1 Multiple Choice [28 points]

Each of the 4 multiple choice problems is worth 7 points.

(i) Consider the complex number $z = 1/(1+i)$. Which point (labeled A to M, as shown) in the complex plane shown in the figure below is closest to z ?

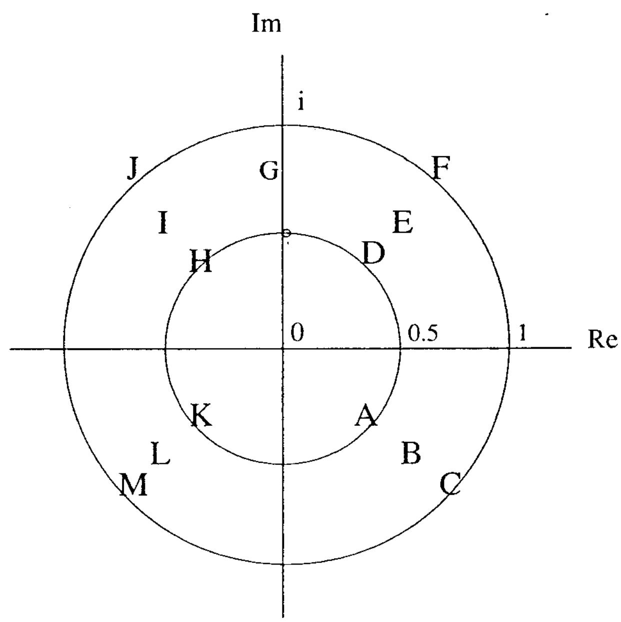


Figure 1: The complex number z on the complex plane.

$$\frac{1}{1+i} \cdot \frac{(1-i)}{(1-i)} = \frac{1-i}{2}$$

$\boxed{B}$

(ii) As we slowly increase the damping term of a slightly damped simple harmonic oscillator, which of the following is correct :

(A) Both the resonance peak and resonance width increase
(B) The resonance width narrows and the resonance peak goes higher
(C) The resonance width increases but the resonance peak lowers
(D) Both the resonance peak and the resonance width decreases
(E) The resonance width broadens but no change in the resonance peak
(F) The resonance peak decreases but no change in the resonance width

$\boxed{C}$

(iii) One end of a horizontal string is attached to an electrically-driven vibrator that vibrates at 200 Hz. The other end passes over a pulley and supports a weight of mass 2 Kg. The distance between two adjacent nodes on the string is 50cm. What is the string mass density μ ?

(Take g to be $10m/s^2$.)

(A) 2×10^{-2} Kg/m
(B) 10^{-2} Kg/m
(C) 10^{-3} Kg/m
(D) 2×10^{-4} Kg/m
(E) 10^{-4} Kg/m
(F) 5×10^{-4} Kg/m
(G) 5×10^{-5} Kg/m

$f = 200$ Hz

$\tau = Mg = 20$ N

$\lambda = 1$ m

$c = f\lambda = 200$ m/s

$\boxed{F}$

$c = \sqrt{\frac{\tau}{\mu}}$

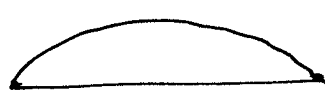
$\mu = \frac{\tau}{c^2} = \frac{20\text{ N}}{(200\text{ m/s})^2} = 5\times10^{-4}$ Kg/m

(iv) Consider a string (length L) with its two ends fixed. It is vibrating at the lowest mode, that is

$$y(x,t) = Asin(kx)cos(wt) \tag{1.1}$$

Consider the mid-point segment of the string (at $x = L/2$). Its maximum speed during the string vibration is

(A) wL/π
(B) $\sqrt{wL/\pi}$
(C) $\sqrt{wL/2\pi}$
(D) wA
(E) $wA/2$
(F) $\pi A/L$
(G) $\sqrt{wLA/\pi}$



$x = \frac{L}{2}$ vibrates in SHM w/ amplitude A.

(i.e. $K\cdot\frac{L}{2} = \frac{\pi}{2}$) ($\sin K\frac{L}{2} = 1$)

$\frac{\partial y}{\partial t} = (A \sin Kx)(-\omega) \sin \omega t$

$\boxed{D}$

$\frac{\partial y}{\partial t} = -A\omega \sin \omega t$

max $= A\omega$ ($\sin = \pm 1$)

2 Standing Wave [10 points]

Two strings of equal length L, with different densities, μ (on the right) and 4μ (on the left), are connected. Their other ends are fixed, as shown in the figure below.

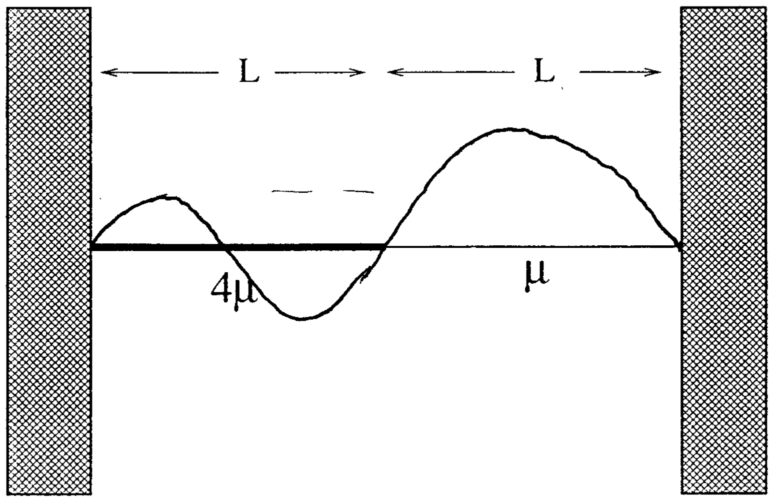


Figure 2: A string composed of two segments.

Sketch the lowest frequency standing wave which has a node at the point where the two strings meet.
You may draw your sketch directly in Figure 2.
(Hint: only simple algebraic calculations needed.)

tensions + frequencies are the same

$c = \sqrt{\frac{\tau}{\mu}}$ so wave speed on left $= \frac{1}{2}$ c on right

$c = \lambda f$ so λ on left is $\frac{1}{2}$ of λ on right

(amplitude also double on the right but
we didn't expect you to worry about that!)

3 Boundary Conditions [20 points]

Consider a string with each of its two ends tied to a spring that can only move in the y-direction. The spring constants are K_1 (at $x = 0$) and K_2 (at $x = L$). The string tension is τ.

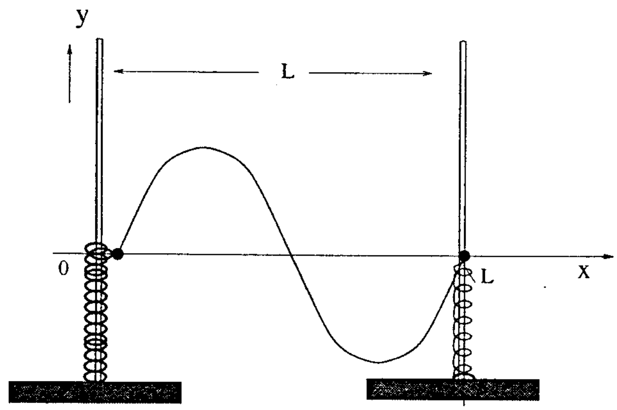


Figure 3: A string with each end tied to a spring

Consider

$$y(x,t) = A cos(kx + \phi) cos(wt) \tag{3.1}$$

Derive the two boundary conditions : one at $x = 0$ and one at $x = L$. Write them in terms of τ, K_1, K_2, L, k and ϕ.

$\underline{x=0}$ $\quad \Sigma F_y = -K_1 y + \tau \frac{\partial y}{\partial x} = m a_y = 0$ $\quad$ subst. in $y(x,t)$ at $x=0$

$$-K_1 \cancel{A} \cos\varphi \cancel{\cos \omega t} + \tau(-k)\cancel{A} \sin\varphi \cancel{\cos \omega t} = 0$$

$$-K_1 \cos\varphi - \tau k \sin\varphi = 0$$

$x = 0$ $\boxed{\tan\varphi = -\frac{K_1}{\tau k}}$

$\underline{x=L}$

$$\Sigma F_y = -K_2 y - \tau \frac{\partial y}{\partial x} = 0 \quad \text{at } x = L$$

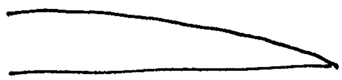
$$-K_2 \cos(\varphi + kL) + \tau k \sin(kL + \varphi) = 0$$

$x = L$ $\boxed{\tan(kL+\varphi) = \frac{K_2}{\tau k}}$

What are the values of k and ϕ in terms of L for the lowest oscillating mode in the cases where

(a) $K_1 = 0$ and K_2 is infinite

free fixed



$\lambda = 4L$

$k = \frac{2\pi}{\lambda} = \frac{\pi}{2L}$

free at $x=0$ so

$\frac{\partial y}{\partial x} = 0 \Rightarrow \phi = 0$

(or any multiple of π)

$$\boxed{k = \frac{\pi}{2L} \qquad \phi = 0}$$

(b) $K_1 = K_2 \to 0$.

free – free

$\lambda = 2L$

$k = \frac{2\pi}{\lambda} = \frac{\pi}{L}$

free at $x=0 \Rightarrow \phi = 0$ (or $n\pi$)

$$\boxed{k = \frac{\pi}{L} \qquad \phi = 0}$$

4 Traveling Wave vs Standing Wave [20 points]

Consider a string of length L with its two ends fixed (at $x = 0$ and $x = L$).

Suppose the resulting standing wave is described by

$$y(x,t) = Asin(kx + wt + \phi) + f(x - ct) \tag{4.1}$$

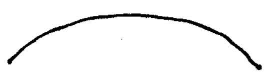
where A is a real constant.

Express w in terms of c and k.

$\boxed{\omega = ck}$

Give the lowest two values of k in terms of L.

$\boxed{\frac{\pi}{L}, \frac{2\pi}{L}}$



$\lambda = 2L \quad k = \frac{2\pi}{2L}$

$\lambda = L \quad k = \frac{2\pi}{L}$

Give the explicit form of the function $f(x - ct)$ in terms of A, k, ϕ and c.

(Hint : Either go to the complex plane, or use a trigonometric relation given in the formula sheet.)

$f(x-ct)$ must be cinusoidal with same amplitude, λ, and frequency.

$\Rightarrow f(x-ct) = A \sin(kx - \omega t + \varphi')$

what's φ'? Apply B.C. at $x=0$

$y(0,t) = 0$

$\boxed{A \sin(kx - \omega t - \varphi)}$

$A \sin(\omega t + \varphi) + A \sin(-\omega t + \varphi') = 0$

true if $\varphi' = -\varphi$

5 Superposition of Traveling Waves [20 points]

The string has a length of $4d$, extending from $-2d$ (with free boundary condition) to $+2d$ (with fixed boundary condition). At $t = 0$, the string is plucked in a way so that it has an initial shape as shown in the figure, and is initially motionless. At $t = 0$, the string is released.

Sketch the shape of the string at $t = d/c, 2d/c, 3d/c$. Here c is the wave speed.

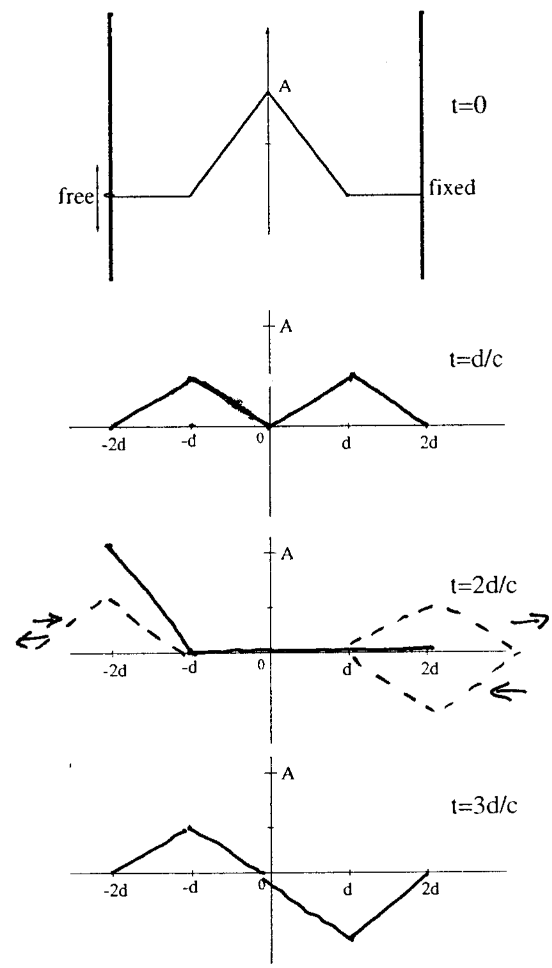


Figure 4: The plucked string.

What is the earliest time T (i.e., the period) that the string will again have its original shape (i.e., that at $t = 0$). Express T in terms of d and c.

at $t = \frac{4d}{c}$, pulses at center but 1 inverted

" $t = \frac{8d}{c}$, " " " " both inverted

" $t = \frac{12d}{c}$, " " " " 1 inverted

" $t = \frac{16d}{c}$, " " " and neither inverted

$\boxed{\frac{16d}{c}}$

END OF EXAM