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Phys 86-650: Homework Assignment # 2

Question 1 :Two level system

In class we discussed the idea that particles are just excitations of fields, which are, in a way, just SHOs. In this question you are asked to calculate transition probabilities in a case of a system with two SHOs that are coupled. This is a simple model of particle interaction.

We consider a system with two DOFs with

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{m\omega_1^2 x_1^2}{2} + \frac{m\omega_2^2 x_2^2}{2} + c\, x_1 x_2^2, \tag{1}$$

and we assume that c is small. We also assume that $\omega_1 = 2\omega_2$. In the following we consider only cases where the system is close to the ground state. Neglecting c we can write the spectrum as $|n_1, n_2\rangle$.

We recall that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad a|n\rangle = \sqrt{n}\,|n-1\rangle. \tag{2}$$

1. When can we treat c as small? Write your answer as a ratio $\ll 1$.

2. When $c = 0$, the states $|1,0\rangle$ and $|0,2\rangle$ are degenerate, and so small c will cause a splitting of these states. Use degenerate perturbation theory to show that the energy eigenstates are

$$\psi_{1,2} = \frac{1}{\sqrt{2}}\left(|1,0\rangle \pm |0,2\rangle\right), \tag{3}$$

   and calculate ΔE.

3. We now assume that $\psi(t=0) = |1,0\rangle$. Calculate the time dependant probability to make a measurement and find the system in the $|0,2\rangle$ state.

4. Consider small times and then write $P = (\Gamma t)^2$. In the case you can think of the situation as an $x_1 \to 2x_2$ decay with lifetime of order $\tau \equiv 1/\Gamma$. What is this “lifetime”?

Question 2 :Harmonic oscillator perturbation theory

Consider a system with 3 DOFs with the following Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{m\omega_x^2 x^2}{2} + \frac{p_y^2}{2m} + \frac{m\omega_y^2 y^2}{2} + \frac{p_z^2}{2m} + \frac{m\omega_z^2 z^2}{2} + \lambda_1 xyz + \lambda_2 x^2 z. \tag{4}$$

We further assume that $\omega_y = 3\omega_x$, $\omega_z \gg \omega_y$ and that λ_1 and λ_2 are small and thus can be treated as perturbation. We denote a state of the system as $|n_x, n_y, x_z\rangle$. In this question you are asked to use two ways to calculate the transition matrix element

$$\mathcal{A}(|0,1,0\rangle \to |3,0,0\rangle). \tag{5}$$

1. Use second order perturbation theory to show that

$$\mathcal{A} = c \times \frac{\lambda_1 \lambda_2}{\omega_z^2 - (2\omega_x)^2} \tag{6}$$

   and find what is c.

2. Use the Feynman diagram for Harmonic oscillator method to get the same result. For that, first draw the diagram and calculate it using the $\hbar = m = 1$ units. The amplitude is the product of the following factors

   (a) For each vertex multiply the amplitude by the corresponding coupling constant.

   (b) For each propagator use $-1/(E_z^2 - q^2)$ where E_z is the energy of the internal state and q is the energy that go out of it.

   (c) Use the correct normalization: (i) For each external state use $1/\sqrt{2\omega_i}$ and (ii) for each final state that appear n times multiply the amplitude by $\sqrt{n!}$ (this factor is called symmetry factor).

   Compare this result to the one you got in the first item.

Question 3 :More Harmonic oscillators

Consider a case with four oscillators, called w, x, y, z and assume that $y \to 2x2w$ is allowed by energy conservation. We use the following interaction

$$\lambda_1 y z^2 + \lambda_2 z x^2 + \lambda_3 z w^2. \tag{7}$$

1. Draw the diagram and calculate the transition matrix element using the Feynman rules and the correct normalization.

2. (Optional) Calculate the amplitude using perturbation theory. For that find the 6 intermediate states that contribute and calculate the 6 matrix elements and add them up. Verify that the result of the two methods agree.