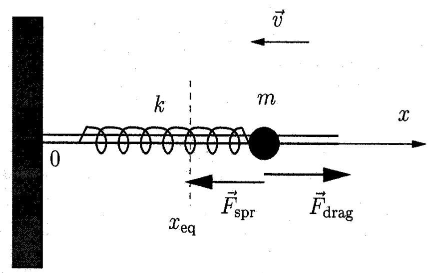

Figure 1: A mass-spring realization of a damped harmonic oscillator.

1 Problem 1: Damped Harmonic Oscillator [28 points]

A damped oscillator is modeled as a mass m with equilibrium position $x = x_{\text{eq}}$ acted on by an ideal spring of spring constant k and a viscous drag force proportional to the velocity, $\vec{F}_{\text{drag}} = -bm\vec{v}$. (See Figure 1.)

(a) Equation of Motion (7 points)

Show that the Equation of Motion for the damped harmonic oscillator is:

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega_0^2(x - x_{\text{eq}}) = 0 \,, \qquad \omega_0 \equiv \sqrt{\frac{k}{m}} \,. \tag{1.1}$$

Writing $\vec{F}_{net} = m\vec{a}$ for the damped harmonic oscillator we get:

$$F_{net,x} = m a_x$$

$$F_{spr,x} + F_{drag,x} = m a_x$$

$$-k(x - x_{eq.}) - bm v_x = m a_x \,|: m$$

$$-\frac{k}{m}(x - x_{eq.}) - b v_x = a_x \,,$$

with $k/m \equiv \omega_0^2$.

Finally, expressing v_x, a_x in terms of the D.of F. $x(t)$, we obtain eq.(1.1):

$$\boxed{\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega_0^2(x - x_{eq.}) = 0}$$

(b) Real Part of $\underline{A}$ (7 points)

In the case $b < 2\omega_0$ the expression

$$x(t) = x_{\text{eq}} + \Re\mathfrak{e}\left[\underline{A}e^{(-b/2+i\omega')t}\right] \ , \ \ \omega' \equiv \sqrt{\omega_0^2 - b^2/4} \tag{1.2}$$

is a general solution to the Equation of Motion (1.1).

Find the real part $\Re\mathfrak{e}\,[\underline{A}]$, given the initial conditions $x_0 = x(t=0)$ and $v_0 = \frac{dx}{dt}(t=0)$. ***Express your answer using no quantities other than*** x_0, v_0, x_{eq}, b, ***and*** ω_0.

Imposing the initial condition $x_0 = x(t=0)$ on the general solution (1.2), we get:

$$x_0 = x(t=0) = x_{eq.} + Re\left[\underline{A}\,\cancelto{1}{e^{(-b/2+i\omega')\cdot 0}}\right] = x_{eq.} + Re[\underline{A}]$$

$$\Longrightarrow \boxed{Re[\underline{A}] = x_0 - x_{eq.}}$$

(c) Imaginary Part of $\underline{A}$ (7 points) *Challenge Problem!*

Using the same initial conditions as in part (b), ***find the imaginary part*** $\Im\mathfrak{m}\,[\underline{A}]$. ***Express your answer using no quantities other than*** x_0, v_0, x_{eq}, b, ***and*** ω_0.

Hint: *Write* $\underline{A} = A_r + iA_i$, *where* $A_r = \Re\mathfrak{e}\,[\underline{A}]$ *and* $A_i = \Im\mathfrak{m}\,[\underline{A}]$, *and solve for* A_i.

Imposing the initial condition $v_0 = \frac{dx}{dt}(t=0)$ on the general solution (1.2) and writing $\underline{A} = A_r + iA_i$:

$$v_0 = \frac{dx}{dt}(t=0) = 0 + \frac{d}{dt} Re\left[\underline{A}\,e^{(-b/2+i\omega')t}\right]_{t=0}$$

$$= Re\left[\underline{A}\left(-\frac{b}{2}+i\omega'\right)\cancelto{1}{e^{(-b/2+i\omega')\cdot 0}}\right] = Re\left[(A_r+iA_i)\left(-\frac{b}{2}+i\omega'\right)\right]$$

$$= Re\left[\left(-\frac{b}{2}A_r - \omega' A_i\right) + i\left(-\frac{b}{2}A_i + \omega' A_r\right)\right] = -\frac{bA_r}{2} - \omega' A_i$$

$$\Longrightarrow A_i = \frac{v_0}{\omega'} - \frac{bA_r}{2\omega'}.$$ Using $A_r = Re[\underline{A}] = x_0 - x_{eq.}$ from (b) and $\omega' \equiv \sqrt{\omega_0^2 - b^2/4}$, we obtain:

$$\boxed{A_i = Im[\underline{A}] = \frac{1}{\sqrt{\omega_0^2 - b^2/4}}\left[v_0 - \frac{b}{2}(x_0 - x_{eq.})\right]}$$

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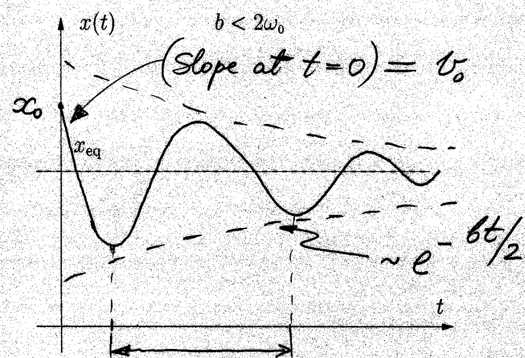
(d) Rough Sketch of the Solution (7 points)

Roughly sketch the solution (1.2) ***assuming*** $x_0 > x_{eq}$ ***and*** $v_0 < 0$. ***(*****Pay attention** ***to the slope and the value of the plot at*** $t = 0$.***)***



$T = \frac{2\pi}{\omega'}$; $\omega' = \sqrt{\omega_0^2 - b^2/4}$

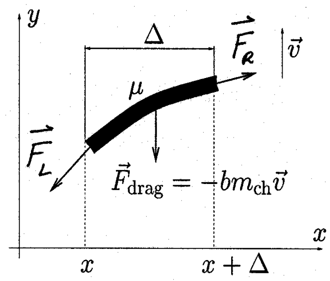


Figure 2: A chunk of string with air resistance.

2 Problem 2: Wave Equation for a String [8 points]

Note: *In this problem we use the same notation as in lecture.*

Consider a chunk of string of length Δ and mass per unit length μ. The string is under tension τ. The displacement in the vertical direction is given by the variable y which is a function of both position x and time t. As opposed to the derivation in lecture, where air resistance is neglected, we now include it. Note that air resistance on a chuck of string moving with velocity $\vec{v}$ creates a force *opposing* the direction of the velocity, i.e., with a y-component

$$F_{\mathrm{drag},y} = -bm_{\mathrm{ch}}v_y \ , \tag{2.1}$$

where b is a positive constant and $m_{\rm ch}$ is the mass of the chunk. A free body diagram on Figure 2 indicates all the forces acting on the chunk. Note that the positive y direction is upward and that we ignore the (small) effects of gravity in this problem.

Which of the following formulas best approximates the wave equation for small amplitude waves on the string with air resistance? (Provide your answer in the box on the next page.)

(A) $\mu\frac{\partial^2 y(x,t)}{\partial x^2} - b\frac{\partial y(x,t)}{\partial t} = \tau\frac{\partial^2 y(x,t)}{\partial t^2}$

(B) $\tau\frac{\partial^2 y(x,t)}{\partial x^2} = \mu\frac{\partial^2 y(x,t)}{\partial t^2}$

(C) $\tau\frac{\partial^2 y(x,t)}{\partial x^2} - b\mu\frac{\partial y(x,t)}{\partial t} = \mu\frac{\partial^2 y(x,t)}{\partial t^2}$

(D) $\tau\frac{\partial^2 y(x,t)}{\partial x^2} - b\mu\frac{\partial y(x,t)}{\partial x} = \mu\frac{\partial^2 y(x,t)}{\partial t^2}$

(E) $\tau\frac{\partial^2 y(x,t)}{\partial x^2} - b\frac{\partial y(x,t)}{\partial t} = \mu\frac{\partial^2 y(x,t)}{\partial t^2}$

(F) $\tau\frac{\partial y(x,t)}{\partial x} - b\mu\frac{\partial^2 y(x,t)}{\partial t^2} = \mu\frac{\partial y(x,t)}{\partial t}$

As argued in lecture, the y-components of the forces $\vec{F}_L$ and $\vec{F}_R$, with which the remaining string pulls the LEFT and the RIGHT end of the chunk respectively, are:

$$F_{L,y} = -\tau\frac{\partial y}{\partial x}(x,t)$$

$$F_{R,y} = +\tau\frac{\partial y}{\partial x}(x+\Delta,t)$$

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Writing $\vec{F}_{net} = m_{ch}\vec{a}$ for the chunk, we obtain:

$$F_{net,y} = m_{ch} a_y$$

$$F_{L,y} + F_{R,y} + F_{drag,y} = m_{ch} a_y$$

$$-\tau \frac{\partial y}{\partial x}(x,t) + \tau \frac{\partial y}{\partial x}(x+\Delta,t) - b m_{ch} v_y = m_{ch} a_y$$

Writing $m_{ch} = \mu\Delta$, we get:

$$\tau\left[\frac{\partial y}{\partial x}(x+\Delta,t) - \frac{\partial y}{\partial x}(x,t)\right] - b\mu\Delta . v_y = \mu\Delta . a_y \,\Big|\, : \Delta$$

$$\tau \frac{\frac{\partial y}{\partial x}(x+\Delta,t) - \frac{\partial y}{\partial x}(x,t)}{\Delta} - b\mu v_y = \mu a_y$$

Taking the limit $\Delta \to 0$ and writing v_y and a_y in terms of the D. of θ. $\boxed{C}$ $\left(v_y = \frac{\partial y}{\partial t}(x,t),\ a_y = \frac{\partial^2 y}{\partial t^2}(x,t)\right)$, we obtain:

$$\boxed{\tau \frac{\partial^2 y}{\partial x^2}(x,t) - b\mu \frac{\partial y}{\partial t}(x,t) = \mu \frac{\partial^2 y}{\partial t^2}(x,t)}$$

Obviously, this is the Wave Equation of answer $\boxed{C}$.

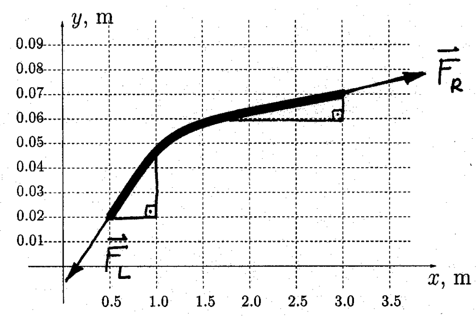


Figure 4: A chunk of vibrating string.

4 Problem 4: Forces, Slopes and Curvature [14 points]

A chunk of vibrating string (tension $\tau = 1000$ N and linear mass density $\mu = 0.6$ kg/m) is shown on the graph below: Neglecting air resistance and gravity, and using the graphical information on Figure 4, ***complete the tasks below:***

(a) Forces (7 points)

For each of the parts below choose the value, **(A)**, **(B)**, **(C)**, **(D)**, **(E)**, or **(F)**, which is closest to the answer of the question: (Note that your answer may not match any of these exactly.)

(A) 60 N , **(B)** 6 N , **(C)** 20 N , **(D)** 15 N , **(E)** 1000 N , **(F)** 53 N .

Provide your answers in the boxes below each question.

- ***What is the magnitude of the y-component of the force on the right hand end of the chunk?***

Reading the slope off the graph, we find:

$$F_{R,y} = \tau \frac{\partial y}{\partial x}(3.5m) \approx 1000N \times \left(\frac{0.01m}{1.5m}\right) \approx \underline{\underline{6.67N}}$$

$\Rightarrow$ The closest answer is (B) 6 N

$\boxed{B}$

- ***What is the magnitude of the y-component of force on the left hand end of the chunk?***

Reading the slope off the graph, we find:

$$F_{L,y} = -\tau \frac{\partial y}{\partial x}(0.5m) \approx -1000N \times \left(\frac{0.03m}{0.5m}\right) \approx \underline{\underline{-60N}}$$

$\Rightarrow$ For the MAGNITUDE $|F_{L,y}|$ we ought to choose answer (A) 60N.

$\boxed{A}$

- ***What is the magnitude of the net force on the chunk?***

The NET FORCE on the chunk points along the y-axis, since the x-components cancel ($F_{net,x} = F_{R,x} + F_{L,x} \approx +\tau - \tau = 0$). Therefore, we find:

$$F_{net} = F_{net,y} = F_{R,y} + F_{L,y} = 6.7N - 60N \approx -53.3N$$

$\Rightarrow$ The CLOSEST answer for the MAGNITUDE $|F_{net}|$ is (F) 53 N.

$\boxed{F}$

(b) Acceleration and Curvature (7 points) *Challenge Problem!*

Using your answer from part (a) (we won't penalize propagation of your errors):

- ***Find the*** magnitude of the y-component of the ***acceleration*** ~~a_y~~ ***of the chunk. (Write out your numerical answer including units. This part is not multiple choice!)***

Using $\vec{F}_{net} = m_{ch}\vec{a}$ for the chunk, we find:

$$|a_y| = \frac{|F_{net,y}|}{m_{ch}} = \frac{|F_{net}|}{\mu \cdot \Delta} \approx \frac{53\,N}{(0.6\,kg/m) \times 2.5\,m}$$

$$\approx \boxed{35.3\ m/s^2}$$

- ***Using the wave equation, estimate the curvature*** $\left|\frac{\partial^2 y}{\partial x^2}\right|$ ***of the chunk. (Write out your numerical answer including units. This part is not multiple choice!)***

From the wave equation for the chunk we find:

$$\tau \frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2} \Rightarrow \left|\frac{\partial^2 y}{\partial x^2}\right| = \frac{\mu}{\tau}|a_y|$$

$$\approx \frac{0.6\,kg/m}{1000\,N} \times 35.3\,m/s^2$$

$$\approx \boxed{0.021\ m^{-1}}$$

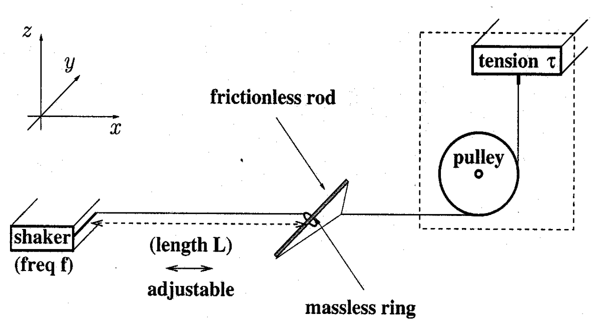


Figure 5: Lab Experiment I.

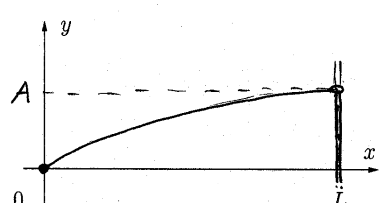
5 Problem 5: Lab Experiment I [21 points]

In your first laboratory experiment, you studied a vibrating string with two fixed ends. Consider now a modification of the experiment in which the right hand end of the string is attached to a small massless ring which slides up and down a horizontal (in and out of the page) post without friction. The post is attached through additional strings to a pulley on the other side of which spring scales (like the ones in lab) provide constant tension (see Figure 5).

As in lecture, L is the length of the string between the rod and the shaker, f is the frequency of the shaker, τ is the string tension and μ is the mass per unit length of the string. Assume μ is constant throughout the problem. The shaker vibrates the string *in and out* of the page. Assume that the left hand end is *fixed* at the shaker while the horizontal motion (along the y-direction) of the right hand end has no constraint, i.e., the right hand end is *free*.

(a) The Fundamental Mode (7 points)

Sketch the shape of the standing wave corresponding to the ***lowest (fundamental) mode***. ***Find an expression for the lowest (fundamental) frequency*** f_1 in terms of L, τ and μ.



From the sketch we see that the for the FUNDAMENTAL MODE:

$$\frac{\lambda_1}{4} = L \Rightarrow \underline{\lambda_1 = 4L}$$

$$\Rightarrow f_1 = \frac{v}{\lambda_1} = \boxed{\frac{1}{4L}\sqrt{\frac{\tau}{\mu}}}$$

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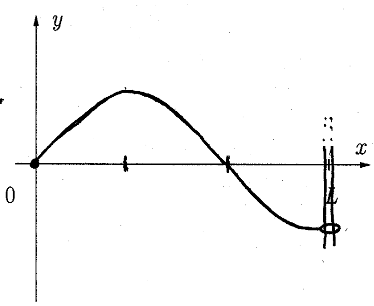
(b) Higher Modes at Fixed Length (7 points)

Sketch the shape of standing wave on the string corresponding to the ***second mode***. ***What should the frequency f_2 of the shaker be*** for this standing wave mode to be formed? (Express f_2 in terms of L, τ and μ.)

From the sketch we see that for the SECOND mode:

$$\frac{3}{4}\lambda_2 = L$$

$$\Rightarrow \lambda_2 = \frac{4}{3}L$$



Thus:

$$f_2 = \frac{v}{\lambda_2} = \frac{1}{\left(\frac{4L}{3}\right)}\sqrt{\frac{\tau}{\mu}}$$

$$\boxed{f_2 = \frac{3}{4L}\sqrt{\frac{\tau}{\mu}}}$$

(c) Higher Modes at Fixed Frequency (7 points)

Now suppose that you can vary the length of the the string. ***Find the string length ℓ at which the 2nd standing wave mode will be formed***, if the frequency of the shaker is fixed at f_1 and the tension τ remains constant. (Express ℓ in terms of the original length L.)

Since we are interested in the SECOND mode again, we can use directly the sketch of part (b): $\lambda_2 = \frac{4}{3}\ell$, where ℓ is the new length of the string. Demanding that $f = \frac{v}{\lambda_2}$ be equal to the frequency found in (a), we get:

$$\frac{1}{\cancel{4}L}\cancel{\sqrt{\frac{\tau}{\mu}}} = f = \frac{3}{\cancel{4}\ell}\cancel{\sqrt{\frac{\tau}{\mu}}} \Rightarrow \boxed{\ell = 3L}$$