physics 214 Fall 2002
Assignment #7 (due 10/24)

1. Consider a spring that obeys Hooke's law with spring constant κ, mass m, and relaxed (unstretched) length L_0. The spring is stretched to a length $L > L_0$. We want to derive the wave equation for *longitudinal* waves on the spring and to find the wavespeed. The derivation is quite similar to the derivation of the wave equation for sound waves.

(a) What is the tension τ_0 in the spring when there is no wave?

(b) As for sound waves, the function $s(x)$ gives the displacement of any point on the spring from its equilibrium position x. Suppose that a wave causes a little section of spring between x and $x + \Delta x$, which has equilibrium length Δx, to be stretched to a length $\Delta x + [s(x + \Delta x) - s(x)]$. What is the tension in the little section of spring? [First figure out the effective spring constant for a length Δx of spring. Hint: if you cut a spring with spring constant κ into three equal pieces, the spring constant of each piece is 3κ.]

(c) Use your result from (b) to show that when a longitudinal wave propagates on the spring, the tension at point x is $\tau(x) = \tau_0 + \kappa L\ \partial s/\partial x$. [Hint: let Δx be very small.]

(d) Using the expression from (c), apply Newton's second law to a short section of spring between x and $x + dx$ to derive the wave equation $\partial^2 s/\partial t^2 = c^2\ \partial^2 s/\partial x^2$. What is the wavespeed c in terms of κ, m, L_0, and/or L?

2. The three-dimensional wave equations for EM waves in vacuum are

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \tag{1}$$

$$\nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial x^2} + \frac{\partial^2 \vec{B}}{\partial y^2} + \frac{\partial^2 \vec{B}}{\partial z^2} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2} \tag{2}$$

(a) Show that the expression

$$\vec{E}(x, y, z, t) = -\hat{z}\Re[\breve{E}_0 \cos(\omega t + kx)] \tag{3}$$

is a solution to the wave equation for $\vec{E}$ (Eq. (1)), where $\hat{z}$ is a unit vector in the $+z$ direction.

(b) In what direction does the wave described by Eq. (3) travel?

(c) Apply Faraday's law

$$\nabla \times \vec{E} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4}$$

to the electric field of Eq. (3). What can you conclude about the magnitude and direction of $\vec{B}(x, y, z, t)$?

(d) Write an expression for the magnetic field of this EM wave.

3. In this problem we examine transverse waves on a thin, flat, stretched membrane (such as a drumhead). The membrane lies in the xy-plane; the transverse displacement is described by $z(x, y, t)$. The membrane has surface tension η and mass per unit area σ. The wave equation for the transverse wave is

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\sigma}{\eta} \frac{\partial^2 z}{\partial t^2} \tag{5}$$

(a) Show that $z(x, y, t) = \Re[\breve{A} e^{i(k_x x + k_y y - \omega t)}]$ is a solution to the wave equation. What are the conditions on k_x, k_y, and/or ω?

(b) What is the wavenumber $k = 2\pi/\lambda = \omega/c$ in terms of k_x and k_y?

(c) Suppose the membrane has a rectangular shape with fixed edges along the lines $x = 0$, $y = 0$, $x = a$, and $y = b$. Show that the standing wave solution $z(x, y, t) = \sin(k_x x)\ \sin(k_y y)\ \Re[\breve{A} e^{i\omega t}]$ satisfies the wave equation and the boundary conditions for certain values of k_x and k_y. What are the allowed values of k_x and k_y?

(d) A particular standing wave mode on this rectangular membrane has the 3rd lowest possible value of k_x and the 2nd lowest possible value of k_y. Assuming $a = 1.5b$, sketch the pattern of vibration and indicate the nodal lines (lines of no displacement).

(e) A microwave oven is a rectangular box with metal shielding all around. The metal shielding ensures that $\vec{E} = 0$ at the edges of the box. Electromagnetic standing waves are set up inside this box. Based on analogy with the 2-dimensional rectangular membrane, explain why it is a good idea to have the food on a rotating platform rather than to have it stay in one position.