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Phys 86-650: Homework Assignment about Lie groups

Question 1 :S_3

In this question we study the group S_3. It is the simplest finite non-Abelian group. You can think about it as all possible permutation of three elements. The group has 6 elements. Thinking about the permutations we see that we get the following representation of the group:

$$\begin{aligned} () &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ (13) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad (23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ (123) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad (321) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned} \tag{1}$$

The names are instructive. For example, (12) represents exchanging the first and second elements. (123) and (321) are cyclic permutation to the right or left.

1. Write explicitly the 6×6 multiplication table for the group.
2. Show that the group is non-Abelian. Hint, it is enough to find one example.
3. Z_3 is a sub group of S_3. Find the three generators that correspond to Z_3.
4. In class we mentioned the following theorem for finite groups

$$\sum_{R_i} \left[\dim(R_i)\right]^2 = N, \tag{2}$$

   where N is the number of elements in the group and R_i are all the irreps. Based on this, proof that the representation in Eq. (1) is reducible.

5. The representation in Eq. (1) is reducible. Write it explicitly in a (1+2) block diagonal representation. (Hint: find a vector which is an eigenvector of all the above matrices.)

6. In the last item you found a two dimensional and a one dimensional representations of S_3. Based on (2) you know that there is only one more representation and that it is one dimensional. Find it.

Question 2 :Lie algebras

Consider two general elements of a Lie groups,

$$A \equiv \exp(i\lambda X_a), \qquad B \equiv \exp(i\lambda X_b). \tag{3}$$

where X_i is a generator. We think about λ as a small parameter. Then, consider a third element

$$C = BAB^{-1}A^{-1} \equiv \exp(i\beta_c X_c). \tag{4}$$

Expand C in powers of λ and show that at lowest order you get the Lie algebra

$$[X_a, X_b] = i f_{abc} X_c, \qquad f_{abc} \equiv \frac{\beta_c}{\lambda^2}. \tag{5}$$

Question 3 :Dynkin diagrams

1. Draw the Dynkin diagram of SO(10).

2. What is the rank of SO(10)?

3. How many generators there are for SO(10)? (We did not proof a general formula for the number of generators for SO(N). It should be simple for you to find such a formula using your understanding of rotations in real N-dimensional spaces.)

4. Based on the Dynkin diagram show that SO(10) has the following subalgebras

$$\mathrm{SO}(8), \qquad \mathrm{SU}(5), \qquad \mathrm{SU}(4) \times \mathrm{SU}(2), \qquad \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(2). \tag{6}$$

   In each case show which simple root you can remove from the SO(10) Dynkin diagram.

Question 4 :SU(3)

1. The three Gell–Mann matrices, $a\lambda_1$, $a\lambda_2$ and $a\lambda_3$ satisfy an SU(2) algebra, where a is a constant. What is a?

2. Does this fact mean that SU(3) is not a simple Lie group?

3. There are two other independent combinations of Gell–Mann matrices that satisfy SU(2) algebras. What are they? Hint: Look at the root diagram.

Question 5 :representations

Here we practice finding the number of degrees of freedom in a given irrep.

1. In SU(5), how many particles there are in the following irreps

$$(1,0,0,0), \quad (0,1,0,0), \quad (1,1,0,0). \tag{7}$$

2. In SU(3) how many particles there are in the following irreps

$$(1,0), \quad (2,0), \quad (1,1), \quad (3,0), \quad (1,2), \quad (2,2). \tag{8}$$

Question 6 :Combining irreps

Here we are going to study the use of Young Tableaux. The details of the method can be found in the PDG, pdg.lbl.gov/2007/reviews/youngrpp.pdf (there is a link in the website of the course). Study the algorithm and do the following calculations. Make sure you check that the number of particles on both sides is the same. Write your answer both in the k-tuple notation and the number notation. For example, in SU(3) you should write

$$(1,0)\times(0,1) = (0,0)+(1,1), \qquad 3\times\bar{3} = 1+8. \tag{9}$$

1. In SU(3) calculate

$$3\times 3, \qquad 3\times 8, \qquad \overline{10}\times 8. \tag{10}$$

2. Given that the quarks are $\mathrm{SU}(3)_C$ triplets, 3, the anti-quarks are $\bar{3}$ and the gluons are color octets, 8, which of the following could be an observable bound state?

$$q\bar{q}, \qquad qq, \qquad qg, \qquad gg, \qquad q\bar{q}g, \qquad qqq. \tag{11}$$

   Note that an observable bound state must be a color singlet.

3. Find what is $\bar{5}$ and 10 in SU(5) in a k-tuple notation.

4. Calculate is SU(5)

$$\bar{5}\times\bar{5}, \qquad 10\times 10, \qquad \bar{5}\times 10. \tag{12}$$