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Bar-Ilan University, Department of Physics

Phys 86-650: Homework Assignment # 1

Question 1 :Axiomatic Newtonian mechanics

In this question you are asked to develop Newtonian mechanics from simple axioms. We first define the system, and assume that we have point like particles that "live" in 3d space and 1d time. The very first axiom is the principle of minimal action that stated that the system follow the trajectory that minimize S and that

$$S = \int L(x, \dot{x}, t)dt \tag{1}$$

Note that we assume here that L is only a function of x and v and not of high derivative. This assumption is not justified from first principle, but at this stage we just assume it.

We now like to find L. We like L to be the most general one that is invariant under some symmetries. These symmetries would be the axioms. For Newtonian mechanics the axiom is that the that system is invariant under:

a) Time translation.

b) Space translation.

c) Rotation.

d) Galilean transformations (I.e. shifting your reference frame by a constant velocity in non-relativistic mechanics.).

Note that when we say "the system is invariant" we mean that the EOMs are unchanged under the transformation. This is the case if the new Lagrangian is the same as the old one up to a function that is a total time derivative, $L' = L + df(x,t)/dt$.

1. Formally define these four transformations. I will start you off: time translation is: the equations of motion are invariant under $t \to t + C$ for any C, which is equivalent to the statement $L(\mathbf{x}, \mathbf{v}, t + C) = L(\mathbf{x}, \mathbf{v}, t) + df/dt$ for some f.

2. We first consider a system with one particle. Let us make the stronger assumption that the Lagrangian itself is invariant under the symmetry transformations (a), (b), (c) (i.e. $f = 0$ for these transformations). Show that $L(x, v, t) = L(v^2)$ where $v^2 \equiv v_i v^i$.

3. Next we would like to show that $L(v^2) = C \times v^2$. For this we first consider an infinitesimal Galilean transformation, $v' = v + \varepsilon$, such that

$$L' \equiv L[v'^2] = L[(v+\varepsilon)^2]. \tag{2}$$

   Expand L' around v^2 to first order in ε.

4. Explain why the requirement of invariance under Galilean transformations implies that the first order term in ε should be a total time derivative.

5. Show that the first order term in ε is a total time derivative only when $L(v^2)$ is linear in v^2, that is, that $L(v^2) = Cv^2$.

6. We now move to a two body problem. We further assume that when the particles are infinitely far apart they can be treated as free particles. Based on this extra assumption, and on the four axioms, show that:

$$L = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} - V(x_1 - x_2) \tag{3}$$

7. Usually in mechanics we pose questions that start like "consider a particle in the following potential, $V(x)$." What is the approximation that is done by moving from the two body problem discussed above to a set-up of a particle in a potential?

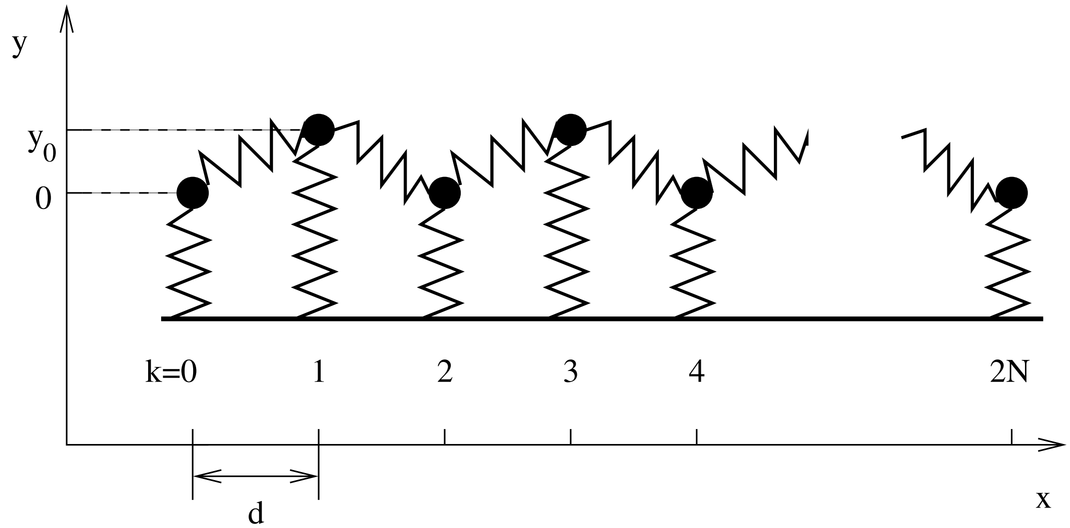
Question 2 :Mattress Theory

In class we discuss a simple mechanical model of a field with linear dispersive relation. (I suggest that you review it in Goldstein chapter 13.1.) In this question we discuss a similar model where the excitation of the field are massive particles. Consider a one-dimensional "mattress" consisting of $2N+1$ bodies with mass M each labeled by $\ell = 0 \dots 2N$. They are connected by springs, with spring constant K each, to the foundation and to the neighboring masses on both sides (see figure). The distance between two neighboring masses along the x axis is fixed (equal to d). We further define

$$\omega_0^2 \equiv \frac{K}{M}, \qquad \ell_0 \equiv \sqrt{y_0^2 + d^2}. \tag{4}$$

The masses are free to move in the y direction. When the masses with even ℓ are at $y = 0$ and the masses with odd ℓ are at $y = y_0$, the system is in equilibrium.

1. Write the terms in the Lagrangian for the ℓth mass. Consider both even and odd ℓ, but not the endpoints ($\ell = 0$ and $\ell = 2N$). Use the variable $z_\ell = (y_\ell - y_\ell^{\text{eq}})$, where y_ℓ^{eq} is the equilibrium position of the ith mass. Restrict your attention to the case of small displacements from equilibrium, $z_\ell \ll d, y_0$ and $d > y_0$.



2. Using that Lagrangian, show that the equation of motion for the y coordinate of the ℓth mass is given by
$$\frac{\partial^2 z_\ell}{\partial t^2} = a(z_{\ell+1} + z_{\ell-1} - 2z_\ell) + bz_\ell \tag{5}$$
and find what are a and b.

3. Next solve the equation. For this, "guess" a solution of the form
$$z_\ell = \sum_q e^{iq\ell}\tilde{z}(q) \tag{6}$$
and find the equation of motion for $\tilde{z}(q)$. You should express your result as
$$\frac{\partial^2 \tilde{z}(q)}{\partial t^2} = f(q)\tilde{z}(q) \tag{7}$$
that is, find what is $f(q)$.

4. Assuming that the masses with $\ell = 0$ and $\ell = 2N$ are held rigidly at $y = 0$, derive the spectrum of the waves on the mattress. That is, find the allowed frequencies $\omega(q)$.

5. When we take $N \to \infty$, we still cannot get $\omega \to \infty$. That is, the system has a UV cutoff. What is (roughly) the UV cutoff scale?

6. We now consider the limit $q \ll 1$. Show that in that limit the dispersion relation is given by that of a relativistic particle, that is, $E^2 = p^2 + m^2$. Yet, in Nature the maximum velocity is 1 (or c), while in our case it is given by the parameters of the mattress. Find the equivalent of the mass and the maximum velocity in our case. (Note, here $p^2 = p^\mu p_\mu = E^1 - \vec{p}^2$. Also, since $\hbar = 1$, we often replace E with ω.)

Question 3 :Lagrangian formalism of E&M

In mechanics, one can start from $L = ma$ or it can be derived form $L = mv^2/2 - V(x)$ based on the principle of minimal action. In this problem you will show that Maxwell's equations can be derived for a Lagrangian density and the principle of minimal action.

We start with some definitions. We define the vector potential and the electromagnetic tensor

$$A_\mu(x_\mu) = \{\phi(x_\mu), A_i(x_\mu)\}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{8}$$

$$\mu, \nu = 0, 1, 2, 3; \qquad i, j, k = 1, 2, 3. \tag{9}$$

The electric field (E) and magnetic field (B) are then

$$E_i = F^{i0}, \qquad B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk}, \tag{10}$$

where the ϵ tensor is defined by $\epsilon_{123} = 1$ and that it is antisymmetric under the exchange of any two indices. Explicitly, $F_{\mu\nu}$ is given by:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{11}$$

We further define the dual field strength tensor

$$\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}, \tag{12}$$

where $\epsilon^{\alpha\beta\mu\nu}$ is the completely antisymmetric rank 4 epsilon tensor. Clearly F is antisymmetric. Moreover, you can show (check it!) that

$$F_{\mu\nu}F^{\mu\nu} = 2\left(B^2 - E^2\right), \qquad \tilde{F}_{\mu\nu}F^{\mu\nu} = -4\,(B \cdot E). \tag{13}$$

The above two combinations are manifestly Lorentz invariant. We also define the four current

$$J^\alpha = (\rho, J_i). \tag{14}$$

We consider Maxwell's equations

$$\nabla \cdot E = \rho, \qquad \nabla \times B - \frac{\partial E}{\partial t} = J_i, \qquad \nabla \cdot B = 0, \qquad \nabla \times E + \frac{\partial B}{\partial t} = 0. \tag{15}$$

1. Show that the inhomogeneous equations (that is, the first two) can be written as

$$\partial_\alpha F^{\alpha\beta} = J^\beta. \tag{16}$$

2. We assume the following Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu, \tag{17}$$

and consider each of A_μ to be an independent DOF. Using the E-L equation show that this Lagrangian gives the two inhomogeneous equations.

Question 4 :Fermionic harmonic oscillator

In class we explain why a field is similar to the simple harmonic oscillator (SHO), and that that they are closely related to bosons. Then we mentioned that we should have a Fermionic field where its excitations corresponds to fermions. Here we discuss the simple case of a Fermionic harmonic oscillator.

You must be very familiar with. We define it with

$$H = \hbar\omega(N + 1/2), \qquad N = a^\dagger a, \tag{18}$$

with

$$[a, a^\dagger] = 1, \qquad [a, a] = [a^\dagger, a^\dagger] = 0. \tag{19}$$

Here we study a similar system that describe a Fermionic one. That is, there is no classical analog for it. Consider instead of the commutation relation of the SHO an anti-commutation relation

$$\{b, b^\dagger\} = 1, \qquad \{b, b\} = \{b^\dagger, b^\dagger\} = 0, \tag{20}$$

where $\{a, b\} = ab + ba$. For the Hamiltonian we have

$$H = \hbar\omega(N - 1/2), \qquad N = b^\dagger b. \tag{21}$$

1. Show that $b^2 = 0$.

2. Show that there is a state, which we denote by $|0\rangle$, that is annihilated by b.

3. We define $|1\rangle = b^\dagger|0\rangle$. Show that this state cannot vanish.

4. We assume that all possible operators must be composed of products of b and $b^\dagger$. Show that $|1\rangle$ and $|0\rangle$ span the Hilbert space. (One way to go is to show that there are a total of four operators. There is a theorm that stated that this number is equal to the dimension of the Hilbert space squared.)

5. What is the energy of each state?