January 11, 2016

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Phys 86-650: Homework Assignment # 1

Question 1 :Axiomatic Newtonian mechanics

In this question you are asked to develop Newtonian mechanics from simple axioms. We first define the system, and assume that we have point like particles that "live" in 3d space and 1d time. The very first axiom is the principle of minimal action that stated that the system follow the trajectory that minimize S and that

$$S = \int L(x, \dot{x}, t) dt \tag{1}$$

Note that we assume here that L is only a function of x and v and not of high derivative. This assumption is not justified from first principle, but at this stage we just assume it.

We now like to find L. We like L to be the most general one that is invariant under some symmetries. These symmetries would be the axioms. For Newtonian mechanics the axiom is that the that system is invariant under:

a) Time translation.

b) Space translation.

c) Rotation.

d) Galilean transformations (I.e. shifting your reference frame by a constant velocity in non-relativistic mechanics.).

Note that when we say "the system is invariant" we mean that the EOMs are unchanged under the transformation. This is the case if the new Lagrangian is the same as the old one up to a function that is a total time derivative, $L' = L + df(x,t)/dt$.

1. Formally define these four transformations. I will start you off: time translation is: the equations of motion are invariant under $t \to t + C$ for any C, which is equivalent to the statement $L(\mathbf{x}, \mathbf{v}, t + C) = L(\mathbf{x}, \mathbf{v}, t) + df/dt$ for some f.
   <u>Answer:</u>

   b) Space translation: The EOM are invariant under $x \to x + C$ for any C, i.e.

$$L(\mathbf{x}, \mathbf{v}, t) = L(\mathbf{x} + \mathbf{C}, \mathbf{v}, t) + \frac{df}{dt}. \tag{2}$$

c) Rotation: The EOM are invariant under $\phi \to \phi + C$ for any C (here ϕ is any angle), i.e.

$$L(\mathbf{x}, \mathbf{v}, t) = L(R(C)\mathbf{x}, R(C)\mathbf{v}, t) + \frac{df}{dt}. \tag{3}$$

d) Galilean transformation: The EOM are invariant under $x \to x + Vt$, where V is the relative velocity between any two frames and it does not change with time. I.e.

$$L(\mathbf{x}, \mathbf{v}, t) = L(\mathbf{x} + \mathbf{V}t, \mathbf{v} + \mathbf{V}, t) + \frac{df}{dt} \tag{4}$$

2. We first consider a system with one particle. Let us make the stronger assumption that the Lagrangian itself is invariant under the symmetry transformations (a), (b), (c) (i.e. $f = 0$ for these transformations). Show that $L(x, v, t) = L(v^2)$ where $v^2 \equiv v_i v^i$.
<u>Answer:</u> Due to a) we know that L cannot be a function of t. Based on b) we know it is not a function of x, and based on c) we know it can only depend on the magnitude of the velocity, that is on v^2.

3. Next we would like to show that $L(v^2) = C \times v^2$. For this we first consider an infinitesimal Galilean transformation, $v' = v + \varepsilon$, such that

$$L' \equiv L[v'^2] = L[(v + \varepsilon)^2]. \tag{5}$$

Expand L' around v^2 to first order in ε.
<u>Answer:</u>

$$L[(v + \varepsilon)^2] = L[(v^2 + 2\mathbf{v} \cdot \varepsilon + \varepsilon^2] = L(v^2) + \left.\frac{\partial L}{\partial v^2}\right|_{v^2} 2v \cdot \epsilon + ... \tag{6}$$

4. Explain why the requirement of invariance under Galilean transformations implies that the first order term in ε should be a total time derivative.
<u>Answer:</u> Because we want to get the same EOMs in any reference frame. This is ensured if the Lagrangian changes only by the addition of a total time derivative under the symmetry transformation.

5. Show that the first order term in ε is a total time derivative only when $L(v^2)$ is linear in v^2, that is, that $L(v^2) = Cv^2$.
<u>Answer:</u> We will require that:

$$\left.\frac{\partial L}{\partial v^2}\right|_{v^2} 2\mathbf{v} \cdot \varepsilon = \frac{df}{dt} \tag{7}$$

First, recall that the funcitonal form of f is $f(\mathbf{x}, t)$. It cannot depend on higher derivatives of x due to our assumption that the Lagrangian depends only on $x, \dot{x}, t$. Therefore:

$$\left.\frac{\partial L}{\partial v^2}\right|_{v^2} 2\mathbf{v} \cdot \varepsilon = \frac{df(x,t)}{dt} = \nabla f \cdot \mathbf{v} + \frac{\partial f}{\partial t} \tag{8}$$

We then conclude that

$$\mathbf{v} \cdot \left(2\frac{\partial L}{\partial v^2}\varepsilon - \nabla f \right) = \frac{\partial f}{\partial t}. \tag{9}$$

First, since the LHS is dotted with v, while the LHS does not depend on v, it follows that both sides are 0. Therefore, the quantity in brackets on the LHS is zero:

$$2\frac{\partial L}{\partial v^2}\varepsilon = \nabla f(\mathbf{x}, t) \tag{10}$$

The LHS is only a function of v^2, while the RHS is only a function of x, t. We thus conclude that both side must be a constant, that is

$$\frac{\partial L}{\partial v^2} = C \quad \Rightarrow \quad L = Cv^2 \tag{11}$$

6. We now move to a two body problem. We further assume that when the particles are infinitely far apart they can be treated as free particles. Based on this extra assumption, and on the four axioms, show that:

$$L = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} - V(x_1 - x_2). \tag{12}$$

Answer: In fact, I made a mistake in this question. Based on the axioms all we can say is that we have another term that is a function of $(x_1 - x_2)^2$, $(v_1 - v_2)^2$ and $(x_1 - x_2) \cdot (v_1 - v_2)$. All of these terms are invariant under all the symmetries that we imposed. The fact that we want the particle to be free at $x_1 - x_2 \to \infty$ just tells us that this function is such that it goes to zero for large $x_1 - x_2$, independent of the value of the velocities. So, in fact, to get a potential that depends only on the separation, it seems that we need one extra axiom that we do not like: that even for a 2 body system the only v dependce is the one of a free particle. In that case we get the above.[1]

An additional comment is that some students argued that rotational symmetry means

[1] Grader's note: There is no such thing as an 'argument from authority' in physics, something is correct or not on its own merits. So if you think that something stated in the homework is wrong, I would much prefer to see a thoughtful argument explaining why, than a short answer leading to the incorrect conclusion. Be brave and follow the logic! Still, I am always going to be lenient either way in this scenario.

that the only v dependence that can show up is in v_1^2 and v_2^2. In fact, rotational symmetry allows all dot products of vectors, including $\mathbf{v}_1.\mathbf{v}_2$, $\mathbf{v}_1.(\mathbf{x}_1 - \mathbf{x}_2)$, etc.

7. Usually in mechanics we pose questions that start like "consider a particle in the following potential, $V(x)$." What is the approximation that is done by moving from the two body problem discussed above to a set-up of a particle in a potential?
Answer: There are sereval cases in which we use it. In some two body problems we use generelized coordinates where one coordinate describes the free-body motion of the center of mass while the other is of a particle with a "reduced mass". In other cases we assume that we do not care about some of the DOFs, e.g. in a case of a ball that hits the Earth, we assume that the Earth is massive enough that is not affected by the ball.

Question 2 :Mattress Theory

In class we discuss a simple mechanical model of a field with linear dispersive relation. (I suggest that you review it in Goldstein chapter 13.1.) In this question we discuss a similar model where the excitation of the field are massive particles. Consider a one-dimensional "mattress" consisting of $2N+1$ bodies with mass M each labeled by $\ell = 0 \ldots 2N$. They are connected by springs, with spring constant K each, to the foundation and to the neighboring masses on both sides (see figure). The distance between two neighboring masses along the x axis is fixed (equal to d). We further define

$$\omega_0^2 \equiv \frac{K}{M}, \qquad \ell_0 \equiv \sqrt{y_0^2 + d^2}. \tag{13}$$

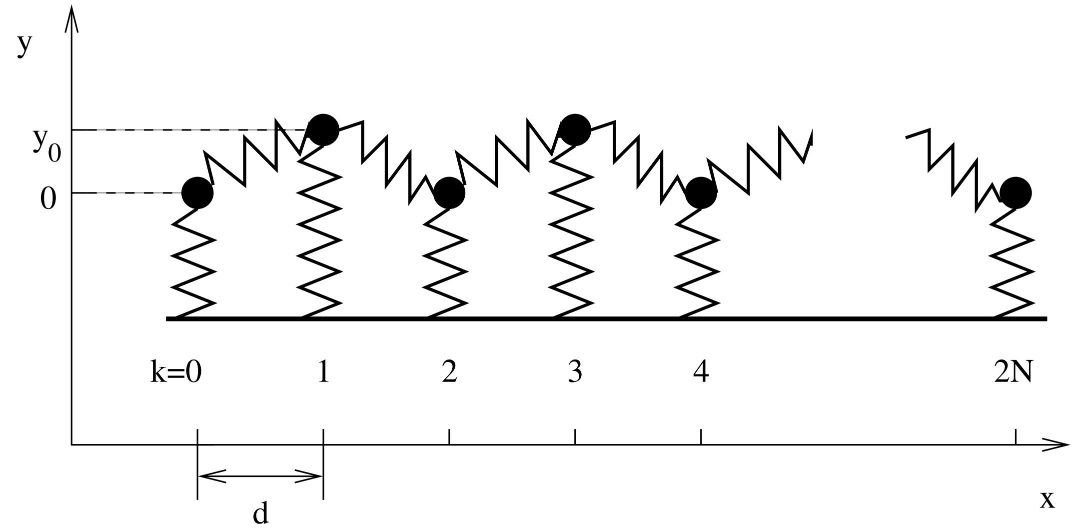
The masses are free to move in the y direction. When the masses with even ℓ are at $y = 0$ and the masses with odd ℓ are at $y = y_0$, the system is in equilibrium.

1. Write the terms in the Lagrangian for the ℓth mass. Consider both even and odd ℓ, but not the endpoints ($\ell = 0$ and $\ell = 2N$). Use the variable $z_\ell = (y_\ell - y_\ell^{\text{eq}})$, where y_ℓ^{eq} is the equilibrium position of the ith mass. Restrict your attention to the case of small displacements from equilibrium, $z_\ell \ll d, y_0$ and $d > y_0$.
Answer: Even numbered mass

$$L_{2l} = \frac{m\dot{z}_{2l}^2}{2} - \frac{1}{2}kz_{2l}^2 \tag{14}$$

$$-\frac{1}{2}k\left(\sqrt{(z_{2l-1} - z_{2l} + y_0)^2 + d^2} - \sqrt{d^2 + y_0^2}\right)^2 \tag{15}$$

$$-\frac{1}{2}k\left(\sqrt{(z_{2l+1} - z_{2l} + y_0)^2 + d^2} - \sqrt{d^2 + y_0^2}\right)^2 \tag{16}$$



Working in the limit of

$$z_{2l-1} - z_{2l} \ll y_0 \tag{17}$$

we get

$$L_{2l} = \frac{m\dot{z}_{2l}^2}{2} - \frac{k}{2}\left[z_{2l}^2 - \frac{y_0^2}{l_0^2}\left(z_{2l-1} - z_{2l}\right)^2 - \frac{y_0^2}{l_0^2}\left(z_{2l+1} - z_{2l}\right)^2\right] \tag{18}$$

We get the same result for the odd numbers.

2. Using that Lagrangian, show that the equation of motion for the y coordinate of the ℓth mass is given by

$$\frac{\partial^2 z_\ell}{\partial t^2} = a(z_{\ell+1} + z_{\ell-1} - 2z_\ell) + bz_\ell \tag{19}$$

and find what are a and b.
<u>Answer:</u> The EOM

$$\begin{aligned} \frac{\partial L}{\partial \dot{z}_{2l}} &= m\dot{z}_{2l} && (20) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{z}_{2l}} &= m\ddot{z}_{2l} && (21) \\ \frac{\partial L}{\partial z_{2l}} &= -kz_{2l} + k\frac{y_0^2}{l_0^2}\left(z_{2l-1} - z_{2l}\right) + k\frac{y_0^2}{l_0^2}\left(z_{2l+1} - z_{2l}\right) && (22) \end{aligned}$$

So the Euler-Lagrange equation is given by

$$\begin{aligned} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{z}_{2l}} = \frac{\partial L}{\partial z_{2l}}\right) \Rightarrow m\ddot{z}_{2l} &= -kz_{2l} + k\frac{y_0^2}{l_0^2}\left(z_{2l-1} - z_{2l}\right) + k\frac{y_0^2}{l_0^2}\left(z_{2l+1} - z_{2l}\right) && (23) \\ \ddot{z}_{2l} &= \frac{k}{m}\frac{y_0^2}{l_0^2}\left(z_{2l-1} + z_{2l+1} - 2z_{2l}\right) - \frac{k}{m}z_{2l} && (24) \end{aligned}$$

Writing

$$\frac{k}{m} = \omega_0^2, \tag{25}$$

we have

$$\ddot{z}_{2l} = \omega^2 \frac{y_0^2}{l_0^2} \left(z_{2l-1} + z_{2l+1} - 2z_{2l}\right) - \omega^2 z_{2l} \tag{26}$$

That is, we found that

$$a = \omega_0^2 \frac{y_0^2}{l_0^2}, \qquad b = -\omega_0^2 \tag{27}$$

3. Next solve the equation. For this, "guess" a solution of the form

$$z_\ell = \sum_q e^{iq\ell} \tilde{z}(q) \tag{28}$$

and find the equation of motion for $\tilde{z}(q)$. You should express your result as

$$\frac{\partial^2 \tilde{z}(q)}{\partial t^2} = f(q)\tilde{z}(q) \tag{29}$$

that is, find what is $f(q)$.

<u>Answer:</u>

$$\begin{aligned}\sum_q e^{iql} \frac{d^2 \overline{z}\,(q)}{dt^2} &= a\left(\sum_q e^{iq(l+1)}\overline{z}\,(q) + \sum_q e^{iq(l-1)}\overline{z}\,(q) - 2\sum_q e^{iql}\overline{z}\,(q)\right) + b\sum_q e^{iql}\overline{z}\,(q) \\ &= \sum_q \left(a\left(e^{iq(l+1)} + e^{iq(l-1)} - 2e^{iql}\right) + be^{iql}\right)\overline{z}\,(q) \end{aligned} \tag{30}$$

This equation has to be satisfied for each q so

$$e^{iql}\frac{d^2\overline{z}\,(q,t)}{dt^2} = \left(a\left(e^{iq(l+1)} + e^{iq(l-1)} - 2e^{iql}\right) + be^{iql}\right)\overline{z}\,(q) \tag{31}$$

$$\frac{d^2\overline{z}\,(q,t)}{dt^2} = \left(a\left(e^{iq} + e^{-iq} - 2\right) + b\right)\overline{z}\,(q,t) \tag{32}$$

thus

$$f\,(q) = 2a\left(\tfrac{e^{iq}+e^{-iq}}{2} - 1\right) + b = 2a\,(\cos q - 1) + b \tag{33}$$

4. Assuming that the masses with $\ell = 0$ and $\ell = 2N$ are held rigidly at $y = 0$, derive the spectrum of the waves on the mattress. That is, find the allowed frequencies $\omega(q)$.

<u>Answer:</u> The $y = 0$ implies that we need to consider only the sin out of the exponent. The $y = 2N$ one quantize the system. That is we get

$$\sin 2Nq = 0 \quad \Longrightarrow \quad q_j = \frac{\pi j}{2N}, \qquad j = 1, 2, ..., 2N-1. \tag{34}$$

Thus, the allowed frequencies are

$$\omega_q = \omega_0 \sqrt{1 + \frac{2y_0^2}{l_0^2}(1 - \cos q_j)}, \qquad q_j = \frac{j\pi}{2N}, \tag{35}$$

5. When we take $N \to \infty$, we still cannot get $\omega \to \infty$. That is, the system has a UV cutoff. What is (roughly) the UV cutoff scale?
   Answer: It is roughly

$$\omega_0 \sqrt{1 + \frac{4y_0^2}{l_0^2}} \tag{36}$$

6. We now consider the limit $q \ll 1$. Show that in that limit the dispersion relation is given by that of a relativistic particle, that is, $E^2 = p^2 + m^2$. Yet, in Nature the maximum velocity is 1 (or c), while in our case it is given by the parameters of the mattress. Find the equivalent of the mass and the maximum velocity in our case. (Note, here $p^2 = p^\mu p_\mu = E^1 - \vec{p}^2$. Also, since $\hbar = 1$, we often replace E with ω.)
   Answer:

$$\omega_q^2 = \omega_0^2 + \frac{2y_0^2\omega_0^2}{l_0^2}(1 - \cos q_j) \approx \omega_0^2 + \frac{y_0^2\omega_0^2 q^2}{l_0^2} \tag{37}$$

   We learn that $m = \omega_0$ and $c = y_0\omega_0/l_0$.

Question 3 :Lagrangian formalism of E&M

In mechanics, one can start from $L = ma$ or it can be derived form $L = mv^2/2 - V(x)$ based on the principle of minimal action. In this problem you will show that Maxwell's equations can be derived for a Lagrangian density and the principle of minimal action.

We start with some definitions. We define the vector potential and the electromagnetic tensor

$$A_\mu(x_\mu) = \{\phi(x_\mu), A_i(x_\mu)\}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{38}$$

$$\mu, \nu = 0, 1, 2, 3; \qquad i, j, k = 1, 2, 3. \tag{39}$$

The electric field (E) and magnetic field (B) are then

$$E_i = F^{i0}, \qquad B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk}, \tag{40}$$

where the ϵ tensor is defined by $\epsilon_{123} = 1$ and that it is antisymmetric under the exchange of any two indices. Explicitly, $F_{\mu\nu}$ is given by:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{41}$$

We further define the dual field strength tensor

$$\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}, \tag{42}$$

where $\epsilon^{\alpha\beta\mu\nu}$ is the completely antisymmetric rank 4 epsilon tensor. Clearly F is antisymmetric. Moreover, you can show (check it!) that

$$F_{\mu\nu}F^{\mu\nu} = 2\left(B^2 - E^2\right), \qquad \tilde{F}_{\mu\nu}F^{\mu\nu} = -4\left(B \cdot E\right). \tag{43}$$

The above two combinations are manifestly Lorentz invariant. We also define the four current

$$J^\alpha = (\rho, J_i). \tag{44}$$

We consider Maxwell's equations

$$\nabla \cdot E = \rho, \qquad \nabla \times B - \frac{\partial E}{\partial t} = J_i, \qquad \nabla \cdot B = 0, \qquad \nabla \times E + \frac{\partial B}{\partial t} = 0. \tag{45}$$

1. Show that the inhomogeneous equations (that is, the first two) can be written as

$$\partial_\alpha F^{\alpha\beta} = J^\beta. \tag{46}$$

<u>Answer:</u> This can be written as two equations:

$$\partial_\alpha F^{\alpha 0} = J^0 = \rho \tag{47}$$

$$\partial_\alpha F^{\alpha i} = J_i \tag{48}$$

The first gives:

$$\begin{aligned} \partial_t F^{00} + \partial_i F^{i0} &= \rho \qquad (49) \\ \nabla \cdot \mathbf{E} &= \rho. \qquad (50) \end{aligned}$$

The second gives:

$$\begin{aligned} \partial_0 F^{0i} + \partial_j F^{ji} &= J_i \qquad (51) \\ -\partial_0 E_i + \epsilon_{ijk}\partial_j B_k &= J_i \qquad (52) \\ -\partial_t \mathbf{E} + \nabla \times \mathbf{B} &= \mathbf{J} \qquad (53) \end{aligned}$$

2. We assume the following Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J^{\mu}A_{\mu}, \tag{54}$$

and consider each of A_μ to be an independent DOF. Using the E-L equation show that this Lagrangian gives the two inhomogeneous equations.

<u>Answer:</u> It is helpful to rewrite:

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F_{\gamma\rho}g^{\alpha\gamma}g^{\beta\rho} + J^{\mu}A_{\mu}, \tag{55}$$

We use the EL equation:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \tag{56}$$

to calculate the first term:

$$\begin{aligned} \frac{\partial}{\partial(\partial_\mu A_\nu)}\mathcal{L} &= -\frac{1}{4}g^{\alpha\gamma}g^{\beta\rho}\frac{\partial}{\partial(\partial_\mu A_\nu)}F_{\alpha\beta}F_{\gamma\rho} \\ &= -\frac{1}{2}g^{\alpha\gamma}g^{\beta\rho}F_{\alpha\beta}\frac{\partial}{\partial(\partial_\mu A_\nu)}(F_{\gamma\rho}) \\ &= -\frac{1}{2}g^{\alpha\gamma}g^{\beta\rho}F_{\alpha\beta}\left(\delta^{\mu}_{\gamma}\delta^{\nu}_{\rho} - \delta^{\nu}_{\gamma}\delta^{\mu}_{\rho}\right) \\ &= -\frac{1}{2}\left(F^{\mu\nu} - F^{\nu\mu}\right) \\ &= -F_{\mu\nu} \end{aligned} \tag{57}$$

We therefore have:

$$\partial_\mu F^{\mu\nu} = -J^{\nu} \tag{58}$$

Question 4 :Fermionic harmonic oscillator

In class we explain why a field is similar to the simple harmonic oscillator (SHO), and that that they are closely related to bosons. Then we mentioned that we should have a Fermionic field where its excitations corresponds to fermions. Here we discuss the simple case of a Fermionic harmonic oscillator.

You must be very familiar with. We define it with

$$H = \hbar\omega(N + 1/2), \qquad N = a^{\dagger}a, \tag{59}$$

with

$$[a, a^{\dagger}] = 1, \qquad [a, a] = [a^{\dagger}, a^{\dagger}] = 0. \tag{60}$$

Here we study a similar system that describe a Fermionic one. That is, there is no classical analog for it. Consider instead of the commutation relation of the SHO an anti-commutation relation

$$\{b, b^\dagger\} = 1, \qquad \{b, b\} = \{b^\dagger, b^\dagger\} = 0, \tag{61}$$

where $\{a, b\} = ab + ba$. For the Hamiltonian we have

$$H = \hbar\omega(N - 1/2), \qquad N = b^\dagger b. \tag{62}$$

1. Show that $b^2 = 0$.
   <u>Answer:</u>
   $$b, b = 2bb = 2b^2 = 0. \tag{63}$$

2. Show that there is a state, which we denote by $|0\rangle$, that is annihilated by b.
   <u>Answer:</u> The Hildert state cannot be empty, so there must be a state, which we denote as $|\psi\rangle$. When apply with b on it with either have $b|\psi\rangle = 0$ which implies that $|\psi\rangle = |0\rangle$ or we have $b|\psi\rangle = |xi\rangle$ wuch that ξ is another state. In that case we apply b again and get $b|\xi\rangle = b^2|\psi\rangle = 0$ where at the last stage we used the fact that $b^2 = 0$.

3. We define $|1\rangle = b^\dagger|0\rangle$. Show that this state cannot vanish.
   <u>Answer:</u>
   $$bb^\dagger|0\rangle = (1 - b^\dagger b)|0\rangle = |0\rangle. \tag{64}$$
   Thus we know that $|1\rangle = b^\dagger|0\rangle \neq 0$.

4. We assume that all possible operators must be composed of products of b and $b^\dagger$. Show that $|1\rangle$ and $|0\rangle$ span the Hilbert space. (One way to go is to show that there are a total of four operators. There is a theorm that stated that this number is equal to the dimension of the Hilbert space squared.)
   <u>Answer:</u> There are a totla of 4 operators, $1, b, b^\dagger$ and $bb^\dagger$. This can be as any other combination can be reduced to it using the anti-commutation relation. We thus show that the Hilbert space is indeed a two level system.

5. What is the energy of each state?
   <u>Answer:</u>
   $$H|0\rangle = \hbar\omega(0 - 1/2)|0\rangle, \qquad H|1\rangle = \hbar\omega(1 - 1/2)|0\rangle \tag{65}$$
   so we conclude that $E_0 = -\hbar\omega/2$ and $E_1 = \hbar\omega/2$.