Physics 214 Fall 2002
Assignment #8 (due 10/31)

Reading related to recent and upcoming lectures: *Young & Freedman* Sections 34-1,2,3,8 and Chapter 35; *Elmore & Heald* Sections 1.9, 2.1, 2.2, 2.4, 5.1, 5.3, 5.6, 5.7, 8.2, 8.3.

1. In this problem we again examine transverse waves on a thin, flat, stretched membrane. The membrane lies in the upper half ($y \geq 0$) of the xy-plane. The membrane has a fixed edge along the line $y = 0$; that is, $z(x, y = 0, t) = 0$. The wave equation for the transverse wave is

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \tag{1}$$

(a) A wave described[1] by $z_1(x, y, t) = Ae^{i(\vec{k}_1 \cdot \vec{r} - \omega t)}$ travels toward the fixed edge, where $\vec{k}_1 = k_x\hat{x} + k_y\hat{y}$ and $\omega = c|\vec{k}| = ck$. Write an expression for $z_2(x, y, t)$, the wave reflected from the fixed edge.

(b) Show that the superposition of the two waves can be written

$$z(x, y, t) = z_1 + z_2 = 2iA \sin k_y y \; e^{i(k_x x - \omega t)} \tag{2}$$

and explain the physical significance of the factor of i.

(c) Where are the nodal lines (lines along which $z = 0$ at all t)?

(d) There are also *moving* lines of zero displacement. At $t = 0$, where are these lines? At what speed and in what direction do they move?

(e) Now we put another fixed edge at $y = a$; the membrane has width a in the y-direction and is very long in the x-direction. We have thus made a channel or waveguide for waves traveling in the x-direction. In order to propagate down the channel without attenuation, the waves must have nodal lines at both fixed edges ($y = 0$ and $y = a$). Suppose the wave has n nodal lines (not counting those at the fixed edges). What is k_y?

(f) Suppose we want only a single mode—the mode with no nodal lines other than the edges—to propagate down the channel. Find the range of values of k for which only this mode propagates. [Hint: k_x must be real in order for the wave to propagate without attenuation.]

(g) Now let's find out what happens if a wave with k smaller than the range found in part (f) is sent down the waveguide. We can think of k_x as imaginary:

$$k_x = i\sqrt{k_y^2 - k^2} = i\kappa \tag{3}$$

where κ is real. Write an equation for the wave $z(x, y, t)$ in terms of the constants κ, a, ω, A, and whatever else you *need*. This is called an *evanescent* wave because it dies out exponentially as a function of x: in a distance $\Delta x = 1/\kappa$, the amplitude of the wave is reduced by a factor $e^{-1} \approx 0.37$.

[1] When we write that a physical quantity such as z, the transverse displacement of the membrane, is equal to a *complex* expression, it is understood that z is *real*ly equal to the *real part* of the expression.

2. In this problem, you will design a converging lens with focal length f. You will find it easier to use cylindrical coordinates (r, Θ, z). The lens will have the following characteristics:

- It will be made of glass with an index of refraction of n.
- It will have rotational symmetry about the line $r = 0$.
- The right hand surface of the glass will be a plane perpendicular to the z-axis.
- The left hand surface will be a convex curve (i.e., the lens will have its maximum thickness at $r = 0$, and the greater the r, the thinner the lens.
- The coordinate system is chosen so that $z = 0$ and $r = 0$ is at the center of the lens. Positive z is towards the left.

Throughout the problem, you may use the small angle approximation $\sin\alpha = \tan\alpha = \alpha$ as well as the thin lens approximation (i.e., the thickness of the lens is always so small compared to the focal length that it may be neglected to simplify the calculations whenever it is helpful.

(a) Put a point source on the z-axis at $z = f$ (i.e., put the source at the focal point of the lens.) Consider the ray that goes along the z-axis towards the lens. What should be the direction of the ray when it exits the lens? How much should this ray be bent by the lens? What does this imply about the slope of the left hand edge of the lens at $r = 0$?

(b) Now consider a ray from the point source that enters the lens at any value of $r > 0$. What should be the direction of the ray when it exits the lens? How much (i.e. through what angle) should this ray be bent by the lens? How much of this bending should be done by the left hand surface? How much by the right hand surface?

(c) You should now be in a position to calculate the slope of the left hand edge of the lens. To satisfy the small angle and the thin lens approximations, the slope (dr/dz) will always turn out to be nearly vertical. Thus, it will be easier to work with the inverse slope (dz/dr). Write down an expression for dz/dr as a function of r.

(d) Now integrate the expression to find an expression for the left hand surface; i.e., $r(z)$ or $z(r)$. This will complete the design of the lens. Describe the shape of your lens in words rather than equations.

(e) The shape of the lens is not spherical. Is it *approximately* spherical near the axis? If so, what is the radius of curvature? Show that the radius is consistent with the "lensmaker's equation" (*Young & Freedman* Eq. (35-19)).

3. *Young & Freedman*, Problem 35-66.